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Deformation and Thermal Stress in a Rectangular Plate Subjected to Aerodynamic Heating

(For the Case of Simply Supported Edges)

By

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Summary. The deformation and thermal stress for a rectangular plate, subjected to an arbitrary symmetrical temperature distribution, are analyzed for the case where the edges of the plate are simply supported, taking the finite deformation into account. The fundamental non-linear simultaneous partial differential equations for the thermoelastic problem are derived from the variational principle and are solved, and it is shown that, if there exists the temperature gradient through the thickness of the plate as seen in the aerodynamic heating, the plate starts to deflect at the moment of heating and does not exhibit the buckling phenomenon according to the mode of temperature distribution and the boundary conditions.

Some numerical examples are given i) for the case where the temperature distribution through the thickness is specified as linear, and ii) for the case of instantaneous heating where the temperature distribution through the thickness is given in a function of time, and then some discussions on the analytical results are given.

1. INTRODUCTION

The problems on deformations and thermal stresses in structural members subjected to aerodynamic heating have become important from the viewpoint of the strength and rigidity of structure and of the aeroelasticity ever since the advent of supersonic airplanes and missiles. It is usual that the deformation problems such as buckling are considered under uniform temperature distribution through the thickness based on the assumption that the structural members are "thermally thin", and that the thermal stresses in considerably thick structural members are simply given only by any temperature distribution through the thickness without taking account of the deformation.

However, the supersonic airplanes and missiles have a tendency to having thick plate structures, and they are rapidly heated aerodynamically, so the temperature is not uniform through the thickness which has the considerable effects on the stress and deformation. For example, if these thermal moments are taken into account, it is supposed that the members do not present the phenomenon of "Euler buckling" but start to deflect at the moment of heating depending on the distribution of temperature and the boundary conditions. But most of the previous investigations [*1*]∼[*4*] have been discussed mainly on the critical buckling values, and the discussions on the deformation under transient heating conditions have been limited to the one-dimensional case [5] [6] only.

However, we consider that it is necessary to take account of the deformation, first of all, which takes place under the transient heating condition depending on the temperature distribution and the boundary conditions, and that the thermal stresses will be thereafter made clear of itself. The above-mentioned fundamental purport of the problems was shown in the one-dimensional case of rectangular beam or flat strip by one of the present authors [6].

In this paper, the two-dimensional flat plate subjected to aerodynamic heating which is of practical importance will be given restricting our consideration to the case of the simply supported boundary condition and the arbitrary, symmetrically distributed temperature. In this analysis, the fundamental equations and the boundary conditions are derived from the variational principle for the case of finite deformation.

Nomenclatures and Symbols

a, b, d	half length, half width and thickness of rectangular plate, respectively.
h	heat transfer coefficient.
p	aerodynamic pressure normal to the plate.
t	time.
u, v, w	displacement components in the median surface in the x, y and z directions, respectively, which are functions of x and y.
A, B, C	integral constants.
D	flexural modulus of rigidity. $D \equiv Ed^3/12(1-\nu^2)$
E	modulus of elasticity.
F	free energy per unit volume.
Q	heat flow into unit volume.
S	entropy per unit volume.
T	temperature rise above the unstrained state.
T_E	adiabatic wall temperature.
U	internal energy per unit volume.
Π	total potential energy.
Π_i	strain energy per unit volume.
α, ρ, c_p, k	coefficient of thermal expansion, density, specific heat and thermal conductivity of the plate material, respectively.
δ	maximum deflection at the midpoint of the plate.
$\varepsilon_{11}, \varepsilon_{22}$	extensional strains in the x and y directions, respectively.
ε_{12}	shearing strain in the xy-plane.
σ_{11}, σ_{22}	extensional stresses in the x and y directions, respectively.
σ_{12}	shearing stress in the xy-plane.
κ_1, κ_2	curvatures of the median surface about the x and y axes, respectively.
κ_{12}	twist of the median surface.
λ	aspect ratio of the rectangular plate. $\lambda \equiv a/b$.

$\mu_m = m\pi/2\lambda$, m: odd integer.

ν	Poisson's ratio.

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\quad \nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}.$$

Subscripts "x" and "y" mean the partial differentiation with respect to x and y, respectively.

Bar over letter refers to the median surface.

2. Derivation of Fundamental Equations and Boundary Conditions

We consider a rectangular flat plate as shown in Fig. 1, and assume that the plate is heated at the upper face and there exists the temperature gradient through the thickness in the initial stage of heating.

