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Hypothetical Theory of Anisotropy and the Bauschinger Effect due to Plastic Strain History*

By

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Summary. The present paper presents the theory of the fundamental problem how anisotropy and the Bauschinger effect, as well as work-hardening, of a metal are correlated with the previous strain history of its plastic deformation. The concept of the strain history tensor indispensable to introducing this theory is provided by the author's preceding paper [3].

The basic ideas and assumptions are all introduced in quite a reasonable way, and they are as follows:

(1) Isotropic state of a material is represented by the spherical unit tensor $\boldsymbol{I}$ of order 2.

(2) The yield function for the isotropic state is derived from $\boldsymbol{I}$ and the deviatoric stress tensor $\boldsymbol{T}'$, and is given, at least in approximation, by the Mises' yield function.

(3) Anisotropic state brought about by the strain history $\boldsymbol{E}$ is represented by the tensor $\boldsymbol{I}+A\boldsymbol{E}$, A being a scalar coefficient dependent not only on the extent of cold-working, but also on the strain history.

(4) The yield function for the anisotropic state resulting from the strain history $\boldsymbol{E}$ is obtained from $\boldsymbol{I}+A\boldsymbol{E}$ and $\boldsymbol{T}'$, by just the same rule as the Mises' yield function for the isotropic state has been derived from $\boldsymbol{I}$ and $\boldsymbol{T}'$.

(5) The Bauschinger effect is introduced into the yield function as the term of the form $B\boldsymbol{E}\cdot\cdot\boldsymbol{T}'$, which contain the strain history $\boldsymbol{E}$ and are linear with respect to $\boldsymbol{T}'$.

Based on this theory, experimental results on yielding and stress-strain relation as the complex of the respective effects of work-hardening, anisotropy and the Bauschinger effect can be analysed with legitimacy, and hence they are obtained individually. The theory was compared with experiments on yielding of mild steel tubes under the stress state of combined tension-torsion after they had been subjected to the strain histories of axial extension and twist, and it was found that the theory was capable of coordinating the yielding phenomena after various strain histories.

1. INTRODUCTION

Most metals come to reveal more or less strain history phenomena as anisotropy and the Bauschinger effect besides work-hardening as plastic deformation proceeds. Work-hardening has almost adequately been incorporated in the mathematical frame work of the plasticity theory, but as for anisotropy and the Bauschinger effect the circumstances are not so satisfactory. The theory of plastic anisotropy proposed by R. Hill [1] is that for a metal which is in a certain state of anisotropy due to some causes, and not the theory which brings the state of anisotropy into correlation with the previous history of deformation as cold working to which the metal has been subjected. And also the theories by other investigators are

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the same in this respect. The only theory concerning the Bauschinger effect in relation to the strain seems to be that by F. Edelman and D.C. Drucker [*2*]. But it cannot be said to be correct in view of the idea of strain used. Because the strain used in the theory is that which is specified by the change in the geometrical configuration of the body, and therefore is fitted for describing elastic deformation, in spite of the fact that the Bauschinger effect should be of the nature of being dependent on the strain history specified by the deformation path, but not on the strain of the above meaning.

The object of the present investigation is to build up a theory, which can explain our experimental facts concerning the dependence of anisotropy and the Bauschinger effect on the previous history of plastic deformation of the material, that is, a theory by which the state of anisotropy and the Bauschinger effect can be predicted from the deformation history. So long as mechanical theory is concerned, anisotropy is regarded as distortion of the yield surface from its shape in the isotropic state, and the Bauschinger effect as transposition and accompanying shape change of the yield surface, due to deformation history. Accordingly, for the purpose of solving our subject mentioned above, it is required that the deformation history is represented by some mechanical quantity, which is possible to be incorporated in the frame work of the plasticity theory. And indeed it is one of the key points for the solution of the problem whether such quantity can be introduced or not, now that the strain usually employed and specified by the change in the geometrical configuration is sure not to serve as such. So we shall now begin with the mechanism, by which anisotropy and the Bauschinger effect are effected, and then basing on it, proceed to the problem of introducing the mechanical quantity representing the history.

To the present day knowledge of general endorsement, anisotropy of metals is considered to be attributable to one or the complex of several causes such as the fibrous texture, the preferred orientation of crystal grains and others due to cold-working. But I can not but doubt whether the preferred orientation is really possible under a uniform deformation of metals in the macroscopic sense, in which individual crystal grains are supposed to be subjected to the almost equal deformation. I think that one of the main causes of anisotropy is the group pattern of dislocations in crystals. At any rate it is sure that *anisotropy occurs as the result of some micro-structural change of the material due to a sequence of successive infinitesimal slips* necessary for the plastic deformation to take place. *And the micro-structure of the material in the deformed state, even if it is of the same geometrical configuration, should not be the same according to the process of slip, i.e. the deformation path, up to the state. Thus the mechanical state of a plastic body specified by its micro-structure is considered to be dependent on the previous slip process, but not on its geometrical configuration directly.* The same thing can be said as for the Bauschinger effect, whether its cause may be the internal stress or the back stress due to the piling up of dislocations.

Considering the matter in this way, it can be seen that the mechanical quantity representing the mechanical state of deformation is given by the integration of the strain increment corresponding to the successive infinitestimal deformations due to slips. According to the plasticity theory, the slip system active in course of the infinitesimal deformation is such as to satisfy the so-called minimum slip principle. But no matter what the active slip system may be, the change in the mechanical state brought about by it is considered to be represented by the strain increment, corresponding to the infinitesimal deformation caused by the slip system. In this case, the strain increment from t to $t+dt$, t being the time or a parameter representing the extent of deformation, is needed to be defined such that the current deformed state t is at the same time an undeformed state with no strain. This is because not only the plastic deformation is essentially of such nature, but also the strain as a result of integration of such strain increment is shown to be the strain history tensor itself dependent on the integration path, i.e. the deformation history. Such strain increment and strain history tensor have already been introduced in my preceding paper [*3*] as $\boldsymbol{DE}$ and $\boldsymbol{E}$ in the form quite legitimate from both the mathematical and physical view-points. *Thus we are naturally led to the belief that it is reasonable to use this strain history tensor* $\boldsymbol{E}$ *as the mechanical quantity representing the mechanical state dependent on deformation history.*

The other reason we must adopt as strain the strain history tensor $\boldsymbol{E}$ *is that, without having recourse to it, we cannot express the plastic work in the form*

$$DW = \boldsymbol{T}\cdot\cdot\boldsymbol{DE} = \sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu} \tag{1.1}$$

inclusive also of the finite deformations, and therefore the plastic potential cannot, at the same time, be the yield function, where $\boldsymbol{T}$ indicates the stress tensor defined in the preceding paper [*3*], which corresponds to the so-called true stress in simple tension, $\boldsymbol{DE}$ the strain increment tensor and $\sigma^{\lambda\mu}$ and $(D\varepsilon)_{\lambda\mu}$ their components respectively referred to an appropriate reference frame. Accordingly, if the yield function $f(\boldsymbol{T}, \boldsymbol{E})$ involving the strain history tensor $\boldsymbol{E}$ were obtained in any way for the material exhibiting anisotropy and the Bauschinger effect, the stree-strain increment relation for the material is seen to be also obtained. In the present paper, I shall attempt to deduce the yield function $f(\boldsymbol{T}, \boldsymbol{E})$ by the aid of the concept of the strain history tensor and some assumptions, which seem quite reasonable. And as this yield function $f(\boldsymbol{T}, \boldsymbol{E})$ is of the nature of being derived, basing on the yield function $f(\boldsymbol{T})$ for the initial isotropic state, it is necessary for us to know the latter first.

2. Yield Criterion of Metals in Isotropic State

Our object in this section is to clarify the yield criterion (or flow criterion) of metals in the initial isotropic state. But this is not necessarily possible, apart from the notable yielding of such special metals as mild steel which obeys the so-called Tresca's criterion, because of the fact that all of the experiments ever performed in the past, and perhaps in future too, concerning yield criterion of usual metals are those for the state more or less cold-worked, on account of their having no clear yield point in the annealed state, and consequently for the state not isotropic in general, owing to the strain history. If there exists any case where the isotropy is maintained still after plastic deformation, that would be by chance. For this reason, the yield criterion for the initial isotropic state must be obtained, by the aid of some theory connecting work-hardening, anisotropy and the Bauschinger effect with strain history, from the experimental results for more or less cold-worked state. And, in order to develop such a theory, it is necessary for us to have conversely had a good knowledge about the yield criterion for the isotropic state, as was stated previously. On account of the mutual interference between these two requirements, we can find no means other than to develop the theory on some assumption on the yield criterion for isotropic state, for its validity to be examined in comparison with experiments. The existing experiments, and accordingly the yield criterion derived from them for the isotropic state, should possibly be reconsidered from this view-point.

Among a number of experiments concerning the yield criterion of metals, those of W. Lode (1926) [4] and of G.I. Taylor and H. Quinney (1931) [5] are most noted. And these also are no exceptions in respect that they are concerned with the state already cold-worked.

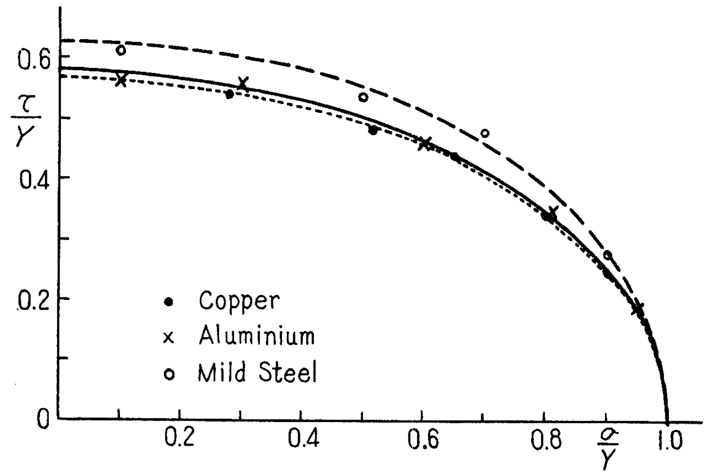

FIGURE 1.

The experiment of G.I. Taylor and H. Quinney is the one concerning the yield criterion of thin circular tubes under combined tension-torsion after subjected to axial elongation, and its result is shown in Fig. 1. In this experiment they paid special attention as to whether anisotropy was produced or not by the initial elongation. Namely they observed that the internal volume of the axially elongated tubes varied during the deformation under the combined tension-torsion, and accordingly showed that the extensional strains in the radial and circumferential directions of the tubes were different in this case. By comparing the principal direction of this strain thus obtained from experiment with that of the applied stress of combined tension-torsion, they found that they showed fairly good agreement, the maximum difference between them being 1.9° and the average 0.64°. Judging from this result, they put forward that the tubes were isotropic even after the initial elongation. But I think it is doubtful whether we can conclude from this that the tubes are really isotropic, because the difference of this extent between the principal directions can also be seen between those of the strains for the two cases when the radial and the circumferential elongations are regarded as equal and when they are regarded as different on account of the volume change.

