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An Investigation of Sampling Measurement of Time Varying Random Signals through Information Theory

By

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Summary. This paper is intended to present a quantitative method of analysis of sampling measurement or conversion of time varying random signals. The analysis is developed by deriving a formula which gives the average information amount per unit time obtained through the sampling measurement of a Gaussian random signal. Considering the limiting case of the formula when the repetition frequency of the sampling measurement is increased extremely, the order of increase of the information amount vs. sampling frequency is examined and interpreted from physical points of view. Comparison between sampling and continuous measurement of the signals is also made in this paper. Using the formula the numerical values of the information amount obtained through the sampling measurement are calculated for several kinds of signal power spectra, and in connection with these examples some interesting phenomena are described.

1. INTRODUCTION

The average information amount obtained through a measurement of a physical quantity is able to be calculated quantitatively using Shannon's theory of information [*1*] just like in the case of communication systems, i.e., the information amount is given as the entropy difference of the probability distribution of the measured quantity (*a priori* probability) and the probalility distribution of uncertainty of that quantity remaining after the measurement (posterior probability), which arises owing to the error of the measuring instrument. In the case of sampling measurement of a time varying random signal at equal time intervals, the average information amount obtained through a set of sampling measurement points is presented as the sum of information amount corresponding to each sampling point, provided that these points are statistically independent of each other. In general, however, it is not the case, because there are mutual dependences among these points as expressed as the auto-correlation function of the signal. In other words, a prediction of the signal value is possible to some extent into the future using the correlation function [*2*] [*3*] [*4*], which changes the *a priori* probability of the signal value before measurement, and the uncertainty of the signal value at a sampling point after the measurement is also changed by the observed values of the neighboring sampling points.

In this paper, the average information amount per unit time obtained through sampling measurement is presented as a function of the power spectrum of the signal to be measured, the sampling frequency and the root mean square value (standard deviation) of measurement errors, which is shown in Eq. (18), under the assumptions that

(1) the signal to be measured is Gaussian and random,

(2) the magnitudes of the measurement errors constitute a Gaussian distribution statistically and are independent of time as well as signal values.

This formula is derived by calculating the entropy difference in the frequency domain. Considering the limiting case of the formula, the order of increase of the information amount vs. sampling frequency is examined and interpreted from physical points of view. It is also shown that, in certain conditions, the sampling measurement of repetition frequency f_s is equivalent, in the sense of information theory, to the continuous measurement by the instrument which has the cut-off frequency $f_c = f_s/2$.

The information amount obtained through sampling measurement is numerically calculated for several kinds of signal power spectra as examples, and some interesting facts such that for the random signals which have some quasi-periodic characteristics, the information amount obtained decreases with the increase of the sampling frequency f_s over some ranges of f_s, are described. Although the formula derived in this paper is not applicable directly to the sampling measurement or conversion of time varying random signals by Analog-to-Digital converters because of the quantization of the signals, these situations are discussed briefly.

2. Properties of Gaussian Random Signals in the Frequency Domain [5]

The Gaussian random signal, which is assumed to be measured in the present study, is a stochastic time function, which is a typical member of an ergodic ensemble and consequently has no frequency component having energy of comparable amount with that of the complete signal, i.e., there is no δ-function in the power spectrum of the signal. A reason for assuming Gaussian random signals as the measured ones is the fact that, according to the central limit theorem, any stationary random signal would aproach to a Gaussian random signal when it passes through linear systems and many kinds of random signals encountered in practice may be classified in this category.

Another reason for this assumption is the properties of Gaussian random signals in the frequency domain convenient for calculation of signal entropies. Sampling theorem of time varying signals states that the signal $x(t)$ which has a time duration of D and a maximum frequency of f_m (maximum frequency of the signal power spectrum) is completely determined by its values of the free points† taken at intervals of $1/2f_m$ in the time domain or by the values of the free points which are taken at frequency intervals of $1/D$ in its Fourier transform as illustrated in Fig. 1. In Gaussian random signals, these free points in the time domain have the properties as follows:

† Although the term "sampling point" is used usually for this meaning, the term "free point" is used intentionally throughout this paper to prevent the confusion with "sampling measurement point".