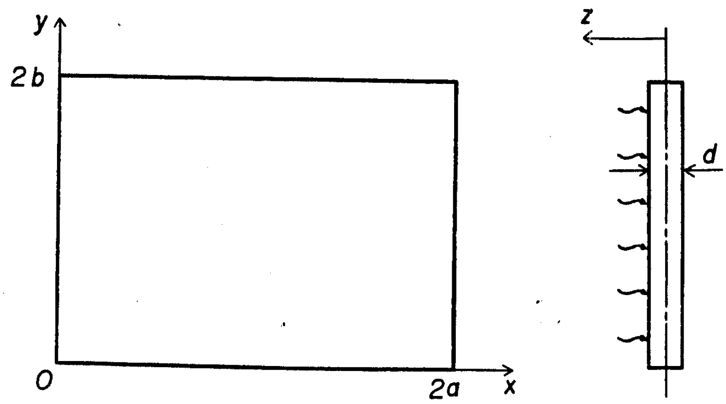

FIGURE 1. Rectangular flat plate.

The two-dimensional stress-strain law can be given by

$$\left.\begin{aligned} E\varepsilon_{11}&=\sigma_{11}-\nu\sigma_{22}+E\alpha T,\\ E\varepsilon_{22}&=\sigma_{22}-\nu\sigma_{11}+E\alpha T,\\ G\varepsilon_{12}&=\frac{E}{2(1+\nu)}\varepsilon_{12}=\sigma_{12}. \end{aligned}\right\} \tag{1}$$

The plate is assumed to be comparatively thin and the plane normal to the median surface is assumed to remain plane after deformation, and the shearing deformation normal to the plate is neglected. Then the strain components can be given by

$$\left.\begin{aligned} \varepsilon_{11}&=\bar{\varepsilon}_{11}-z\kappa_1,\\ \varepsilon_{22}&=\bar{\varepsilon}_{22}-z\kappa_2,\\ \varepsilon_{12}&=\bar{\varepsilon}_{12}-2z\kappa_{12}. \end{aligned}\right\} \tag{2}$$

Introducing the strain components in the median surface and the curvatures which can be expressed by Eqs. (3) and (4) taking account of the first order terms in the finite deformation only,

$$\left.\begin{aligned} \bar{\varepsilon}_{11}&=u_x+\frac{1}{2}w_x^2,\\ \bar{\varepsilon}_{22}&=v_y+\frac{1}{2}w_y^2,\\ \bar{\varepsilon}_{12}&=u_y+v_x+w_xw_y. \end{aligned}\right\} \tag{3}$$

$$\kappa_1 = w_{xx}, \quad \kappa_2 = w_{yy}, \quad \kappa_{12} = w_{xy}, \tag{4}$$

we obtain the following expressions for the strain components of Eqs. (2):

$$\left.\begin{aligned} \varepsilon_{11} &= u_x + \frac{1}{2} w_x^2 - z w_{xx}, \\ \varepsilon_{22} &= v_y + \frac{1}{2} w_y^2 - z w_{yy}, \\ \varepsilon_{12} &= u_y + v_x + w_x w_y - 2 z w_{xy}. \end{aligned}\right\} \tag{5}$$

We will derive the equilibrium equation by the variational method with the aid of the well-known theorem of the stationary potential energy. It has been already pointed out by Hemp [7] that the "free energy" F replaces the usual strain energy for the case of iso-thermal thermoelasticity. The outline for the case of this problem will be shown as below.

The macroscopic state at a point in a body will be defined by the strain components and the temperature. If mechanical or thermal loading is not so abrupt as to excite vibration in the body, a small change of state will be expressed by the First Law of Thermodynamics (Law of Conservation of Energy).

$$\delta U = \delta Q + \delta \Pi_i. \tag{6}$$

Further, the reversibility of change of state being established in a perfect elastic body, the Second Law of Thermodynamics requires that

$$\delta Q = (T + T_0)\delta S. \tag{7}$$

The "free energy" F per unit volume is defined by

$$F = U - (T + T_0)S, \tag{8}$$

and is regarded as a function of the strain components and the temperature. Eqs. (6), (7) and (8) yield

$$\delta F = \delta U - (T + T_0)\delta S - S\delta T = \sum \sigma_{ij}\delta\varepsilon_{ij} - S\delta T, \tag{9}$$

and the following relations can be obtained:

$$\frac{\partial F}{\partial \varepsilon_{ij}} = \sigma_{ij}, \quad \frac{\partial F}{\partial T} = -S. \tag{10}$$

For the case of adiabatic change, we may take either U or Π_i, which are the same with each other.