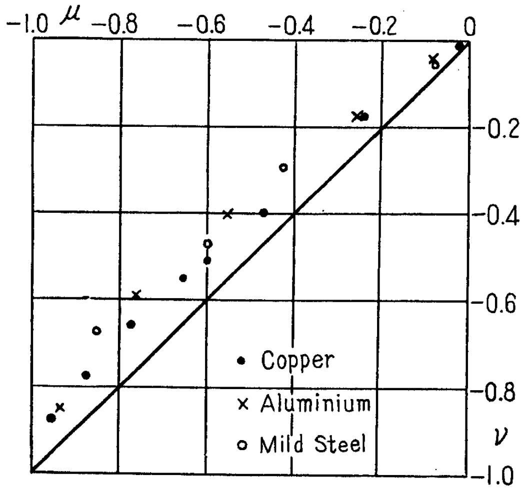

FIGURE 2.

On the other hand, Fig. 2 shows the relation between the Lode's parameters μ and ν obtained by Taylor and Quinney from the measurements of the axial elongation, twist and volume change of the tubes. If the material were isotropic, the relation $\mu=\nu$ would be equivalent to the statement that the yielding obeys the so-called Mises' criterion, and the deviation of the $\mu\sim\nu$ relation from $\mu=\nu$ corresponds to that of the actual yield criterion from the Mises' one. For non-isotropic state of materials, it is meaningless to consider the $\mu\sim\nu$ relation.

From the observed $\mu\sim\nu$ relation for copper, G.I. Taylor (1947) [*6*] calculated its yield locus according to the maximum work principle, assuming that it is isotropic, i.e. that the principal axes of both strain and stress are coincident, and obtained the result shown by the dotted line in Fig. 1, which shows good agreement with the yield points obtained directly from the experiments. But, for alminium, the two loci, one from the $\mu\sim\nu$ relation and the other from the direct method, show small discrepancy, and for mild steel they have quite different tendencies. This is considered to be no other than show that the condition of isotropy is not fulfilled for the mild steel as against the copper. And if this is really true, it may perhaps be due to the fact that the pure metals, as copper, do not so work-harden as to show a notable anisotropy, whereas the alloys, as mild steel, exhibit anisotropy as well as work-hardening on account of foreign atoms operating as obstacles to the motion of dislocations.

From the above reasoning it seems probable that metals, particularly of the face-centered cubic crystals, have in their isotropic state the yield criterion represented by the locus of dotted line in Fig. 1, which lies inside of, but very close to, the Mises' criterion. This result seems to be also supported by the calculated result on the yield surface obtained for polycrystalline face-centered metals by

J.F.W. Bishop and R. Hill [7]. But since the calculation of this kind cannot be expected to have enough accuracy, it is considered to have a qualitative meaning alone. At any rate, the discrepancy of the yield criterion embodied by the dotted line in Fig. 1 from the Mises' criterion being very small, it may possibly be replaced by the Mises' criterion itself as the first approximation.

3. THE MISES' YIELD CRITERION AND ITS RELATION TO ISOTROPIC PROPERTY OF MATERIALS

According to the result deduced in the preceding section, we can assume the Mises' criterion as that for the yielding (or plastic flow) of isotropic metals. The yield criterion for non-isotropic state of metals due to strain history is possible to be derived on the basis of this assumption. The validity of this assumption and of the strain history theory thus introduced should be decided by the comparison of deduced results of the theory with experimental facts.

For the purpose of introducing the strain history theory for the yield criterion, it is first necessary to give the Mises' criterion an appropriate mathematical expression. And further for this purpose, some coordinate system is required to be introduced. And as such, of course, any one may serve. For the theoretical treatment of the problem, however, is most fitted the Lagrangian coordinate system which was already introduced in the preceding paper [3]. According to the Lagrangian method, the Mises' yield function is given by the second scalar invariant of the deviatoric stress tensor

$$\boldsymbol{T}'=\sigma'^{\lambda\mu}\boldsymbol{e}_\lambda\boldsymbol{e}_\mu=\sigma'_{\lambda\mu}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu \tag{3.1}$$

i.e. by

$$\begin{aligned} f(\sigma^{\lambda\mu})&=\frac{1}{2}\boldsymbol{T}'\cdot\cdot\boldsymbol{T}'=\frac{1}{2}\sigma'^{\lambda\mu}\sigma'_{\lambda\mu}\\ &=\frac{1}{2}g_{\lambda\iota}g_{\mu\kappa}\sigma'^{\lambda\mu}\sigma'^{\iota\kappa}=\frac{1}{2}g_{\lambda\kappa}g_{\mu\iota}\sigma'^{\lambda\mu}\sigma'^{\iota\kappa} \end{aligned} \tag{3.2}$$

where the dots "$\cdot\cdot$" mean the double scalar product of two dyadics, which may otherwise be expressed as Spur of product of the two tensors and $g_{\lambda\mu}$ the metric fundamental tensor. Here, it must be noticed that the usual expression

$$f(\sigma^{\lambda\mu})=\frac{1}{2}\sigma'_{\lambda\mu}\sigma'_{\lambda\mu}$$

given in many text books, which makes no distinction between the contra- and co-variant components of the stress tensor, is applicable only to the case of rectangular Cartesian coordinate system, and accordingly it does not serve as any clue to our theory. Setting

$$g'_{\lambda\mu\iota\kappa}=\frac{1}{2}(g_{\lambda\iota}g_{\mu\kappa}+g_{\lambda\kappa}g_{\mu\iota}) \tag{3.3}$$

we can write (3.2) in the form

$$f(\sigma^{\lambda\mu})=\frac{1}{2}g'_{\lambda\mu\iota\kappa}\sigma'^{\lambda\mu}\sigma'^{\iota\kappa}. \tag{3.4}$$

Since $g_{\lambda\mu}$ is symmetric with respect to λ, μ, the relations

$$g'_{\lambda\mu\iota\kappa}=g'_{\mu\lambda\iota\kappa}=g'_{\lambda\mu\kappa\iota}=g'_{\mu\lambda\kappa\iota}=g'_{\iota\kappa\lambda\mu} \tag{3.5}$$

are seen to hold. It is also found from (3.3) that

$$g'_{\lambda\mu\iota\kappa}g^{\lambda\mu}=g_{\iota\kappa}, \quad g'_{\lambda\mu\iota\kappa}g^{\iota\kappa}=g_{\lambda\mu} \tag{3.6}$$

hold.

Now we introduce the tensor of order 4

$$\boldsymbol{J}'=g'_{\lambda\mu\iota\kappa}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\boldsymbol{e}^{\iota}\boldsymbol{e}^{\kappa} \tag{3.7}$$

which has $g'_{\lambda\mu\iota\kappa}$ given by (3.3) as the components referred to the bacic tensors $\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\boldsymbol{e}^{\iota}\boldsymbol{e}^{\kappa}$. Then it is easily found that $\boldsymbol{J}'$ is a tensor which transforms any tensor of order 2 to itself, i.e. such that

$$\boldsymbol{J}'\cdot\cdot\boldsymbol{T}'=\boldsymbol{T}' \tag{3.8}$$

for $\boldsymbol{T}'$ for example. This is no other than show that the tensor $\boldsymbol{J}'$ is the spherical unit tensor of order 4 just as

$$\boldsymbol{I}=g_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\lambda} \tag{3.9}$$

consisting of the metric fundamental tensor $g_{\lambda\mu}$ is the spherical unit tensor of order 2. Making use of such $\boldsymbol{J}'$, we can write (3.4) in the form

$$f(\sigma^{\lambda\mu})=\frac{1}{2}\boldsymbol{T}'\cdot\cdot\boldsymbol{J}'\cdot\cdot\boldsymbol{T}'. \tag{3.10}$$

That the Mises' function is constructed from $\boldsymbol{T}'$ by means of $\boldsymbol{J}'$ which is of spherical symmetry is considered to be the cause that it represents the yield criterion for the isotropic state. Of course, some yield functions other than the Mises' one are also possible for isotropic state, as was mentiond previously. And if these were given mathematical expressions, they would also be composed only of the stress tensor $\boldsymbol{T}'$ and of the spherical tensor, though have more complicated form than (3.10). At any rate, *the property of spherical symmetry of* $\boldsymbol{J}'$*, which specifies the isotropic features of* (*3.10*)*, is attributable to that of tensor* $\boldsymbol{I}$*, given by* (*3.9*) *from which* $\boldsymbol{J}'$ *is derived according to the rule* (*3.3*). That is, the isotropic state of the material defined by the Mises' function (3.10) is reducible to the spherical tensor $\boldsymbol{I}$ itself: $\boldsymbol{I}$ is the very mathematical expression of the state of isotropy of the material. Thus we can see it to be clarified that *the yield function* (*3.10*) *for the isotropic state of the material is derived from the tensor* $\boldsymbol{I}$ *representing the very state, according to the rule* (*3.3*). And this conclusion will be seen, in the following section, to serve as a guiding principle for introducing the theory, we aim at, which describes the state of anisotropy and the Bauschinger effect in their dependence on the strain history of the material.

Before proceeding to the main subject, we will now give the expression of the Mises' yield function in terms of the stress tensor

$$\boldsymbol{T}=\sigma^{\lambda\mu}\boldsymbol{e}_{\lambda}\boldsymbol{e}_{\mu}, \tag{3.11}$$

not of the deviatoric stress tensor $\boldsymbol{T}'$, which will become necessary afterwards for introducing the mechanical equation of state. The hydrostatic tension is defined by

$$\bar{\sigma}=\frac{1}{3}\sigma^{\lambda\mu}g_{\lambda\mu}, \tag{3.12}$$

and the deviatoric stress tensor by

$$\sigma'^{\lambda\mu}=\sigma^{\lambda\mu}-\bar{\sigma}g^{\lambda\mu}. \tag{3.13}$$

Substituting this into (3.4), we have

$$f(\sigma^{\lambda\mu})=\frac{1}{2}(g'_{\lambda\mu\iota\kappa}\sigma^{\lambda\mu}\sigma^{\iota\kappa}-3\bar{\sigma}^2), \tag{3.14}$$

therefore, by using (3.12),

$$f(\sigma^{\lambda\mu})=\frac{1}{2}\left(g'_{\lambda\mu\iota\kappa}-\frac{1}{3}g_{\lambda\mu}g_{\iota\kappa}\right)\sigma^{\lambda\mu}\sigma^{\iota\kappa}. \tag{3.15}$$

Putting

$$g_{\lambda\mu\iota\kappa}=g'_{\lambda\mu\iota\kappa}-\frac{1}{3}g_{\lambda\mu}g_{\iota\kappa}, \tag{3.16}$$

we can write (3.15) in the form

$$f(\sigma^{\lambda\mu})=\frac{1}{2}g_{\lambda\mu\iota\kappa}\sigma^{\lambda\mu}\sigma^{\iota\kappa}. \tag{3.17}$$

If we consider the tensor

$$\boldsymbol{J}=g_{\lambda\mu\iota\kappa}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\boldsymbol{e}^{\iota}\boldsymbol{e}^{\kappa} \tag{3.18}$$

as in the case of $\boldsymbol{J}'$, (3.17) can be written in the form

$$f(\sigma^{\lambda\mu})=\frac{1}{2}\boldsymbol{T}\cdot\cdot\boldsymbol{J}\cdot\cdot\boldsymbol{T} \tag{3.19}$$

irrelevant to the coordinate system. It is the same as for the case of $g'_{\lambda\mu\iota\kappa}$ that the relations

$$g_{\lambda\mu\iota\kappa}=g_{\mu\lambda\iota\kappa}=g_{\lambda\mu\kappa\iota}=g_{\mu\lambda\kappa\iota}=g_{\iota\kappa\lambda\mu} \tag{3.20}$$

and

$$g_{\lambda\mu\iota\kappa}g^{\lambda\mu}=0,\quad g_{\lambda\mu\iota\kappa}g^{\iota\kappa}=0 \tag{3.21}$$

hold as to $g_{\lambda\mu\iota\kappa}$.