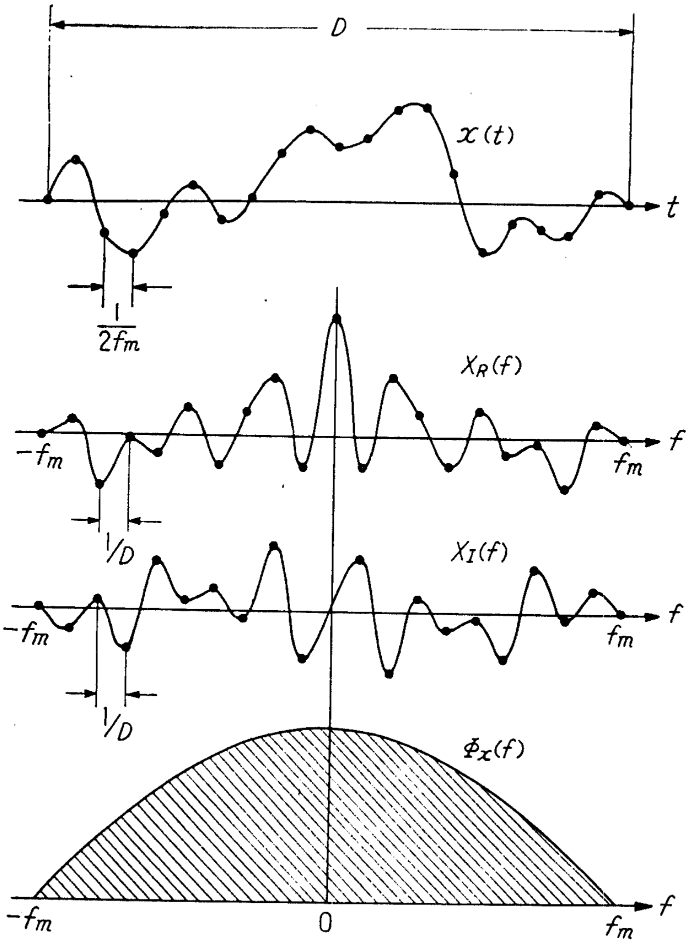

FIGURE 1. A Gaussian random signal which has time duration D and a maximum frequency f_m. The free points in the frequency domain are statistically independent of each other.

(1) The signal values $x(n/2f_m)$, corresponding to each free point in the time domain, constitute statistically a Gaussian distribution which has the standard deviation of σ_x, where n is an integer and σ_x is equal to the root mean square value of the signal.

(2) In General, these free points in the time domain are correlated to each other. In the frequency domain, denoting real and imaginary parts of Fourier transform of the signal $x(t)$ as

$$X(f)=\int_{-\infty}^{\infty} x(t)e^{-2\pi jft}dt=X_R(f)+jX_I(f)\,, \tag{1}$$

the free points in the frequency domain have the properties as follows:

(3) The values $X_R(n/D)$, $X_I(n/D)$ corresponding to each free point in the frequency domain belong to Gaussian distributions and the standard deviations of these distributions are given as

$$\begin{aligned}\sigma_{XR}(n/D)=\sigma_{XI}(n/D)&=\sqrt{\frac{1}{2}\overline{|X(n/D)|^2}}^{\,\dagger}\\&=\sqrt{\frac{D}{2}\Phi_x(n/D)}\,,\end{aligned} \tag{2}$$

† The bar over a symbol represents ensemble average of that quantity.

where $\Phi_x(f)$ is the power spectrum of the signal, which is related to the auto-correlation function $\phi_x(\tau)$ as

$$\phi_x(\tau)=\lim_{D\to\infty}\frac{1}{2D}\int_{-D}^{D}x(t)\cdot x(t+\tau)dt\,, \tag{3}$$

$$\Phi_x(f)=\int_{-\infty}^{\infty}\phi_x(\tau)e^{-2\pi jf\tau}d\tau\,, \tag{4}$$

and

$$\phi_x(0)=\sigma_x^2=\int_{-\infty}^{\infty}\Phi_x(f)df. \tag{5}$$

(4) These free points in the frequency domain are independent of each other.

The foregoing statements come from the definition of Gaussian random signals, and the property (4) indicates that the calculation of the signal entropy in the frequency domain is greatly advantageous over that in the time domain, since when there are no correlations, the entropies of different distributions are additive. It is further noted that, if the free points of a random signal in the time domain are also independent of each other, the power spectrum of the signal would be flat or "white" within the frequency range $-f_m\sim f_m$, and the Gaussian random signal which has a flat power spectrum is called as a pure random signal.

3. A Fictitious Noise Representing Measurement Errors

The observed values obtained by the sampling measurement may be represented by a train of δ-functions or impulses, each of which is equal in magnitude respectively to the sum of values of the measured signal at a sampling instant and the measurement error of that sampling point. As mentioned in Sec. 1, it is assumed that the measurement errors are independent of time as well as signal values and constitute statistically a Gaussian distribution. Then the operation of sampling measurement may be considered as the impulse modulation of the signal $y(t)$ which is the superposition of the measured signal $x(t)$ and a fictitious noise $n(t)$, that is

$$y(t)=x(t)+n(t) \tag{6}$$

and

$$y^*(t)=x^*(t)+n^*(t)\,, \tag{7}$$

where the asterisks denote the impulse modulated signals (Fig. 2). The fictitious noise $n(t)$ should be a Gaussian random noise having a flat power spectrum in the frequency range $-f_s/2\sim f_s/2$ (f_s: sampling frequency) as shown in Fig. 3, or it should be pure random noise, since it must represent the measurement errors whose values at sampling instants are statistically independent of each other.

The Fourier transform of Eq. (7)

$$Y^*(f)=X^*(f)+N^*(f) \tag{8}$$

is expressed by those of the continuous signals [6] [7] as

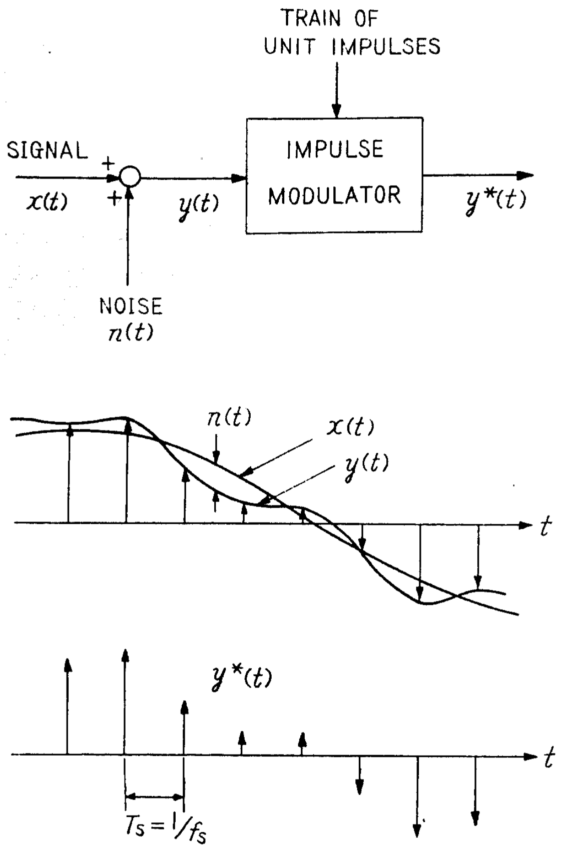

FIGURE 2. The operation of sampling measurement is considered as the impulse modulation of the signal $y(t)$ which is the superposition of the measured signal $x(t)$ and a fictitious noise $n(t)$.