The formula for F can be obtained with the aid of Eqs. (1) and (10).

$$F = \frac{E}{2(1-\nu^2)}\left[\varepsilon_{11}^2 + 2\nu\varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}^2 + \frac{1}{2}(1-\nu)\varepsilon_{12}^2\right] - \frac{E\alpha T}{(1-\nu)}(\varepsilon_{11} + \varepsilon_{22}) + C_T(T), \tag{11}$$

where $C_T(T)$ is a function of T only and can be expressed by introducing the specific heat.

Now, we consider a rectangular plate subjected to the aerodynamic heating and to the aerodynamic pressure normal to the plate only as the external force, then the total potential energy Π can be expressed by

$$\Pi = \int_V F dV - \int_0^{2b}\int_0^{2a} p(x, y) w(x, y)\, dx dy. \tag{12}$$

Substituting Eqs. (11), (2) and (4) into the first term of Eq. (12), we finally obtain the following expression for Π:

$$\Pi=\frac{Ed}{2(1-\nu^2)}\Big[\int_0^{2b}\int_0^{2a}\Big\{\bar{\varepsilon}_{11}^2+2\nu\bar{\varepsilon}_{11}\bar{\varepsilon}_{22}+\bar{\varepsilon}_{22}^2+\frac{1}{2}(1-\nu)\bar{\varepsilon}_{12}^2\Big\}dxdy$$
$$+\frac{d^2}{12}\int_0^{2b}\int_0^{2a}\{w_{xx}^2+2\nu w_{xx}w_{yy}+w_{yy}^2+2(1-\nu)w_{xy}^2\}\,dx\,dy\Big]$$
$$-\frac{E\alpha}{(1-\nu)}\Big[\int_0^{2b}\int_0^{2a}(\bar{\varepsilon}_{11}+\bar{\varepsilon}_{22})\bar{T}dxdy-\int_0^{2b}\int_0^{2a}(w_{xx}+w_{yy})\tilde{T}dxdy\Big]$$
$$+d\int_0^{2b}\int_0^{2a}C_T(T)dxdy-\int_0^{2b}\int_0^{2a}p(x,y)w\,dxdy, \tag{13}$$

where

$$\bar{T}=\int_{-d/2}^{d/2}T(x,y,z)dz,\quad \tilde{T}=\int_{-d/2}^{d/2}zT(x,y,z)dz. \tag{14}$$

Substituting Eqs. (3) into the terms of $\bar{\varepsilon}_{11}$, $\bar{\varepsilon}_{22}$ and $\bar{\varepsilon}_{12}$ of Eq. (13), and by the variational operation on u, v and w with the aid of the usual theorem of the stationary potential energy, that is, from $\delta\Pi=0$, we can obtain the following equilibrium equations in the three directions.

$$\frac{Ed}{2(1-\nu^2)}\Big[2\frac{\partial}{\partial x}\Big\{\Big(u_x+\frac{1}{2}w_x^2\Big)+\nu\Big(v_y+\frac{1}{2}w_y^2\Big)\Big\}$$
$$+(1-\nu)\frac{\partial}{\partial y}(u_y+v_x+w_xw_y)\Big]=\frac{E\alpha}{(1-\nu)}\bar{T}_x, \tag{15}$$

$$\frac{Ed}{2(1-\nu^2)}\Big[2\frac{\partial}{\partial y}\Big\{\Big(v_y+\frac{1}{2}w_y^2\Big)+\nu\Big(u_x+\frac{1}{2}w_x^2\Big)\Big\}$$
$$+(1-\nu)\frac{\partial}{\partial x}(u_y+v_x+w_xw_y)\Big]=\frac{E\alpha}{(1-\nu)}\bar{T}_y, \tag{16}$$