Substituting (3.16) into (3.18), and then using (3.9), we can write $\boldsymbol{J}$ in the form

$$\boldsymbol{J}=\boldsymbol{J}'-\frac{1}{3}\boldsymbol{I}\boldsymbol{I} \tag{3.22}$$

corresponding to (3.16). The tensor $\boldsymbol{J}$ is seen to be of spherical symmetry, though not unit tensor, because of the fact that it concerns only the spherical unit tensor $\boldsymbol{I}$ of order 2. For this reason, the Mises' yield function (3.19) expressed in terms of the stress $\boldsymbol{T}$ is also shown to be that for the isotropic state of the material represented by the tensor $\boldsymbol{I}$ of order 2.

4. Yield Function for the State of Anisotropy and the Bauschinger Effect due to Strain History

What has been obtained as a conclusion in the preceding section is that the state of isotropy is represented by the unit tensor $\boldsymbol{I}$ of order 2, and the Mises' yield function for the state is given by (3.10) or (3.19) by means of the spherical tensors $\boldsymbol{J}'$ or $\boldsymbol{J}$ of order 4, derived from $\boldsymbol{I}$ by (3.3) or (3.16). We will now enterprise in the following to extend this result to the general case where the material exhibits anisotropy and the Bauschinger effect due to strain history.

The first problem for solution is by what mechanical quantity a plastically deformed state is expressed. It was already mentioned in Introduction that what specifies the mechanical state as anisotropy etc. of materials after plastic deformation is not the change in the geometrical configuration due to the deformation, but the change in their micro-structure such as the group pattern of dislocations, which depends on the path of deformation up to the state. Accordingly the mechanical quantity which represents such mechanical state ought to be the strain history tensor $\boldsymbol{E}$, introduced in my preceding paper [*3*], which depends on the deformation path. Thus *it may be quite natural to consider that the mechanical state deviating by* $A\boldsymbol{E}$ (*A: scalar*) *from the isotropic state with no strain, due to the strain history* $\boldsymbol{E}$, *is expressed by the tensor*

$$\boldsymbol{I}+A\boldsymbol{E}\equiv(g_{\lambda\mu}+A\varepsilon_{\lambda\mu})\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu} \tag{4.1}$$

just as the isotropic state was expressed by $\boldsymbol{I}$. What must be particularly remarked here is that the expression (4.1) is impossible by the usual strain which finds legitimate application in the elasticity theory, and therefore is specified by the change in the geometrical configuration. The scalar A is a coefficient representing the rate of development of anisotropy with plastic deformation. It is fortunate if A be a scalar function only of the extent of plastic deformation, i.e. the plastic work W or the second scalar invariant of $\boldsymbol{E}$, say. But A may or may not be so; it may possibly be a scalar function of the strain history tensor $\boldsymbol{E}$ itself. If so, the extent of anisotropy becomes dependent on the deformation path as well as the amount of deformation.

The next problem which confronts us is what is the yield function which corresponds to the state of anisotropy represented by (4.1). This question can be answered by the following assumption. That is, we postulate that *the yield function for anisotropic state is derived from the anisotropic tensor* $\boldsymbol{I}+A\boldsymbol{E}$ *in just the same procedure as in the case where Mises' yield function for isotropic state has been derived from the isotropic tensor* $\boldsymbol{I}$. This assumption is really an assumption, but it seems so reasonable from logic that it can not be said as such. On the belief of this conception, *the tensor* $\boldsymbol{C}'$ *of order 4 specifying the yield function for anisotropic state, which corresponds to the tensor* $\boldsymbol{J}'$ *for isotropic state, is obtained from* $\boldsymbol{I}+A\boldsymbol{E}$ ((*4.1*)) *by the same rule as* (*3.3*), *in the form*

$$\left.\begin{aligned}\boldsymbol{C}'&=c'_{\lambda\mu\iota\kappa}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\boldsymbol{e}^{\iota}\boldsymbol{e}^{\kappa},\\ c'_{\lambda\mu\iota\kappa}&=\frac{1}{2}[(g_{\lambda\iota}+A\varepsilon_{\lambda\iota})(g_{\mu\kappa}+A\varepsilon_{\mu\kappa})\\ &\qquad+(g_{\lambda\kappa}+A\varepsilon_{\lambda\kappa})(g_{\mu\iota}+A\varepsilon_{\mu\iota})].\end{aligned}\right\} \tag{4.2}$$

By virtue of this tensor, the yield function for the anisotropic state $\boldsymbol{I}+A\boldsymbol{E}$ *is considered to be given by*

$$f(\sigma^{\lambda\mu},\varepsilon_{\lambda\mu})=\frac{1}{2}\boldsymbol{T}'\cdot\cdot\boldsymbol{C}'\cdot\cdot\boldsymbol{T}'=\frac{1}{2}c'_{\lambda\mu\iota\kappa}\sigma'^{\lambda\mu}\sigma'^{\iota\kappa}. \tag{4.3}$$

The tensor components $c'_{\lambda\mu\iota\kappa}$ in (4.2) is seen to be composed of the terms, independent of A and involving A and A^2 respectively, such that

$$c'_{\lambda\mu\iota\kappa}=g'_{\lambda\mu\iota\kappa}+AL'_{\lambda\mu\iota\kappa}+A^2M'_{\lambda\mu\iota\kappa}, \tag{4.4}$$

where $g'_{\lambda\mu\iota\kappa}$ is given by (3.3), and

$$L'_{\lambda\mu\iota\kappa}=\frac{1}{2}(g_{\lambda\iota}\varepsilon_{\mu\kappa}+g_{\mu\kappa}\varepsilon_{\lambda\iota}+g_{\lambda\kappa}\varepsilon_{\mu\iota}+g_{\mu\iota}\varepsilon_{\lambda\kappa}), \tag{4.5}$$

$$M'_{\lambda\mu\iota\kappa}=\frac{1}{2}(\varepsilon_{\lambda\iota}\varepsilon_{\mu\kappa}+\varepsilon_{\lambda\kappa}\varepsilon_{\mu\iota}). \tag{4.6}$$

If $A=0$ in particular, the yield function (4.3) is reduced to the Mises' function (3.10), being

$$c'_{\lambda\mu\iota\kappa}=g'_{\lambda\mu\iota\kappa}. \tag{4.7}$$

The tensors $L'_{\lambda\mu\iota\kappa}$ and $M'_{\lambda\mu\iota\kappa}$ are seen to consist of the linear and the second order terms with respect to the strain history $\varepsilon_{\lambda\mu}$. Being given the strain history $\varepsilon_{\lambda\mu}$, the yield function (4.3) is seen to be determined, apart from the scalar coefficient A representing the extent of anisotropy. It will be needless to mention that the components $L'_{\lambda\mu\iota\kappa}$ and $M'_{\lambda\mu\iota\kappa}$ are also referred to the basic tensors $\boldsymbol{e}^\lambda\boldsymbol{e}^\mu\boldsymbol{e}^\iota\boldsymbol{e}^\kappa$ as in the case of $c'_{\lambda\mu\iota\kappa}$. Namely, putting

$$\boldsymbol{L}'=L'_{\lambda\mu\iota\kappa}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu\boldsymbol{e}^\iota\boldsymbol{e}^\kappa, \tag{4.8}$$

$$\boldsymbol{M}'=M'_{\lambda\mu\iota\kappa}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu\boldsymbol{e}^\iota\boldsymbol{e}^\kappa, \tag{4.9}$$

we can write $\boldsymbol{C}'$ in the form

$$\boldsymbol{C}'=\boldsymbol{J}'+A\boldsymbol{L}'+A^2\boldsymbol{M}'. \tag{4.10}$$

The tensor $\boldsymbol{C}'$ is seen to satisfy the symmetry relations such that

$$c'_{\lambda\mu\iota\kappa}=c'_{\mu\lambda\iota\kappa}=c'_{\lambda\mu\kappa\iota}=c'_{\mu\lambda\kappa\iota}=c'_{\iota\kappa\lambda\mu} \tag{4.11}$$

because of $g_{\lambda\mu}$ and $\varepsilon_{\lambda\mu}$ being symmetric.

The foregoing considerations as for strain history effects have been restricted to anisotropy alone. And now we are in the stage to introduce the Bauschinger effect as other one of the strain history effects. The Bauschinger effect exhibits itself not only in the case when the direction of loading is reversed, but also when it is altered. While anisotropy is the change in the shape of the yield surface, the Bauschinger effect is considered to be the change in both its center and size. *Accordingly in the yield function $f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})$ it ought to be expressed by a scalar invariant of first order with respect to the stress tensor $\sigma'^{\lambda\mu}$, i.e. by*

$$BB'_{\lambda\mu}\sigma'^{\lambda\mu} \tag{4.11}$$

with some tensor $B'_{\lambda\mu}$ and scalar B. B may generally be a scalar function of the strain history $\boldsymbol{E}$. *In order that (4.11) may be a scalar invariant and further $B'_{\lambda\mu}$ be a function of the strain history $\varepsilon_{\lambda\mu}$, we cannot but put*

$$B'_{\lambda\mu}=\varepsilon_{\lambda\mu}. \tag{4.12}$$

Thus the terms representing the Bauschinger effect in the yield function are seen to be given by

$$B\varepsilon_{\lambda\mu}\sigma'^{\lambda\mu}. \tag{4.13}$$

This expression apparently seems identical with that already introduced by F. Edelman and D.C. Drucker [2] as representing the Bauschinger effect. But it must be particularly noticed that our expression (4.13) is distinguished from theirs in the point that our $\varepsilon_{\lambda\mu}$ is the strain history tensor dependent on deformation path, while theirs is the usual strain independent of it. And what is important is that the strain history effect such as the Bauchinger effect is possible to be described only by the strain history tensor. By virtue of (4.13), the yield function taking account of the Bauschinger effect as well as anisotropy is given by

$$\begin{aligned} f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu}) &= \frac{1}{2}\boldsymbol{T}' \cdot\cdot \boldsymbol{C}' \cdot\cdot \boldsymbol{T}' - B\boldsymbol{E} \cdot\cdot \boldsymbol{T}' \\ &= \frac{1}{2}c'_{\lambda\mu\iota\kappa}\sigma'^{\lambda\mu}\sigma'^{\iota\kappa} - B\varepsilon_{\lambda\mu}\sigma'^{\lambda\mu}. \end{aligned} \tag{4.14}$$

The negative sign before B in (4.14) is to make $B>0$.