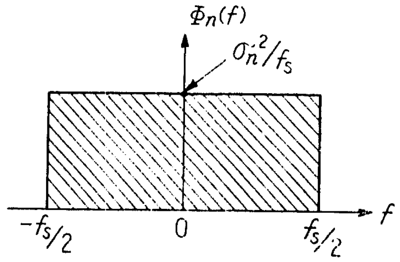

FIGURE 3. The power spectrum of the fictitious noise $n(t)$. The noise should be a pure random noise having a flat power spectrum. (f_s: repetition frequency of the sampling measurement, σ_n: the root mean square value of measurement errors)

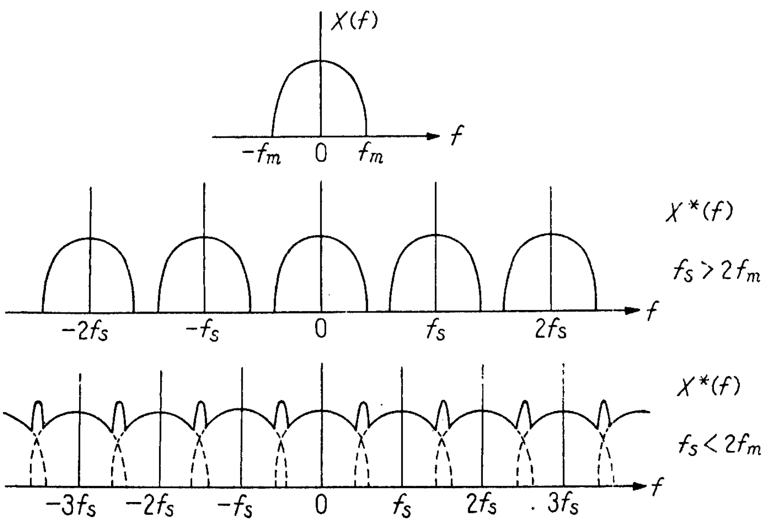

FIGURE 4. The Fourier transform of a impulse modulated signal is a periodic function having a period f_s.

$$\left.\begin{aligned} X^*(f)&=f_s\sum_{n=-\infty}^{\infty}X(f+nf_s)\,,\\ N^*(f)&=f_s\sum_{n=-\infty}^{\infty}N(f+nf_s)\,,\\ Y^*(f)&=f_s\sum_{n=-\infty}^{\infty}Y(f+nf_s)\,,\end{aligned}\right\} \tag{9}$$

where $X(f)$, $N(f)$, $Y(f)$ are Fourier transforms of $x(t)$, $n(t)$, $y(t)$ respectively and n is an integer. It should be noted here that all of these are periodic functions having a period f_s as shown in Fig. 4, and within the frequency range $-f_s/2<f<f_s/2$, Eq. (9) may be reduced to

$$\left.\begin{aligned} N^*(f)&=f_sN(f)\,,\\ Y^*(f)&=f_s\left\{\sum_{n=-\infty}^{\infty}X(f+nf_s)+N(f)\right\}.\end{aligned}\right\} \tag{10}$$

4. Information Amount Obtained through Sampling Measurement

According to the illustration in the preceding section, the observed values obtained through the sampling measurement may be represented by $y^*(t)$ and the signal $x^*(t)$ may be considered as the measured quantity. Applying Shannon's formula of information amount [1] to this case, the ensemble average of the information amount obtained through a set of sampling measurement points is given as

$$I=H(y^*)-Avx^*Hx^*(y^*)\,, \tag{11}$$

where $H(y^*)$, $Avx^*Hx^*(y^*)$ are entropy of $y^*(t)$, conditional entropy of $y^*(t)$ when $x^*(t)$ are known, respectively. The conditional probability distribution of $y^*(t)$, when $x^*(t)$ are known, is the probability distribution of the measurement errors $n^*(t)$, which are independent of the signal values as assumed, so Eq. (11) may be rewritten as

$$I=H(y^*)-H(n^*)\,, \tag{12}$$

where $H(n^*)$ is entropy of the fictitious noise $n^*(t)$. Moreover, Eq. (12) may be represented in the frequency domain as

$$I=H(Y^*)-H(N^*)\,. \tag{13}$$

The entropies in Eq. (13), calculated in the frequency domain, are different from those calculated in the time domain by a constant, but the information amount is given as the difference of the two entropies and the constant is canceled out, so Eq. (13) is valid. Although the terms in Eq. (13) must be calculated by considering the entropies at each free point in the Fourier transforms of the impulse modulated signals, the free points included in the frequency range $0\sim f_s/2$ are sufficient so far as the calculation of the entropies is concerned, because $Y^*(f)$ and $N^*(f)$ are periodic functions having a period f_s and $Y^*(-f)$ and $N^*(-f)$ are complex conjugates of $Y^*(f)$ and $N^*(f)$ respectively.