$$\frac{Ed}{2(1-\nu^2)}\Big(2\frac{\partial}{\partial x}\Big[\Big\{\Big(u_x+\frac{1}{2}w_x^2\Big)+\nu\Big(v_y+\frac{1}{2}w_y^2\Big)\Big\}w_x\Big]+2\frac{\partial}{\partial y}\Big[\Big\{\Big(v_y+\frac{1}{2}w_y^2\Big)$$
$$+\nu\Big(u_x+\frac{1}{2}w_x^2\Big)\Big\}w_y\Big]+(1-\nu)\frac{\partial}{\partial x}\Big[(u_y+v_x+w_xw_y)w_y\Big]$$
$$+(1-\nu)\frac{\partial}{\partial y}\Big[(u_y+v_x+w_xw_y)w_x\Big]\Big)$$
$$-\frac{Ed^3}{12(1-\nu^2)}[w_{xxxx}+2w_{xxyy}+w_{yyyy}]$$
$$=\frac{E\alpha}{(1-\nu)}\Big[\frac{\partial}{\partial x}(\bar{T}w_x)+\frac{\partial}{\partial y}(\bar{T}w_y)+(\tilde{T}_{xx}+\tilde{T}_{yy})\Big]-p(x,y). \tag{17}$$

If the extensional stresses $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ and the shearing stress $\bar{\sigma}_{12}$ in the median surface corresponding to $\bar{\varepsilon}_{11}$, $\bar{\varepsilon}_{22}$ and $\bar{\varepsilon}_{12}$ are introduced with the following relations:

$$\left.\begin{aligned}\bar{\varepsilon}_{11}&=\frac{1}{E}(\bar{\sigma}_{11}-\nu\bar{\sigma}_{22})+\frac{\alpha\bar{T}}{d},\\ \bar{\varepsilon}_{22}&=\frac{1}{E}(\bar{\sigma}_{22}-\nu\bar{\sigma}_{11})+\frac{\alpha\bar{T}}{d},\\ \bar{\varepsilon}_{12}&=\frac{2(1+\nu)}{E}\bar{\sigma}_{12},\end{aligned}\right\}\tag{18}$$

then Eqs. (15) and (16) are reduced to

$$\left.\begin{aligned}\frac{\partial\bar{\sigma}_{11}}{\partial x}+\frac{\partial\bar{\sigma}_{12}}{\partial y}=0,\\ \frac{\partial\bar{\sigma}_{12}}{\partial x}+\frac{\partial\bar{\sigma}_{22}}{\partial y}=0.\end{aligned}\right\}\tag{19}$$

This pair of equations can be satisfied by introducing the well-known Airy's stress function $\chi(x, y)$, defined by the relations

$$\bar{\sigma}_{11}=\chi_{yy},\quad \bar{\sigma}_{22}=\chi_{xx},\quad \bar{\sigma}_{12}=-\chi_{xy}.\tag{20}$$

Then, the equilibrium equation of Eq. (17) in the z direction is expressed by

$$D\nabla^4 w=d(\chi_{yy}w_{xx}-2\chi_{xy}w_{xy}+\chi_{xx}w_{yy})-\frac{E\alpha}{(1-\nu)}\nabla^2\tilde{T}+p.\tag{21}$$

Eliminating u and v from the three of Eqs. (3) and using Eqs. (18) and (20), the following compatibility equation can be obtained.

$$\nabla^4\chi=E(w_{xy}^2-w_{xx}w_{yy})-\frac{E\alpha}{d}\nabla^2\bar{T}.\tag{22}$$

These equations of (21) and (22) coincide with the ones given by Mar [8].

All the possible boundary conditions obtained from the operation of $\delta\Pi=0$ are summarized in Eqs. (23).

At $x=0$ and $2a$,

$$\delta u=0 \text{ or } \frac{Ed}{(1-\nu^2)}\left[\left(u_x+\frac{1}{2}w_x^2\right)+\nu\left(v_y+\frac{1}{2}w_y^2\right)\right]-\frac{E\alpha}{(1-\nu)}\bar{T}=0,\tag{23-1}$$

$$\delta v=0 \text{ or } u_y+v_x+w_xw_y=0,\tag{23-2}$$

$$\begin{aligned}\delta w=0 \text{ or } &\frac{Ed}{(1-\nu^2)}\left[\left\{\left(u_x+\frac{1}{2}w_x^2\right)+\nu\left(v_y+\frac{1}{2}w_y^2\right)\right\}w_x+\frac{(1-\nu)}{2}(u_y+v_x+w_xw_y)w_y\right.\\ &\left.-\frac{d^2}{12}\{w_{xxx}+(2-\nu)w_{xyy}\}\right]-\frac{E\alpha}{(1-\nu)}(\bar{T}w_x+\tilde{T}_x)=0,\end{aligned}\tag{23-3}$$