Since the yield function (4.3), or more generally (4.14), is expressed in an invariant form, it holds for any coordinate system in the same form. For example, for the local coordinate system introduced in the preceding paper [3], it assumes the form

$$f(\boldsymbol{T}, \boldsymbol{E}) = \frac{1}{2}c'_{ijkl}\sigma'^{ij}\sigma'^{kl} - B\varepsilon_{ij}\sigma'^{ij}, \tag{4.15}$$

where

$$c'_{ijkl} = g'_{ijkl} + AL'_{ijkl} + A^2M'_{ijkl} \tag{4.16}$$

and

$$g'_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}), \tag{4.17}$$

$$L'_{ijkl} = \frac{1}{2}(g_{ik}\varepsilon_{jl} + g_{jl}\varepsilon_{ik} + g_{il}\varepsilon_{jk} + g_{jk}\varepsilon_{il}), \tag{4.18}$$

$$M'_{ijkl} = \frac{1}{2}(\varepsilon_{ik}\varepsilon_{jl} + \varepsilon_{il}\varepsilon_{jk}). \tag{4.19}$$

If the local coordinate system is rectangular Cartesian in particular, then $g_{ij}=\delta_{ij}$, and the above components are considerably simplified. We have the similar expression of the yield function for the Eulerian coordinate system too, so long as it is concerned with the same material element.

The above expressions of the yield function are all referred to the deviatoric stress tensor $\boldsymbol{T}'$, and now it is needed to give its expression in terms of the stress tensor $\boldsymbol{T}$, for the purpose of obtaining the mechanical equation of state later on. Substituting (3.13) and (3.12) into (4.14), and then setting

$$\left.\begin{aligned} \boldsymbol{C} &= c_{\lambda\mu\iota\kappa}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\boldsymbol{e}^{\iota}\boldsymbol{e}^{\kappa}, \\ c_{\lambda\mu\iota\kappa} &= c'_{\lambda\mu\iota\kappa} - \frac{1}{3}c'_{\lambda\mu\alpha\beta}g^{\alpha\beta}g_{\iota\kappa} - \frac{1}{3}c'_{\alpha\beta\iota\kappa}g^{\alpha\beta}g_{\lambda\mu} \\ &\quad + \frac{1}{9}c'_{\alpha\beta\gamma\delta}g^{\alpha\beta}g^{\gamma\delta}g_{\lambda\mu}g_{\iota\kappa}, \end{aligned}\right\} \tag{4.20}$$

we have, by virtue of $\varepsilon_{\lambda\mu}g^{\lambda\mu}=0$ the expression

$$f(\boldsymbol{T},\boldsymbol{E})=\frac{1}{2}\boldsymbol{T}\cdot\cdot\boldsymbol{C}\cdot\cdot\boldsymbol{T}-B\boldsymbol{E}\cdot\cdot\boldsymbol{T}$$

$$=\frac{1}{2}c_{\lambda\mu\iota\kappa}\sigma^{\lambda\mu}\sigma^{\iota\kappa}-B\varepsilon_{\lambda\mu}\sigma^{\lambda\mu}. \tag{4.21}$$

Here the tensor $\boldsymbol{C}$ satisfies the same symmetry relations

$$c_{\lambda\mu\iota\kappa}=c_{\mu\lambda\iota\kappa}=c_{\lambda\mu\kappa\iota}=c_{\mu\lambda\kappa\iota}=c_{\iota\kappa\lambda\mu} \tag{4.22}$$

as in the case of $\boldsymbol{C}'$. Substituting the $c'_{\lambda\mu\iota\kappa}$ of (4.4), into the right-hand side of (4.20), and then putting the respective parts consisting of $g'_{\lambda\mu\iota\kappa}$, $L'_{\lambda\mu\iota\kappa}$ and $M'_{\lambda\mu\iota\kappa}$ by $g_{\lambda\mu\iota\kappa}$, $L_{\lambda\mu\iota\kappa}$ and $M_{\lambda\mu\iota\kappa}$, we obtain

$$c_{\lambda\mu\iota\kappa}=g_{\lambda\mu\iota\kappa}+AL_{\lambda\mu\iota\kappa}+A^2M_{\lambda\mu\iota\kappa}\,, \tag{4.23}$$

$$g_{\lambda\mu\iota\kappa}=g'_{\lambda\mu\iota\kappa}-\frac{1}{3}g_{\lambda\mu}g_{\iota\kappa}\,, \tag{4.24}$$

$$L_{\lambda\mu\iota\kappa}=L'_{\lambda\mu\iota\kappa}-\frac{2}{3}(\varepsilon_{\lambda\mu}g_{\iota\kappa}+\varepsilon_{\iota\kappa}g_{\lambda\mu})\,, \tag{4.25}$$

$$\begin{aligned}M_{\lambda\mu\iota\kappa}=M'_{\lambda\mu\iota\kappa}&-\frac{1}{3}M'_{\lambda\mu\alpha\beta}g^{\alpha\beta}g_{\iota\kappa}-\frac{1}{3}M'_{\alpha\beta\iota\kappa}g^{\alpha\beta}g_{\lambda\mu}\\&+\frac{1}{9}M'_{\alpha\beta\gamma\delta}g^{\alpha\beta}g^{\gamma\delta}g_{\lambda\mu}g_{\iota\kappa}\,.\end{aligned} \tag{4.26}$$

The $g_{\lambda\mu\iota\kappa}$ of (4.24) is identical with that which has already been given by (3.16), and represents the part of $c_{\lambda\mu\iota\kappa}$ corresponding to the Mises' yield function. The expression (4.21) of the yield function by means of the stress tensor being independent of the hydrostatic pressure, the relation

$$f(\boldsymbol{T},\boldsymbol{E})=\frac{1}{2}c_{\lambda\mu\iota\kappa}\sigma^{\lambda\mu}\sigma^{\iota\kappa}-B\varepsilon_{\lambda\mu}\sigma^{\lambda\mu}$$

$$=\frac{1}{2}c_{\lambda\mu\iota\kappa}(\sigma^{\lambda\mu}-pg^{\lambda\mu})(\sigma^{\iota\kappa}-pg^{\iota\kappa})-B\varepsilon_{\lambda\mu}(\sigma^{\lambda\mu}-pg^{\lambda\mu})$$

is shown to hold. It must be remarked that the relation of this form does not hold for the expression (4.14) by the deviatoric stress tensor.

The expression (4.21) of the yield function being of invariant form, it holds for any coordinate system in the same form. For the local coordinate system for instance, it is expressed as

$$f(\boldsymbol{T},\boldsymbol{E})=\frac{1}{2}c_{ijkl}\sigma^{ij}\sigma^{kl}-B\varepsilon_{ij}\sigma^{ij}, \tag{4.27}$$

where

$$c_{ijkl}=g_{ijkl}+AL_{ijkl}+A^2M_{ijkl}\,, \tag{4.28}$$

$$g_{ijkl}=g'_{ijkl}-\frac{1}{3}g_{ij}g_{kl}\,, \tag{4.29}$$

$$L_{ijkl}=L'_{ijkl}-\frac{2}{3}(\varepsilon_{ij}g_{kl}+\varepsilon_{kl}g_{ij}), \tag{4.30}$$

$$M_{ijkl}=M'_{ijkl}-\frac{1}{3}M'_{ijrs}g^{rs}g_{kl}-\frac{1}{3}M'_{rskl}g^{rs}g_{ij}$$
$$+\frac{1}{9}M'_{rsmn}g^{rs}g^{mn}g_{ij}g_{kl}. \tag{4.31}$$

In the case when the material reveals neither anisotropy nor the Bauschinger effect in particular, for all plastic deformation, we can put $A=0$, $B=0$ and all the expressions (4.3), (4.14), (4.15), (4.21) and (4.27) are reduced to the Mises' yield function.

For the sake of convenience for the later application of our present theory to some practical cases, the tensor components $g'_{\lambda\mu\iota\kappa}$, $L'_{\lambda\mu\iota\kappa}$ and $M'_{\lambda\mu\iota\kappa}$ which are necessary for the calculation of $c'_{\lambda\mu\iota\kappa}$ are shown in Tables 1, 2 and 3. These components

TABLE 1. $g'_{\lambda\mu\iota\kappa}=\frac{1}{2}(g_{\lambda\iota}g_{\mu\kappa}+g_{\lambda\kappa}g_{\mu\iota})$

	σ'^{11}	σ'^{22}	σ'^{33}	σ'^{23}	σ'^{31}	σ'^{12}
σ'^{11}	$(g_{11})^2$ (1)	$(g_{12})^2$ (2)	$(g_{31})^2$ (2)	$g_{12}g_{31}$ (4)	$g_{31}g_{11}$ (4)	$g_{11}g_{12}$ (4)
σ'^{22}		$(g_{22})^2$ (1)	$(g_{23})^2$ (2)	$g_{22}g_{23}$ (4)	$g_{23}g_{12}$ (3)	$g_{12}g_{22}$ (4)
σ'^{33}			$(g_{33})^2$ (1)	$g_{23}g_{33}$ (4)	$g_{33}g_{31}$ (4)	$g_{31}g_{23}$ (4)
σ'^{23}				$\frac{1}{2}[g_{22}g_{33}+(g_{23})^2]$ (4)	$\frac{1}{2}(g_{23}g_{31}+g_{12}g_{33})$ (8)	$\frac{1}{2}(g_{12}g_{23}+g_{22}g_{31})$ (8)
σ'^{31}					$\frac{1}{2}[g_{33}g_{11}+(g_{31})^2]$ (4)	$\frac{1}{2}(g_{31}g_{12}+g_{23}g_{11})$ (8)
σ'^{12}						$\frac{1}{2}[g_{11}g_{22}+(g_{12})^2]$ (8)

TABLE 2. $L'_{\lambda\mu\iota\kappa}=\frac{1}{2}(g_{\lambda\iota}\varepsilon_{\mu\kappa}+g_{\mu\kappa}\varepsilon_{\lambda\iota}+g_{\lambda\kappa}\varepsilon_{\mu\iota}+g_{\mu\iota}\varepsilon_{\lambda\kappa})$