The signal $y(t)=x(t)+n(t)$ is the sum of two mutually independent Gaussian random signals, so $y(t)$ is also a Gaussian random signal and the values of the free points in the frequency domain $Y(m/D)$, where m is an integer, are independent of each other according to the properties of Gaussian random signals mentioned in Sec. 2. Consequently, the values of free points $Y^*(m/D)$, as well as $N^*(m/D)$, are also independent of each other, provided that the sampling frequency f_s is integral multiples of $1/D$ (this restriction will be removed later by making $D \to \infty$). The real and imaginary parts of these values constitute Gaussian distributions since they are the sum of independent variables which belong to Gaussian distributions. The vaules of $Y^*(f)$ and $N^*(f)$ other than these free points are completely determined by $Y^*(m/D)$ and $N^*(m/D)$. Accordingly, using the notations $H_m(Y^*)$ and $H_m(N^*)$ for the entropies concerning the variables $Y^*(m/D)$ and $N^*(m/D)$, Eq. (13) may be transformed to

$$I=\sum_m \{H_m(Y^*)-H_m(N^*)\}\,, \tag{14}$$

where m takes all integers in the range $0<m<f_sD/2$.

The variance (square of standard deviation) of the Gaussian distribution of the sum of indepent variables, each of which constitutes a Gaussian distribution, is equal to the sum of variances of each distribution, and the entropy of a Gaussian distribution which has standard deviation σ is $\frac{1}{2}\log_2 2\pi e\sigma^2$ bits. Consequently, from Eq. (2) and Eq. (10), the entropy $H_m(Y^*)$ is given by considering the real and imaginary parts of $Y^*(m/D)$ as

$$\begin{aligned} H_m(Y^*)&=2\times\frac{1}{2}\log_2 2\pi e f_s^2\left\{\sum_{n=-\infty}^{\infty}\frac{1}{2}\overline{\left|X\left(\frac{m}{D}+nf_s\right)\right|^2}+\frac{1}{2}\overline{\left|N\left(\frac{m}{D}\right)\right|^2}\right\}\\ &=\log_2 \pi e D f_s^2\left\{\sum_{n=-\infty}^{\infty}\Phi_x\left(\frac{m}{D}+nf_s\right)+\frac{\sigma_n^2}{f_s}\right\}. \end{aligned} \tag{15}$$

Similarly

$$H_m(N^*)=\log_2 \pi e D f_s^2\left(\frac{\sigma_n^2}{f_s}\right). \tag{16}$$

Hence

$$I=\sum_m \log_2\left\{\frac{f_s}{\sigma_n^2}\sum_{n=-\infty}^{\infty}\Phi_x\left(\frac{m}{D}+nf_s\right)+1\right\}. \tag{17}$$

Denoting the average information amount obtained per unit time as I_0, the formula to be derived becomes

$$\begin{aligned} I_0&=\lim_{D\to\infty}\frac{1}{D}I\\ &=\lim_{D\to\infty}\frac{1}{D}\sum_m \log_2\left\{\frac{f_s}{\sigma_n^2}\sum_{n=-\infty}^{\infty}\Phi_x\left(\frac{m}{D}+nf_s\right)+1\right\}\\ &=\int_0^{f_s/2}\log_2\left\{\frac{f_s}{\sigma_n^2}\sum_{n=-\infty}^{\infty}\Phi_x(f+nf_s)+1\right\}df \qquad \text{bits per unit time,} \end{aligned} \tag{18}$$

where f_s is the repetition frequency of the sampling measurement; σ_n, the root mean square value of the measurement errors; $\Phi_x(f)$, the power spectrum of the signal $x(t)$; and n is an integer. It should not be overlooked here that $\sum_n \Phi_x(f+nf_s)$

in (18) is nothing other than the pulse spectral density which appears in the statistical treatment of sampled-date control systems [4] [8].

5. Order of Increase of Information Amount vs. Sampling Frequency

Considering the limiting cases when $f_s \to 0$ and $f_s \to \infty$ in Eq. (18), the order of increase of I_0 vs. f_s will be investigated in this section. If the range of power spectrum of the signal to be measured is limited within $-f_m \sim f_m$ as shown in Fig. 1, then for the range of $f_s \ll f_m$†

$$f_s \sum_{n=-\infty}^{\infty} \Phi_x(f+nf_s) \doteqdot \int_{-\infty}^{\infty} \Phi_x(f)df$$

$$= \sigma_x^2 \text{ (= mean square value of the signal } x(t)). \tag{19}$$

Hence

$$I_0 \doteqdot \int_0^{f_s/2} \log_2\left(\frac{\sigma_x^2}{\sigma_n^2}+1\right)df = \frac{f_s}{2}\log_2\left(\frac{\sigma_x^2}{\sigma_n^2}+1\right). \tag{20}$$

Eq. (20) indicates that I_0 increases in proportion to f_s. This phenomenon is also deduced by the fact that when f_s is very small compared with the maximum frequency f_m, the entropies at the sampling points are additive because these points are far separated each other and the correlations among them are decreased substantially to zero.

In the range of $f_s \geq 2f_m$, Eq. (18) is reduced to

$$I_0 = \int_0^{f_m} \log_2\left\{\frac{f_s}{\sigma_n^2}\Phi_x(f)+1\right\}df. \tag{21}$$

When f_s is further increased, the term $\frac{f_s}{\sigma_n^2}\Phi_x(f)$ in Eq. (21) becomes very large compared with 1 in the frequency range $0 \leq f < f_m - \varepsilon$ ($\varepsilon > 0$ and $\varepsilon \to 0$ when $f_s \to \infty$) and

$$I_0 \doteqdot \int_0^{f_m-\varepsilon} \log_2\left\{\frac{f_s}{\sigma_n^2}\Phi_x(f)\right\}df$$

$$= \int_0^{f_m-\varepsilon} \log_2 \frac{\Phi_x(f)}{\sigma_n^2}df + \int_0^{f_m-\varepsilon} \log_2 f_s df$$

$$\doteqdot \text{Const.} + f_m \log_2 f_s\,. \tag{22}$$

That is, I_0 increases with respect to f_s by the order of $\log f_s$ as shown in Fig. 5. This indicates that when f_s is very large, the sampling points are aggregated densely and by increasing the sampling frequency the signal values are measured more accurately. This phenomenon bears some resemblance to the case of taking the average of the observed values of a physical quantity, in which the accuracy of the averaged value is increased with the increase of the number of times of the independent measurements.

† More accurately, for the range of $f_s \ll W$, where W is the bandwidth of the signal as shown in Fig. 9.

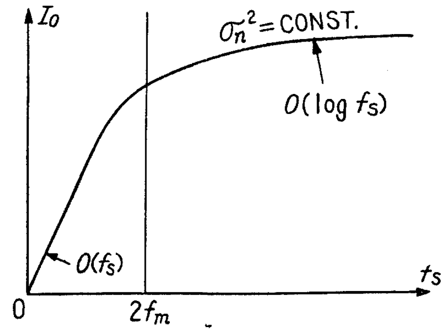

FIGURE 5. The order of increase of information amount obtained per unit time (I_0) vs. the repetition frequency of the sampling measurement (f_s) when the range of power spectrum of the signal to be measured is limited within $-f_m \sim f_m$ as shown in Fig. 1.

6. COMPARISON OF SAMPLING AND CONTINUOUS MEASUREMENT

The continuous measurement of a random signal by a recording instrument such as a pen-writing oscillograph, may be considered fundamentally as Fig. 6,

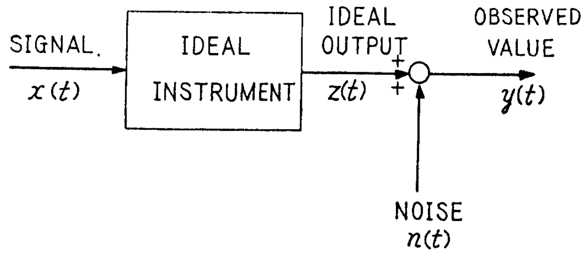

FIGURE 6. Observed value of continuous measurement can be considered as the superposition of ideal error-free output and a fictitious noise representing the errors.

provided that the measurement errors are independent of the signal values as well as time and the distribution of the magnitudes of the errors is Gaussian. In the figure, the signal $x(t)$ is measured by an ideal error-free instrument and the output $z(t)$ of this instrument is added to a fictitious noise which represents the measurement errors of the actual instrument. Then the following equation which gives the average information amount obtained per unit time through continuous measurement is derived by calculating the entropy difference in the frequency domain just like the manner for the sampling measurement.

$$I_0 = \int_0^\infty \log_2\left\{\frac{|G(f)|^2 \Phi_x(f)}{\Phi_n(f)} + 1\right\} df \quad \text{bits per unit time,} \qquad (23)$$

where $G(f)$ is the frequency response of the measuring instrument; $\Phi_x(f)$, power spectrum of the signal $x(t)$; $\Phi_n(f)$, power spectrum of the fictitious noise. It should be recognized that $|G(f)|^2\Phi_x(f)$ in Eq. (23) is no more than the power spectrum of the signal $z(t)$, the output of the ideal instrument. Accordingly, Eq. (23) can be regarded as a special case of Shannon's formula for the capacity of a communication channel [*1*]

$$C = W \log \frac{P+N}{N}, \tag{24}$$

(C: channel capacity, W: bandwidth of the channel,
P: signal power, N: noise power)

in which the power spectrum of the signal and noise are assigned as well as their power.

When it is assumed that the frequency response of the instrument is flat shaped within the frequency range $-f_c \sim f_c$ and the power spectrum of the noise which represents the measurement errors is also flat within this range as shown in Fig. 7. Then Eq. (23) is reduced to

$$I_0 = \int_0^{f_c} \log_2 \left\{ \frac{2f_c}{\sigma_n^2} \Phi_x(f) + 1 \right\} df, \tag{25}$$

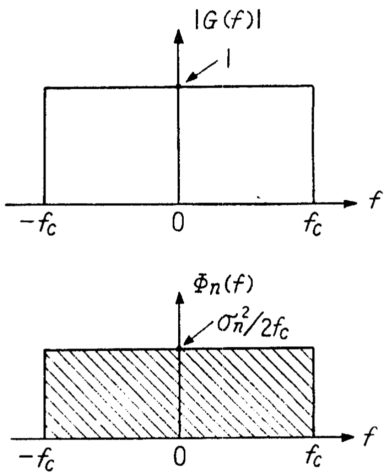

FIGURE 7. Assumed characteristics of the measuring instrument and the power spectrum of the fictitious noise.

where σ_n is the root mean square value of the errors. If it is also assumed that the power spectrum of the signal to be measured is limited within $-f_m \sim f_m$ as shown in Fig. 1 and $f_c \geq f_m$, then

$$I_0 = \int_0^{f_m} \log_2 \left\{ \frac{2f_c}{\sigma_n^2} \Phi_x(f) + 1 \right\} df. \tag{26}$$

Comparing Eq. (26) with Eq. (21) of the case of sampling measurement, the both are identically equal if $f_s = 2f_c$. In other words, the sampling measurement of the repetition frequency f_s is equivalent to the continuous measurement by the instrument which has the cut-off frequency of $f_c = f_s/2$, provided that the foregoing assumptions are valid and the root mean square value of the errors are equal in both cases.