	σ'^{11}	σ'^{22}	σ'^{33}	σ'^{23}	σ'^{31}	σ'^{12}
σ'^{11}	$2g_{11}\varepsilon_{11}$ (1)	$2g_{12}\varepsilon_{12}$ (2)	$2g_{31}\varepsilon_{31}$ (3)	$g_{12}\varepsilon_{31}+g_{31}\varepsilon_{12}$ (4)	$g_{31}\varepsilon_{11}+g_{11}\varepsilon_{31}$ (4)	$g_{11}\varepsilon_{12}+g_{12}\varepsilon_{11}$ (4)
σ'^{22}		$2g_{22}\varepsilon_{22}$ (1)	$2g_{23}\varepsilon_{23}$ (2)	$g_{22}\varepsilon_{23}+g_{23}\varepsilon_{22}$ (4)	$g_{23}\varepsilon_{12}+g_{12}\varepsilon_{23}$ (4)	$g_{12}\varepsilon_{22}+g_{22}\varepsilon_{12}$ (4)
σ'^{33}			$2g_{33}\varepsilon_{33}$ (1)	$g_{23}\varepsilon_{33}+g_{33}\varepsilon_{23}$ (4)	$g_{33}\varepsilon_{31}+g_{31}\varepsilon_{33}$ (4)	$g_{31}\varepsilon_{23}+g_{23}\varepsilon_{31}$ (4)
σ'^{23}				$\frac{1}{2}(g_{22}\varepsilon_{33}+g_{33}\varepsilon_{22}+2g_{23}\varepsilon_{23})$ (4)	$\frac{1}{2}(g_{23}\varepsilon_{31}+g_{31}g_{23}+g_{12}\varepsilon_{33}+g_{33}\varepsilon_{12})$ (8)	$\frac{1}{2}(g_{12}\varepsilon_{23}+g_{23}\varepsilon_{12}+g_{22}\varepsilon_{31}+g_{31}\varepsilon_{22})$ (8)
σ'^{31}					$\frac{1}{2}(g_{33}\varepsilon_{11}+g_{11}\varepsilon_{33}+2g_{31}\varepsilon_{31})$ (4)	$\frac{1}{2}(g_{31}\varepsilon_{12}+g_{12}\varepsilon_{31}+g_{23}\varepsilon_{11}+g_{11}\varepsilon_{23})$ (8)
σ'^{12}						$\frac{1}{2}(g_{11}\varepsilon_{22}+g_{22}\varepsilon_{11}+2g_{12}\varepsilon_{12})$ (4)

TABLE 3. $M'_{\lambda\mu\iota\kappa}=\frac{1}{2}(\varepsilon_{\lambda\iota}\varepsilon_{\mu\kappa}+\varepsilon_{\lambda\kappa}\varepsilon_{\mu\iota})$

	σ'^{11}	σ'^{22}	σ'^{33}	σ'^{23}	σ'^{31}	σ'^{12}
σ'^{11}	$(\varepsilon_{11})^2$ (1)	$(\varepsilon_{12})^2$ (2)	$(\varepsilon_{31})^2$ (2)	$\varepsilon_{12}\varepsilon_{31}$ (4)	$\varepsilon_{31}\varepsilon_{11}$ (4)	$\varepsilon_{11}\varepsilon_{12}$ (4)
σ'^{22}		$(\varepsilon_{22})^2$ (1)	$(\varepsilon_{23})^2$ (2)	$\varepsilon_{22}\varepsilon_{23}$ (4)	$\varepsilon_{23}\varepsilon_{12}$ (4)	$\varepsilon_{12}\varepsilon_{22}$ (4)
σ'^{33}			$(\varepsilon_{33})^2$ (1)	$\varepsilon_{23}\varepsilon_{33}$ (4)	$\varepsilon_{33}\varepsilon_{31}$ (4)	$\varepsilon_{31}\varepsilon_{23}$ (4)
σ'^{23}				$\frac{1}{2}[\varepsilon_{22}\varepsilon_{33}+(\varepsilon_{23})^2]$ (4)	$\frac{1}{2}(\varepsilon_{23}\varepsilon_{31}+\varepsilon_{12}\varepsilon_{33})$ (8)	$\frac{1}{2}(\varepsilon_{12}\varepsilon_{23}+\varepsilon_{22}\varepsilon_{31})$ (8)
σ'^{31}					$\frac{1}{2}[\varepsilon_{33}\varepsilon_{11}+(\varepsilon_{31})^2]$ (4)	$\frac{1}{2}(\varepsilon_{31}\varepsilon_{12}+\varepsilon_{23}\varepsilon_{11})$ (8)
σ'^{12}						$\frac{1}{2}[\varepsilon_{11}\varepsilon_{22}+(\varepsilon_{12})^2]$ (4)

representing the coefficients of $\sigma'^{\lambda\mu}\sigma'^{\iota\kappa}$ being symmetric with respect to the diagonal of the tables, only those on one side of the diagonal are shown in the tables. The figures in the parentheses under each coefficient indicate the number of terms identical with the term with the coefficient. For example, the terms identical with that which gives the product $\sigma'^{23}\sigma'^{31}$ are those of $\sigma'^{23}\sigma'^{13}$, $\sigma'^{32}\sigma'^{31}$, $\sigma'^{32}\sigma'^{13}$ besides that, and further those in which the order of product is reversed, those in all numbering eight. Thus $\sigma'^{23}\sigma'^{31}$ may be multiplied by the coefficient in this position, and then increased by eight times, in order to obtain the corresponding terms in the yield function. This multiple is shown in the parenthesis. In these tables are given the coefficients by the Lagrangian coordinate system, and those by the local coordinate system, g'_{ijkl}, L'_{ijkl}, M'_{ijkl} are, of course, given quite similarly, only $g_{\lambda\mu}$, $\varepsilon_{\lambda\mu}$ being replaced by g_{ij}, ε_{ij}. Since the local coordinate system can be maintained orthogonal irrespective of deformation, the coefficients are simplified, many terms in them vanishing.

5. YIELD FUNCTION FOR THE STATE OF ANISOTROPY AND THE BAUSCHINGER EFFECT DUE TO STRAIN HISTORY—*Continued.*

In the preceding section, we derived the yield function for non-isotropic state due to strain history, basing on an irresistible logic, on the assumption of the Mises' yield function for isotropic state which is also regarded as justifiable within errors, if exist, of very small amount. This result is of general validity, and yielding of most metals seems to be well explained by the yield criterion specified by this function. But according to the kind of metals and of strain histories it seems that there exists some case where the complete form of the function does not hold, but it is needed to put the first order term of A equal to zero [*8*]. One may be contented with the explanation that this is such a special case of the yield function obtained in the preceding section. But why does the first order term of A alone vanish? Now this question will be answered in a rather reasonable manner, by considering another possibility of anisotropy and the Bauschinger effect being produced by strain history.

We consider, as against the yield function $f=\frac{1}{2}\boldsymbol{T}'\cdot\cdot\boldsymbol{J}'\cdot\cdot\boldsymbol{T}'$ for isotropic case, that the tensor $\boldsymbol{C}^*$ of order 4 characterizing the yield function

$$f=\frac{1}{2}\boldsymbol{T}'\cdot\cdot\boldsymbol{C}^*\cdot\cdot\boldsymbol{T}' \tag{5.1}$$

for anisotropic state, will show a deviation $\boldsymbol{A}^*$ from $\boldsymbol{J}'$, and accordingly can be put

$$\boldsymbol{C}^*=\boldsymbol{J}'+\boldsymbol{A}^*. \tag{5.2}$$

And further we suppose that the deviatoric tensor $\boldsymbol{A}^*$ is derived from the strain history tensor $\boldsymbol{E}$ in just the same way as $\boldsymbol{J}'$ has been derived from $\boldsymbol{I}$, i.e. that

$$\left.\begin{aligned} \boldsymbol{A}^* &= A^* M'_{\lambda\mu\iota\kappa}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu\boldsymbol{e}^\iota\boldsymbol{e}^\kappa, \\ M'_{\lambda\mu\iota\kappa} &= \frac{1}{2}(\varepsilon_{\lambda\iota}\varepsilon_{\mu\kappa}+\varepsilon_{\lambda\kappa}\varepsilon_{\mu\iota}). \end{aligned}\right\} \tag{5.3}$$

This result is seen to be identical with that obtained by droping the linear term of A in the general yield function derived in the preceding section, only the scalar coefficient A^* being different from A^2. Thus the case where the linear term of A can be neglected, i.e. where it can be put that $L'_{\lambda\mu\iota\kappa}=0$, is seen to be justifiable from the view-point presented in this section.

6. Yield Criterion and Law of Work-Hardening

In Section 2, we have made the statement that it is impossible to know the yield criterion and the rate of work-hardening in the hypothetical case when metarials were isotropic, and hence also to know the state of anisotropy and the Bauschinger effect which they really reveal, without some theory relating anisotropy and the Bauschinger effect as well as work-hardening to strain history. This is because most metals show behaviors as the combination of all these strain history effects. For instance, the stress-strain curve in extension of a rod specimen is a result of not only work-hardening, but also anisotropy due to the strain history of the extension itself. The same thing can be said as for the stress-strain curve in the case of twist of a thin circular tube. On the other hand, in order to investigate yield condition of a metal, it is needed to consider about its state more or less cold-worked, because it generally does not have clear yield point in the annealed state. The yield locus for the stress state of combined tension-torsion of a tube after an axial elongation, for instance, is not the locus for isotropic state, but that for the state revealing anisotropy and the Bauschinger effect on account of the previous elongation. As for the yield locus of a tube after twisting, the matter is quite the same. Thus we can see that the stress-strain relation and the yield condition as a result of experiments is none other than a complex of such various effects as work-hardening, anisotropy and the Bauschinger effect. It must be said unfortunate that the investigations ever performed in this domain all have taken no account of these facts. Our present theory, which we are now going to introduce, is that which is possible to resolve the complex of effects into its individual elements, and accordingly to explain the experimental results from a unified viewpoint. Such yield criterion and the law of work-hardening as to serve for this purpose is derivable basing on the yield function so far introduced.