7. Numerical Calculation of the Information Amount Obtained by Sampling Measurement

In the case of the sampling measurement of the Gaussian random signal which has the flat power spectrum between the frequencies $-f_m$ and f_m as shown in Fig. 8, i.e., in the case of a pure random signal, the average information amount per

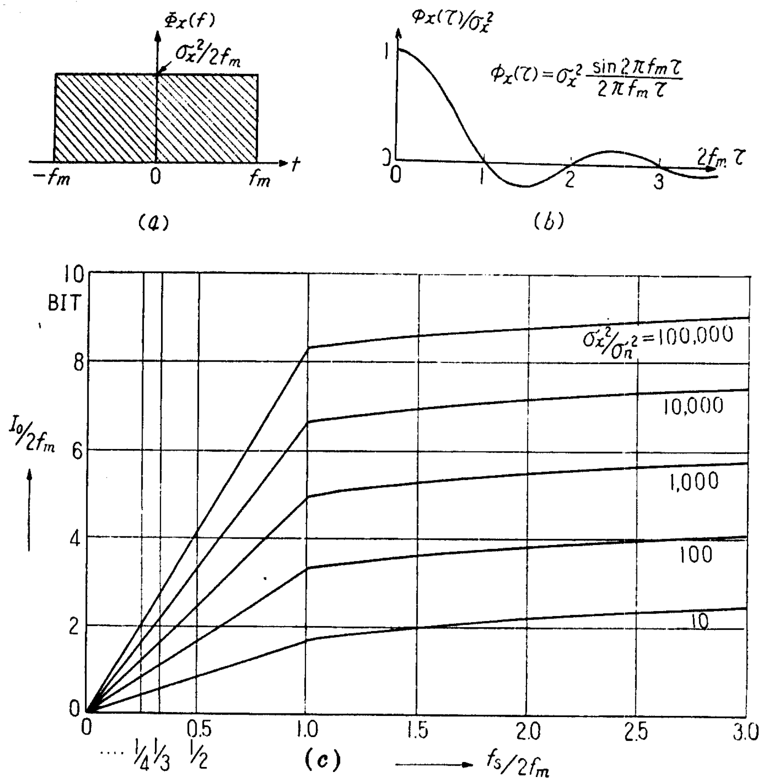

FIGURE 8. The power spectrum of a pure random signal (a), the auto-correlation function (b), and the average information amount obtained through the sampling measurement of this signal (c).

unit time is calculated numerically using Eq. (18). The result is

$$\frac{I_0}{2f_m}=\frac{1}{2}\left[\left\{\frac{\alpha f_s}{2f_m}-1\right\}\log_2\left\{\frac{(\alpha-1)f_s\sigma_n^2}{2f_m\sigma_n^2}+1\right\}\right.$$
$$\left.+\left\{1-(\alpha-1)\frac{f_s}{2f_m}\right\}\log_2\left\{\frac{\alpha f_s\sigma_x^2}{2f_m\sigma_n^2}+1\right\}\right] \quad \text{bits} \tag{27}$$

for the section

$$\frac{1}{\alpha}\leq\frac{f_s}{2f_m}\leq\frac{1}{\alpha-1}, \tag{28}$$

where α is any positive integer, and it is shown graphically in Fig. 8, in which $I_0/2f_m$ is shown as the function of $f_s/2f_m$, non-dimensional sampling frequency, with the parameter σ_x^2/σ_n^2, signal-to-noise energy ratio. The auto-correlation function $\phi_x(\tau)$ of the signal is also shown in the figure.

Although the curves in Fig. 8 are slightly concave in each section defined by Eq.

(28), they are almost linear within the range $f_s/2f_m<1$. When $f_s/2f_m=1$, i.e., at the critical sampling frequency appointed by the sampling theorem, the curves show sharp break off. In the range beyond this critical sampling frequency, Eq. (27) is reduced to

$$\frac{I_0}{2f_m}=\frac{1}{2}\log_2\left(\frac{f_s\sigma_x^2}{2f_m\sigma_n^2}+1\right)\qquad \text{bits} \tag{29}$$

by putting $\alpha=1$, and I_0 increases approximately by the order of $\log f_s$ with respect to f_s as mentioned in Sec. 5.

The second example is the case when the frequency band of the signal power spectrum does not start at zero as shown in Fig. 9. The auto-correlation of the signal,

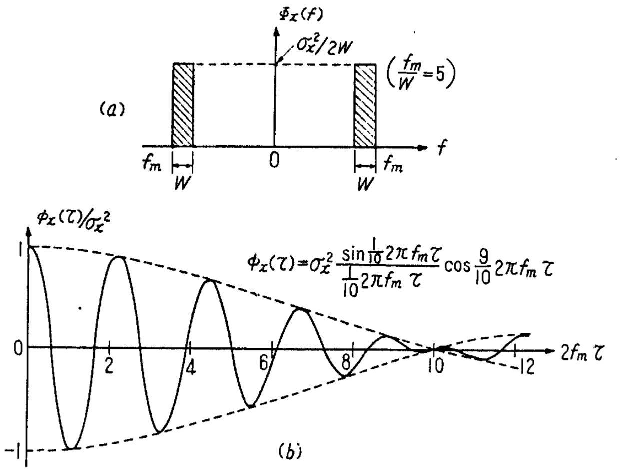

FIGURE 9. The power spectrum (a), and the auto-correlation function (b) of a Gaussena random signal, the frequency band of which does not start at zero.

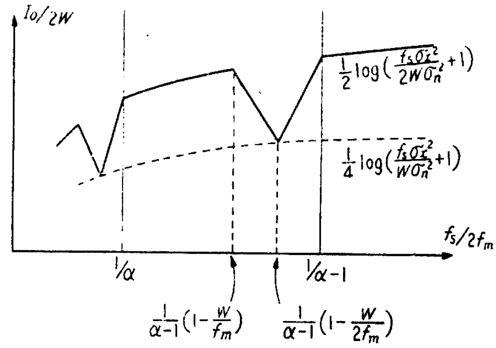

FIGURE 10. The information amount obtained through the sampling measurement of the signal of Fig. 9. The curves exhibit several minimum points.

$$\phi_x(\tau)=\sigma_x^2\frac{\sin\pi W\tau}{\pi W\tau}\cos 2\pi f_0\tau\,, \tag{30}$$

where

$$f_0=f_m-\frac{W}{2} \tag{31}$$

or the center frequency of the band, is also shown in the figure. These kinds of Gaussian random signals reveal some quasi-periodic characteristics in the wave-forms in the time domain. The application of Eq. (18) to this case is so complicated that it is examined only for the range $f_s\geq 2W$ and the results are shown graphically in Fig. 10. It should be noted here that, differing from Fig. 8, the curves of I_0 exhibit minimum points when

$$\frac{f_s}{2f_m}=\frac{1}{\alpha-1}\left(1-\frac{W}{2f_m}\right) \tag{32}$$

or

$$f_s=\frac{2f_0}{\alpha-1}\,, \tag{33}$$

where α is any positive integer. Comparing Eq. (33) with Eq. (30), it would be recognized that when Eq. (33) holds, the sampling interval $T_s = 1/f_s$ is equal to an integral multiple of the half period of the auto-correlation function of the signal. Consequently, it would be proper to say that under such circumstances the uncertainty of prediction for the future value of the signal decreases and, in turn, the entropy of the *a priori* probability is also decreased. In Fig. 11 a numerical example of this case is shown where the ratio f_m/W is assumed to be 5.

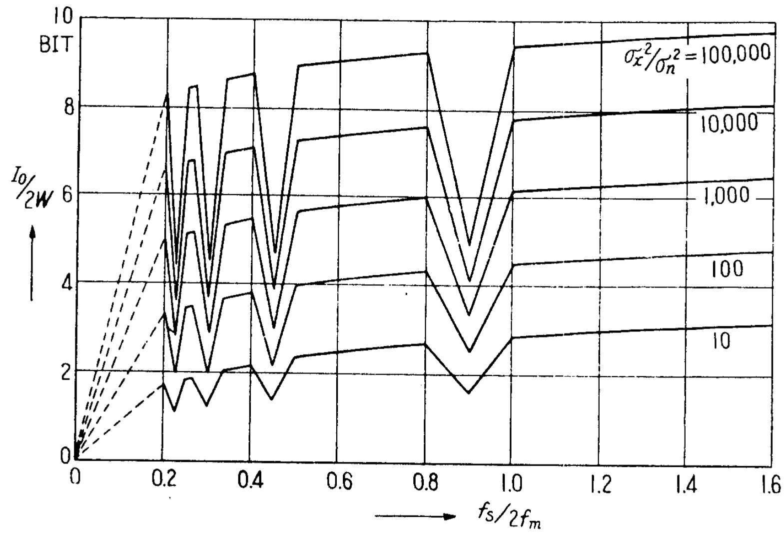

FIGURE 11. A numerical example of Fig. 10 where the ratio f_m/W is assumed to be 5.

Another example is shown in Fig. 12, in which the signal power spectrum is assumed to be a bell-jar type. Such a kind of Gaussian random signal would be obtained by passing a pure random signal through a first order resistance-capacitance combination system. The summation $\sum_n \Phi_x(f+nf_s)$ of this spectrum is easily obtained as a two-sided z-transform of the auto-correlation function by using the tables [7] [9]. Then

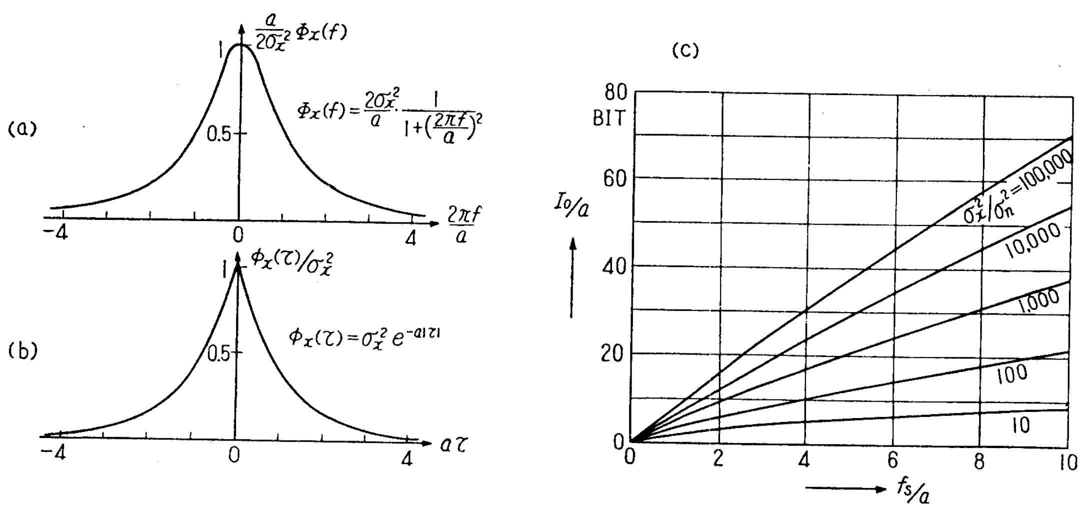

FIGURE 12. A signal power spectrum of bell-jar type (a), and its auto-correlation function (b). The relation of the information amount obtained through the sampling measurement of this signal vs. the sampling frequency is shown in (c).