In the preceding paper [3] we made the assertion that it is unreasonable to regard the work-hardening as a function of plastic work $W=\sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu}$ which cannot be a state variable, but it ought to be a function of the internal energy U stored in the metals by plastic deformation. Admitting this, we can put

$$f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})=H(U) \tag{6.1}$$

where $f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})$ is the yield function derived in the preceding section. In case, in particular, when U varies in the same proportion to W for all the deformation paths, (6.1) can be replaced by

$$f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})=F(W). \tag{6.2}$$

Of course, the yield criterion for a certain state work-hardened is represented by

$$f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})=\text{const.} \tag{6.3}$$

7. Mechanical Equation of State

In the preceding paper [3], we obtained the result that the virtual work principle in the form

$$DW=\int_0^t \sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu} \tag{7.1}$$

just the same as for the small deformation, is possible to hold also for the finite deformation, only when the strain history tensor and the stress defined by (11.2) of the preceding paper [3] are used as strain and stress. In consequence of this and the maximum work principle, the yield function playing the role of plastic potential as well, we have the mechanical equation of state in the form

$$(D\varepsilon)_{\lambda\mu}=\frac{\partial f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})}{\partial\sigma^{\lambda\mu}}D\lambda \tag{7.2}$$

for any magnitude of plastic deformation. Where, denoting by DQ the heat quantity which flow out during dt, $D\lambda$ is given by

$$D\lambda=\frac{Df(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})+H'(U)DQ}{H'(U)\sigma^{\lambda\mu}\dfrac{\partial f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})}{\partial\sigma^{\lambda\mu}}}, \tag{7.3}$$

when (6.1) holds, or by

$$D\lambda=\frac{Df(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})}{F'(W)\sigma^{\lambda\mu}\dfrac{\partial f(\partial^{\lambda\mu}, \varepsilon_{\lambda\mu})}{\partial\sigma^{\lambda\mu}}}, \tag{7.4}$$

when (6.2) holds.

Applying (7.2) to (4.21), we have

$$(D\varepsilon)_{\lambda\mu}=(c_{\lambda\mu\iota\kappa}\sigma^{\iota\kappa}-B\varepsilon_{\lambda\mu})D\lambda\,, \tag{7.5}$$

or

$$D\varepsilon_{ij}=(c_{ijkl}\sigma^{kl}-B\varepsilon_{ij})D\lambda \tag{7.6}$$

for the local coordinate system. Expressed in a form independent of coordinate system, it becomes

$$\boldsymbol{DE}=(\boldsymbol{C}\cdot\cdot\boldsymbol{T}-B\boldsymbol{E})D\lambda. \tag{7.7}$$

Representing $\sigma^{\iota\kappa}$ of (7.5) by the deviatoric stress components by means of (3.13) and taking account of

$$c_{\lambda\mu\iota\kappa}g^{\iota\kappa}=0,$$

we obtain

$$\left.\begin{aligned}(D\varepsilon)_{\lambda\mu}&=(c_{\lambda\mu\iota\kappa}\sigma'^{\iota\kappa}-B\varepsilon_{\lambda\mu})D\lambda,\\ D\varepsilon_{ij}&=(c_{ijkl}\sigma'^{kl}-B\varepsilon_{ij})D\lambda,\\ \boldsymbol{DE}&=(\boldsymbol{C}\cdot\cdot\boldsymbol{T}'-B\boldsymbol{E})D\lambda.\end{aligned}\right\} \tag{7.8}$$

Introducing (4.23), (4.24), (4.25) and (4.26) into (7.8), and using

$$g_{\lambda\mu}\sigma'^{\lambda\mu}=0$$

we can write

$$\left.\begin{aligned}(D\varepsilon)_{\lambda\mu}&=\Big\{\Big[g'_{\lambda\mu\iota\kappa}+A\Big(L'_{\lambda\mu\iota\kappa}-\frac{2}{3}g_{\lambda\mu}\varepsilon_{\iota\kappa}\Big)\\ &\quad+A^2\Big(M'_{\lambda\mu\iota\kappa}-\frac{1}{3}M'_{\alpha\beta\iota\kappa}g^{\alpha\beta}g_{\lambda\mu}\Big)\Big]\sigma'^{\iota\kappa}-B\varepsilon_{\lambda\mu}\Big\}D\lambda,\\ D\varepsilon_{ij}&=\Big\{\Big[g'_{ijkl}+A\Big(L'_{ijkl}-\frac{2}{3}g_{ij}\varepsilon_{kl}\Big)\\ &\quad+A^2\Big(M'_{ijkl}-\frac{1}{3}M'_{rskl}g^{rs}g_{ij}\Big)\Big]\sigma'^{kl}-B\varepsilon_{ij}\Big\}D\lambda,\\ \boldsymbol{DE}&=\Big\{\Big[\boldsymbol{J}'+A\Big(\boldsymbol{L}'-\frac{2}{3}\boldsymbol{I}\boldsymbol{E}\Big)\\ &\quad+A^2\Big(\boldsymbol{M}'-\frac{1}{3}\boldsymbol{I}\,\boldsymbol{I}\cdot\cdot\boldsymbol{M}'\Big)\Big]\cdot\cdot\boldsymbol{T}'-B\boldsymbol{E}\Big\}D\lambda.\end{aligned}\right\} \tag{7.9}$$

If there exist no anisotropy and no Bauschinger effect in particular, then being $A=0$, $B=0$, (7.5), (7.7) and (7.9) are reduced to

$$\left.\begin{aligned}(D\varepsilon)_{\lambda\mu}&=g_{\lambda\mu\iota\kappa}\sigma^{\iota\kappa}D\lambda,\\ \boldsymbol{DE}&=\boldsymbol{J}\cdot\cdot\boldsymbol{T}D\lambda,\end{aligned}\right\} \tag{7.10}$$

and

$$\left.\begin{aligned}(D\varepsilon)_{\lambda\mu}&=g'_{\lambda\mu\iota\kappa}\sigma'^{\iota\kappa}D\lambda,\\ \boldsymbol{DE}&=\boldsymbol{J}'\cdot\cdot\boldsymbol{T}'D\lambda\end{aligned}\right\} \tag{7.11}$$

respectively. For the rectangular Cartesian local coordinate system, the second equation (7.11) is seen to be read as the so-called Lévy-Mises equations

$$\left.\begin{aligned}D\varepsilon_x&=\sigma'_x D\lambda,\cdots,\\ D\gamma_{yz}&=\tau_{yz}D\lambda,\cdots.\end{aligned}\right\} \tag{7.12}$$

8. Yield Locus for Circular Tubes under Combined Tension-Torsion after the Strain History of Axial Elongation

The strain history theory of anisotropy and the Bauschinger effect introduced in the above is regarded so legitimate as to be irresistible in respect of its assumption and the logic of deduction. But so long as it is a postulate, it must be examined by experimental facts. Now, for this purpose, we will apply our theory to the special case of yielding of circular tubes under combined tension-torsion, after they have been subjected to a pre-strain of axial elongation.

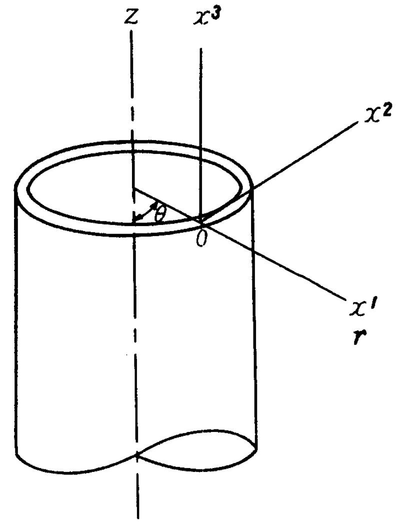

Fig. 3.

As shown in Fig. 3, we choose the x^1, x^2, x^3 axes of a rectangular Cartesian local coordinate system coincident with the radial, circumferential and axial directions of the circular tube. Since the deformation under consideration is uniform, we may take x^i instead of dx^i which alone has a meaning generally in local coordinate system. By taking such coordinate system, the pre-strain of axial elongation of the tube by n_0 times is represented by

$$\varepsilon_{11}=\varepsilon_{22}=-\frac{1}{2}\varepsilon_0,\quad \varepsilon_{33}=\varepsilon_0,\quad \text{the other}\quad \varepsilon_{ij}=0, \tag{8.1}$$

where

$$\varepsilon_0=\log n_0 \tag{8.2}$$

and the combined stress state of axial tension σ and torsion τ by

$$\left.\begin{aligned} &\sigma'^{11}=\sigma'^{22}=-\frac{1}{3}\sigma,\quad \sigma'^{33}=\frac{2}{3}\sigma,\\ &\sigma'^{23}=\tau,\quad \text{the other}\quad \sigma'^{ij}=0. \end{aligned}\right\} \tag{8.3}$$

From (8.1) and Tables 2 and 3, we obtain

$$\left.\begin{aligned}
&L'_{1111}=L'_{2222}=-\varepsilon_0, \quad L'_{3333}=2\varepsilon_0,\\
&L'_{2323}=L'_{2332}=L'_{3223}=L'_{3232}=\frac{1}{4}\varepsilon_0,\\
&L'_{3131}=L'_{3113}=L'_{1331}=L'_{1313}=\frac{1}{4}\varepsilon_0,\\
&L'_{1212}=L'_{1221}=L'_{2112}=L'_{2121}=\frac{1}{2}\varepsilon_0,\\
&\qquad\text{the other } L'_{ijkl}=0,
\end{aligned}\right\} \tag{8.4}$$

and

$$\left.\begin{aligned}
&M'_{1111}=M'_{2222}=\frac{1}{4}\varepsilon_0^2, \quad M'_{3333}=\varepsilon_0,\\
&M'_{2323}=M'_{2332}=M'_{3223}=M'_{3232}=-\frac{1}{4}\varepsilon_0^2,\\
&M'_{3131}=M'_{3113}=M'_{1331}=M'_{1313}=-\frac{1}{4}\varepsilon_0^2,\\
&M'_{1212}=M'_{1221}=M'_{2112}=M'_{2121}=\frac{1}{8}\varepsilon_0^2,\\
&\qquad\text{the other } M'_{ijkl}=0.
\end{aligned}\right\} \tag{8.5}$$

The tensor components g'_{ijkl} defined by (4.17) are given, in this case, by

$$\left.\begin{aligned}
&g'_{1111}=g'_{2222}=g'_{3333}=1,\\
&g'_{2323}=g'_{3131}=g'_{1212}=\frac{1}{2},\\
&\qquad\text{the other } g'_{ijkl}=0.
\end{aligned}\right\} \tag{8.6}$$

In consequence, the yield function for the combined stress state (8.3) after the pre-straining of (8.1) is obtained in the form

$$f=\frac{1}{3}\left[\sigma^2\left(1+A\varepsilon_0+\frac{3}{4}A^2\varepsilon_0^2\right)+3\tau^2\left(1+\frac{1}{2}A\varepsilon_0-\frac{1}{2}A^2\varepsilon_0^2\right)\right]-B\varepsilon_0\sigma. \tag{8.7}$$

If the stress, when the tube was first elongated by ε_0, is

$$\sigma=\sigma_0, \quad \tau=0, \tag{8.8}$$

the function (8.7) must also be satisfied by (8.8), i.e. the relation

$$\begin{aligned}
f&=\frac{1}{3}\left[\sigma^2\left(1+A\varepsilon_0+\frac{3}{4}A^2\varepsilon_0^2\right)+3\tau^2\left(1+\frac{1}{2}A\varepsilon_0-\frac{1}{2}A^2\varepsilon_0^2\right)\right]-B\varepsilon_0\sigma\\
&=\frac{1}{3}\sigma_0^2\left(1+A\varepsilon_0+\frac{3}{4}A^2\varepsilon_0^2\right)-B\varepsilon_0\sigma_0
\end{aligned} \tag{8.9}$$

holds. This is the yield criterion for the yielding by the combined stress (σ, τ) for the tube subjected to the strain history of extension ε_0.