$$f_s \sum_n \Phi_x(f+nf_s)=\sigma_x^2\frac{1-d^2}{(z-d)(z^{-1}-d)}$$
$$=\sigma_x^2\frac{1-d^2}{1+d^2-2d\cos 2\pi f T_s}, \tag{34}$$

where $z=e^{2\pi jfT_s}$ and $d=e^{-\alpha T_s}=e^{-\alpha/f_s}$. Substituting Eq. (34) into Eq. (18) and performing the integration, the information amount would be calculated. But it is difficult to perform this integration analytically, so a graphical method, in which a planimeter is used to measure the area under the curves for several combinations of f_s/α and σ_x^2/σ_n^2, is adopted. Comparison of Fig. 12 with Fig. 8 and Fig. 11 indicates that the curves in Fig. 12 bend more slowly when f_s is increased than those of the preceding examples. Further analysis reveals that the order of increase of I_0 with respect to f_s for this case is $\sqrt{f_s}$ provided that f_s is very large, in contrast with the order of $\log f_s$ in Fig. 8 and Fig. 11. This phenomenon may be considered as a consequence of the existence of the high frequency conponents in the signal power spectrum, that is, the waveform of the signal contains fine variations and by increasing f_s these fine structures of the signal are disclosed accordingly.

8. Sampling Measurement or Conversion of Time Varying Random Signals by Analog-Digital Converters

An important case of the sampling measurement of time varying signals is the measurement or coversion of the signals by analog-to-digital converters in which the quantization or "round-off" of the signal values takes place simultaneously with the sampling operation. If the quantization error is very large compared with other statistical errors, the signal value $x(t)$ lying somewhere within a quantization "box" of width q will yield a digital output $y(t)$ of the converter corresponding to the center of the box. Accordingly the probablility distribution of this output or the observed values $p(y)$ is discrete one. Each probability constituting the distribution is located at the center of each quantization box and it has a magnitude equal to the area under the probability distribution $p(x)$ of the signal values within the bounds of the corresponding box as shown in Fig. 13. As have

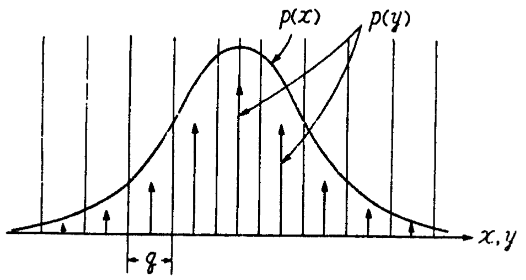

FIGURE 13. Quantization of a time varing random signal. The probability distribution of the magnitude of the quantized signal $p(y)$ is a discrete one.

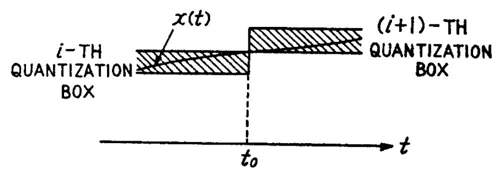

FIGURE 14. The time t_0, the instant of transition of a time varying random signal from a quantization box to the next, can be measured precisely when the sampling frequency of the A-D converter is increased.

already been mentioned Eq. (18) is not applicable to this case, because the measurement errors do not constitute a Gaussian distribution and they are not independent of the signal values, instead, the errors are completely determined when the signal values are assigned provided that the quantization is done perfectly.

The formula giving the information amount for this case will be derived from Eq. (11). The second term of Eq. (11), the conditional entropy $Av x^* H x^*(y^*)$, becomes zero according to the foregoing statement, so the average information amount I is equal to the entropy of the discrete probability distribution of the observed values, that is

$$I=H(y^*)=-\sum_i p_i \log p_i \qquad (\sum_i p_i=1)\,, \tag{35}$$

where i is an integer which specifies the quantization box and p_i is the probability getting the observed value corresponding to i-th quantization box. When a set of n observed value are obtained through the sampling measurement of a time varying random signal by an A-D converter, Eq. (35) must be considered in the signal space of n dimensions, and the calculation of I is so tedeous that it is almost impossible without the aid of electronic digital computers.

Although it will not be investigated further into this situation, the following comments would be added. If the signal to be measured is constant, the measurement of this signal by an A-D converter is trivial because after knowing the quantization box in which the signal is lying, the following measurements are no more than the repetition of the preceding measurements and no diminution of the uncertainty is achieved. But in the case where a time varying random signal is measured, the instant of transition of the signal from a quantization box to the next, which is shown in Fig. 14, can be measured more and more precisely with the increase of the sampling frequency and, in turn, the signal can be decided precisely as a whole. Hence, the average information amount per unit time obtained through an A-D converter would increase boundlessly when the sampling frequency is increased infinitely.

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