It is seen that the yield locus represented by (8.9) shows some deviation from the ellipse representing the Mises' yield locus. If we put

$$\left.\begin{aligned} 1+\alpha&=\frac{1+\frac{1}{2}A\varepsilon_0-\frac{1}{2}A^2\varepsilon_0^2}{1+A\varepsilon_0+\frac{3}{4}A^2\varepsilon_0^2},\\ \beta&=\frac{\frac{3}{2}B\varepsilon_0}{1+A\varepsilon_0+\frac{3}{4}A^2\varepsilon_0^2},\end{aligned}\right\} \tag{8.10}$$

(8.9) is written

$$\frac{(\sigma-\beta)^2}{(\sigma_0-\beta)^2}+\frac{\tau^2}{\left(\frac{\sigma_0-\beta}{\sqrt{3(1+\alpha)}}\right)^2}=1. \tag{8.11}$$

On the other hand, the Mises' yield locus is represented by

$$\frac{\sigma^2}{\sigma_0^2}+\frac{\tau^2}{\left(\frac{\sigma_0}{\sqrt{3}}\right)^2}=1 \tag{8.12}$$

which is identical with the result obtained by putting $\alpha=0$ and $\beta=0$ in (8.11). The comparison between (8.11) and (8.12) makes their difference clear. As shown in Fig. 4 the center of the Mises' ellipse is (0, 0) in the σ, τ plane, and the major and minor radii are σ_0 and $\sigma_0/\sqrt{3}$, while those for the yield locus effected by the strain history are $(\beta, 0)$, $2(\sigma_0-\beta)$ and $2(\sigma_0-\beta)/\sqrt{3(1+\alpha)}$, respectively.

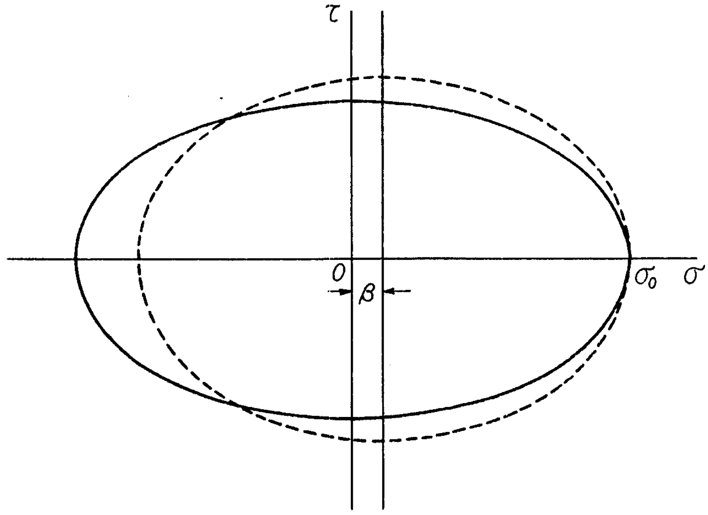

FIGURE 4.

9. Yield Locus for Circular Tubes under Combined Tension-Torsion after the Strain History of Twist

As another example, we will now consider the yielding under combined tension-torsion of a thin circular tube which has been subjected to the strain history of twist. We take also in this case the rectangular Cartesian local coordinate system shown in Fig. 3, with respect to the tube. If the tube is supposed to be so twisted as the tangent of the twist angle becomes γ_0, then the components of this strain history tensor are given by

$$\varepsilon_{23}=\frac{1}{2}\gamma_0,\quad \text{the other}\quad \varepsilon_{ij}=0. \tag{9.1}^*$$

The stress state of combined tension-torsion is expressed by (8.3). The tensor components L'_{ijkl} and M'_{ijkl} for the strain history (9.1) are obtained, from Tables 2 and 3, as

$$\left.\begin{aligned}
&L'_{2223}=L'_{2232}=L'_{2322}=L'_{3222}=\frac{1}{2}\gamma_0,\\
&L'_{33}\ \ =L'_{3332}=L'_{2333}=L'_{3233}=\frac{1}{2}\gamma_0,\\
&L'_{3112}=L'_{3121}=L'_{1312}=L'_{1321}\\
&=L'_{1231}=L'_{1213}=L'_{2131}=L'_{2113}=\frac{1}{4}\gamma_0,\\
&\qquad\text{the other}\ L'_{ijkl}=0,
\end{aligned}\right\} \tag{9.2}$$

$$\left.\begin{aligned}
&M'_{2233}=M'_{3322}=\frac{1}{4}\gamma_0^2,\\
&M'_{2323}=M'_{2332}=M'_{3223}=M'_{3232}=\frac{1}{8}\gamma_0^2,\\
&\qquad\text{the other}\ M'_{ijkl}=0.
\end{aligned}\right\} \tag{9.3}$$

The relations (8.6) hold also in this case. In consequence, the yield function for the combined stress state (8.3) is given by

$$f=\frac{1}{3}\left[\sigma^2\left(1-\frac{1}{6}A^2\gamma_0^2\right)+3\tau^2\left(1+\frac{1}{4}A^2\gamma_0^2\right)+\sigma\tau A\gamma_0\right]-B\gamma_0\tau. \tag{9.4}$$

If the stress necessary for the strain history (9.1) to be produced, is

$$\sigma=0,\quad \tau=\tau_0 \tag{9.5}$$

the function (9.4) is also satisfied by this stress (9.5), i.e. the relation

$$\begin{aligned}
f&=\frac{1}{3}\left[\sigma^2\left(1-\frac{1}{6}A^2\gamma_0^2\right)+3\tau^2\left(1+\frac{1}{4}A^2\gamma_0^2\right)+\sigma\tau A\gamma_0\right]-B\gamma_0\tau\\
&=\tau_0^2\left(1+\frac{1}{4}A^2\gamma_0^2\right)-B\gamma_0\tau_0
\end{aligned} \tag{9.6}$$

holds. This is the yield criterion of the tube for the combined stress state (8.3), after it has been subjected to the strain history (9.1) of twist.

If we put

$$\left.\begin{aligned}
&a=1-\frac{1}{6}A^2\gamma_0^2,\\
&b=3\left(1+\frac{1}{4}A^2\gamma_0^2\right),\\
&h=\frac{1}{2}A\gamma_0,
\end{aligned}\right\} \tag{9.7}$$

* See (8.23) of the preceding paper [3].

$$\left.\begin{aligned} f&=-\frac{3}{2}B\gamma_0,\\ c&=-\left[3\left(1+\frac{1}{4}A^2\gamma_0^2\right)\tau_0^2-3B\gamma_0\tau_0\right]\\ &=-(b\tau_0^2+2f\tau_0), \end{aligned}\right\}$$

the yield criterion (9.6) is written

$$a\sigma^2+2h\sigma\tau+b\tau^2+2f\tau+c=0. \tag{9.8}$$

This equation (9.8) represents an ellipse in the σ, τ plane, whose center is situated at

$$\left.\begin{aligned} \sigma_c&=\frac{hf}{ab-h^2}=-\frac{\frac{3}{4}A\gamma_0B\gamma_0}{3-\frac{1}{8}A^4\gamma_0^4},\\ \tau_c&=\frac{-af}{ab-h^2}=\frac{\frac{3}{2}\left(1-\frac{1}{6}A^2\gamma_0^2\right)B\gamma_0}{3-\frac{1}{8}A^4\gamma_0^4} \end{aligned}\right\} \tag{9.9}$$

and the major axis is inclined by the angle θ such that

$$\tan 2\theta=\frac{2h}{a-b}=-\frac{A\gamma_0}{2+\frac{11}{12}A^2\gamma_0^2}. \tag{9.10}$$

The major and minor axes of the ellipse have the length

$$2\sqrt{-\frac{c'}{a'}} \quad\text{and}\quad 2\sqrt{-\frac{c'}{b'}} \tag{9.11}$$

respectively, where a', b' are the solutions of the equation

$$t^2-(a+b)t+(ab-h^2)=0,$$

i.e.

$$\left.\begin{matrix}a'\\ b'\end{matrix}\right\}=\frac{1}{2}\left\{4+\frac{7}{12}A^2\gamma_0^2\pm\sqrt{4+\frac{14}{3}A^2\gamma_0^2+\frac{121}{144}A^4\gamma_0^4}\right\} \tag{9.12}$$

and

$$c'=f\tau_c+c=\frac{af^2}{ab-h^2}=-\frac{\frac{9}{4}\left(1-\frac{1}{6}A^2\gamma_0^2\right)B^2\gamma_0^2}{3-\frac{1}{8}A^4\gamma_0^4}. \tag{9.13}$$

In Fig. 5 are shown the ellipses represented by (9.6), or (9.8), in the chain and broken lines, as against the Mises' ellipse (full line). It was already mentioned that these ellipses were distinguished from that of Mises in respect of the position of center and the lengths and inclinations of the axes. The two ellipses have the opposite inclinations according to the sign of A. For the yield function, in particular, which has no term of the first order with respect to A, we can put $h=0$ in

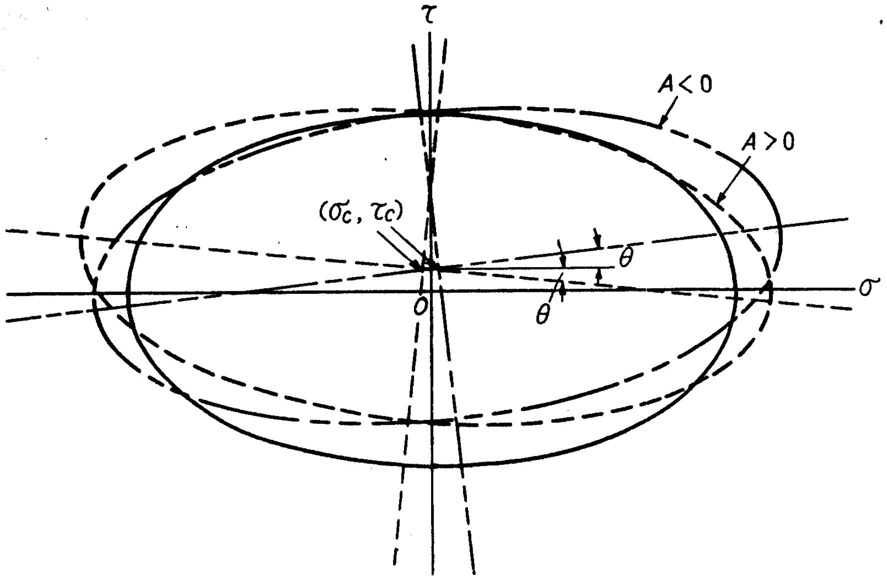

FIGURE 5.

(9.7), and therefore the axes of the ellipse (9.8) are found to show no inclination with the coordinate axes.

10. Comparison of the Present Theory with Experiments

There are scarcely any experiments inquiring the yielding of a thin circular tube under the stress state of combined tension-torsion, after it has been subjected to the strain history of either axial extension or twist. The only one of the experiments of this kind is that of Taylor and Quinney (1931) [*5*] for the case of strain history of axial extension. The result of their experiment has already been shown in Fig. 1. It was stated there that the yield locus of aluminium tubes showed good agreement with the Mises' ellipse, and the locus of copper lay somewhat inside of it, while that of mild steel showed notable discrepancy outside the Mises' ellipse, and further that the difference between these metals as to the yielding properties was attributable to that of the extent of their alloying. That is, the notable deviation of the yield point of mild steel tubes from the Mises' locus is supposed to be due to anisotropy caused by the strain history of the initial elongation. From this view-point, we investigated by experiments the yield locus for mild steel tubes under the stress state of combined tension-torsion after they have been subjected to the strain histories of axial elongation and twist respectively, and obtained the results, as shown in Fig. 6a and 6b, which have a tendency quite converse to each other with regard to the Mises' yield locus. In the figure σ_{iso} and τ_{iso} represent the yield stress in tension and tortion respectively, when it is assumed that the material are isotropic in spite of its prestraining. The results of these experiments and their interpretation have already been published in a Japan journal [*8*], in advance of the present paper. According to the results of the above investigation, it is clearly shown that the yield loci for the both strain histories, shown in Fig. 6a and 6b, can be well explained from the unifying view-point based on our present theory for the case where the first order term of A vanishes. Even from the qualitative comparison alone of the yield loci in Fig. 4 and Fig. 5 with the experimental results in Fig. 6a and 6b, we can see how our theory is reasonable in view of the experimental facts.

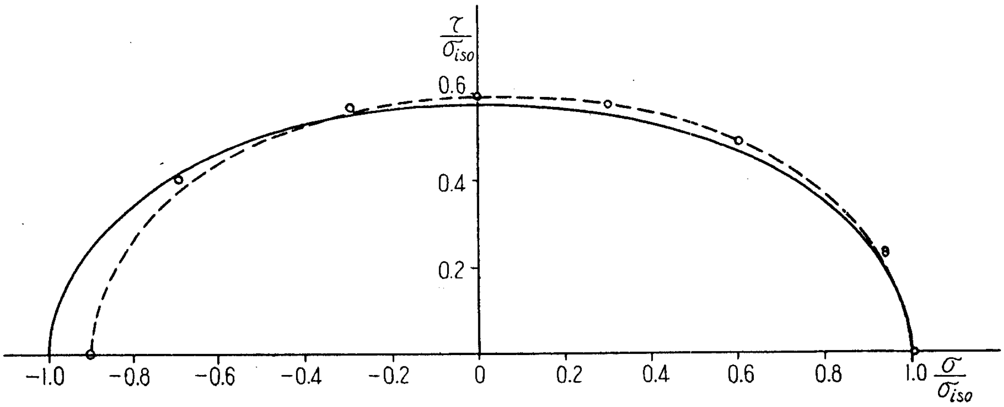

FIGURE 6a.

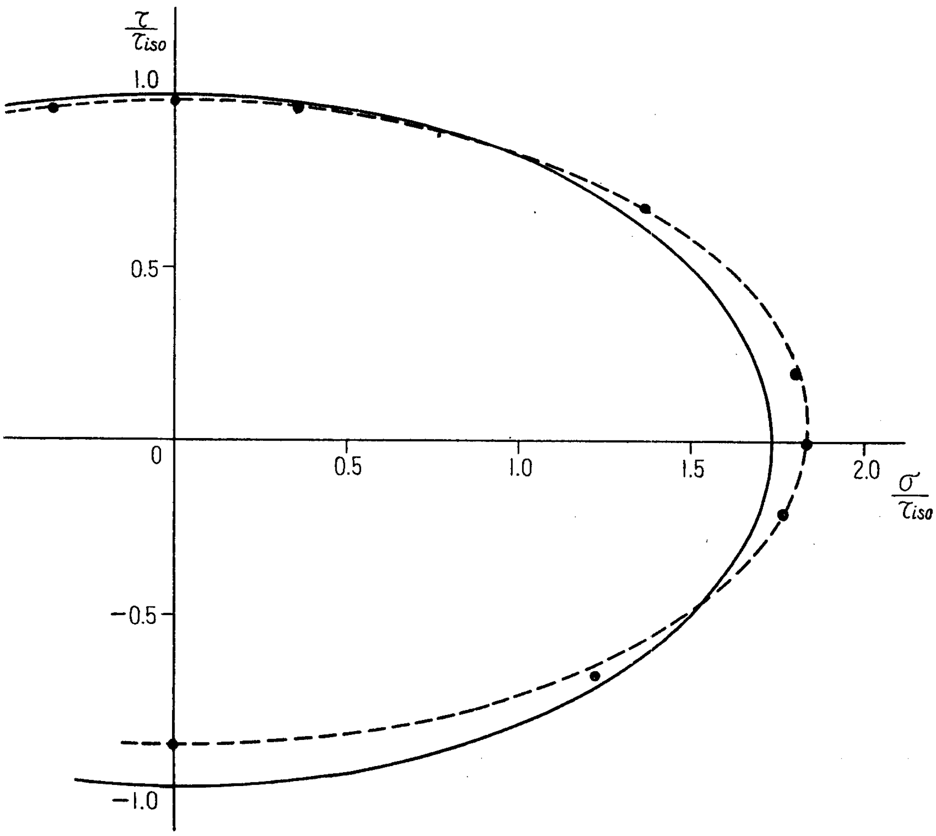

FIGURE 6b.

From the above considerations, we may certainly infer that the deviation from the Mises' criterion, of the experimental results of Taylor-Quinney [5] for the yield point of mild steel tubes is also due to the strain history of the initial elongation. But here it deserves special attention that some change in the internal volume of the tubes was found in their experiments during the deformation under the combined stress, although there exists nothing in our experiments. If this volume change is true, not being due to some error in the measurements, it is regarded as showing that the yield criterion for the mild steel tubes for isotropic state has somewhat inward deviation from the Mises' criterion, as was mentioned previously. The fact that the yield point of the tubes in their experiments nevertheless showed notable deviation towards the outside of the Mises' criterion, as shown in Fig. 1, is considered to be due to the anisotropy resulting from the strain history of axial extension. And this deviation and hence the extent of anisotropy, is regarged so much greater than in the case where the Mises' criterion is assumed to be that for isotropic state. But now, for the sake of applicability of our present theory, we assume that the Mises' criterion holds for isotropic state also in the experiments of Taylor-Quinney for mild steel tubes. Thus applying (8.11), we obtain the curve represented by broken line in Fig. 1, which agrees closely with the experimental results, by assuming

$$A\varepsilon_0=0.199 \quad \text{or} \quad -0.526. \tag{10.1}$$

Here we put

$$B=0 \tag{10.2}$$

in (8.11), since there exists no evidence concerning the Bauschinger effect in the Taylor-Quinney's experiments. Substituting the value of the extensional prestrain

$$\varepsilon_0=0.037 \tag{10.3}$$

in their experiments into (10.1), we have

$$A=5.37 \quad \text{or} \quad -14.2. \tag{10.4}$$

The problem which of the two values of A, one the positive and the other the negative, should be chosen as valid must be determined taking account of other experimental facts. For example, if the yield locus after the strain history of twist is represented by an ellipse of negative θ, or if the tube is as well extended in the axial direction as twisted by pure torsion, then A is positive, and vice versa. Of these matters we will give full account in succeeding papers.

11. CONCLUSION

The theory introduced in this paper is that which correlates plastic anisotropy and the Bauschinger effect with the previous strain history from which these result. The basic assumptions and logics which underlie this theory are regarded as irresistible from various view-points, and are as follows:

(1) Isotropic state of metals is represented by the spherical unit tensor $\boldsymbol{I}$ of order 2.

(2) For isotropic state, yielding occurs according to the Mises' criterion.

(3) Metals subjected to the strain history $\boldsymbol{E}$ are in the anisotropic state of $\boldsymbol{I}+A\boldsymbol{E}$, A being a scalar coefficient.

(4) The yield criterion for the anisotropic state $\boldsymbol{I}+A\boldsymbol{E}$ is derived in just the same way as the Mises' criterion has been derived from the isotropic state $\boldsymbol{I}$.

For these theses to be made possible to be stated, we necessitate the new and fundamental concept of strain history tensor $\boldsymbol{E}$, which was already introduced in the preceding paper [3].

The theory of plasticity based on this yield function which takes into account anisotropy and the Bauschinger effect as well as work-hardening was compared with various experiments and was found to show good agreement. And it is still more interesting that some results, which are deduced from our theory and even impossible to be expected by our common sense, are also given experimental proofs.

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概　　要

塑性変形履歴による異方性および
バウシンガー効果の理論

吉　村　慶　丸

一般に金属は塑性変形の履歴によって加工硬化，異方性，Bauschinger 効果を示し，実際の現象はこれらの複合効果として現われる．これに対して現在の塑性力学には加工硬化だけしか取入れられていないために，それによってはこのような現実の塑性変形を正しく捉えることができない．

この現状に鑑み，筆者はこの研究において塑性力学を，加工硬化，異方性，Bauschinger 効果が共存する場合に拡張することを試みた．このためには異方性，Bauschinger 効果と変形履歴とを関係づける法則を見出すことが必要である．しかしこれを上述のような複合現象から実験的に帰納することは原理的に不可能なことであって，われわれはどうしてもある仮説を必要とする．

筆者の仮説を要約すると次の如くである．物質の変形前の等方状態は球（単位）テンソル $\boldsymbol{I}$ によって表現され，且つ等方状態に対する降伏函数はいわゆる Mises のそれによって与えられ，それは $\boldsymbol{I}$ と偏差応力テンソル $\boldsymbol{T}'$ とから一定の規則によって導かれる．歪履歴 $\boldsymbol{E}$ による異方性状態は $\boldsymbol{I}+A\boldsymbol{E}$ (A はスカラー) によって表現され，それに対する降伏函数は，$\boldsymbol{I}$ と $\boldsymbol{T}'$ から Mises の函数が導かれたと全く同じ規則によって，$\boldsymbol{I}+A\boldsymbol{E}$ と $\boldsymbol{T}'$ から導かれる．このようなことが言えるためには歪履歴テンソル $\boldsymbol{E}$ という概念が必要であり，これは筆者が既に前の研究において導入した，塑性に関しては極めて重要な新しい基本概念である．Bauschinger 効果についても同様であるから省略する．

この仮説は，上に見られるように，極めて合理的なものであり，その成立は疑う余地がないものと思う．事実，今までにそれは多くの実験結果と比較され，よい一致を示している．しかもこの理論から得られた，常識では予期し得べくもない結果がいずれも鮮かに実証されるということは興味あることである．