$$\delta w_x=0 \text{ or } D(w_{xx}+\nu w_{yy})+\frac{E\alpha}{(1-\nu)}\tilde{T}=0,\tag{23-4}$$

at $y=0$ and $2b$,

$$\delta u=0 \text{ or } u_y+v_x+w_xw_y=0,\tag{23-5}$$

$$\delta v=0 \text{ or } \frac{Ed}{(1-\nu^2)}\left[\left(v_y+\frac{1}{2}w_y^2\right)+\nu\left(u_x+\frac{1}{2}w_x^2\right)\right]-\frac{E\alpha}{(1-\nu)}\bar{T}=0,\tag{23-6}$$

$$\delta w=0 \text{ or } \frac{Ed}{(1-\nu^2)}\left[\left\{\left(v_y+\frac{1}{2}w_y^2\right)+\nu\left(u_x+\frac{1}{2}w_x^2\right)\right\}w_y+\frac{(1-\nu)}{2}(u_y+v_x+w_xw_y)w_x\right.$$

$$-\frac{d^2}{12}\{w_{yyy}+(2-\nu)w_{xxy}\}\Big]-\frac{E\alpha}{(1-\nu)}(\bar{T}w_y+\tilde{T}_y)=0, \tag{23-7}$$

$$\delta w_y=0 \text{ or } D(w_{yy}+\nu w_{xx})+\frac{E\alpha}{(1-\nu)}\tilde{T}=0, \tag{23-8}$$

at $x=0$ and $2a$; $y=0$ and $2b$,

$$\delta w = 0 \text{ or } w_{xy}=0. \tag{23-9}$$

The problem will then be attributed to the solution of the nonlinear simultaneous partial differential equations Eqs. (21) and (22) under the reasonable boundary conditions when the distributions of $\bar{T}$ and $\tilde{T}$ are given.

Equations (21) and (22) which take account of large deflection are the fundamental equations for a heated plate. If the term concerning the temperature are omitted, they are reduced to the well-known equations by von Kármán. When the terms of $\tilde{T}$ and $p(x, y)$ in Eq. (21) and the first non-linear term of large deflection in Eq. (22) are absent, these linear simultaneous partial differential equations become the ones which were used by many investigators to obtain the critical buckling values under the uniform temperature distribution through the thickness.

The purpose of this paper is to make clear the behaviours of a heated plate for the complicated case where all the terms in Eqs. (21) and (22) are taken into account. As seen in Eq. (21), the existence of the term of $\nabla^2\tilde{T}$ is equivalent to the action of the normal pressure, and so it is considered that a plate will start to deflect from the beginning of heating without exhibiting the buckling behaviour at the critical temperature. Furthermore, the existence of $\tilde{T}$ on the boundary, which is not clamped but simply supported, is equivalent to the actions of edge moment on the boundary, even if $\nabla^2\tilde{T}$ does not exist over the plate, and so it is expected that a plate will deflect from the beginning of heating.

Such a phenomenon was already shown in the one-dimensional beams or flat strips by one of the present authors [*6*]. It is too difficult to obtain the exact solution of the non-linear simultaneous partial differential equations of Eqs. (21) and (22), and so the successive approximation method called as "Poincaré's method" [*9*], which was found to be effective in the analyses of the vibration of bar [*10*] and of the buckling problem for rectangular plate under compression [*11*], will be used here.

The thermal stresses in a point in a plate are defined by the following equations after determining the displacements in the three directions.

$$\sigma_{11}=\frac{E}{(1-\nu^2)}\Big[\Big(u_x+\frac{1}{2}w_x^2\Big)+\nu\Big(v_y+\frac{1}{2}w_y^2\Big)-z(w_{xx}+\nu w_{yy})\Big]-\frac{E\alpha T}{(1-\nu)}, \tag{24-1}$$

$$\sigma_{22}=\frac{E}{(1-\nu^2)}\Big[\Big(v_y+\frac{1}{2}w_y^2\Big)+\nu\Big(u_x+\frac{1}{2}w_x^2\Big)-z(w_{yy}+\nu w_{xx})\Big]-\frac{E\alpha T}{(1-\nu)}, \tag{24-2}$$

$$\sigma_{12}=\frac{E}{2(1+\nu)}[(u_y+v_x+w_xw_y)-2zw_{xy}]. \tag{24-3}$$

3. SULUTIONS OF PLATE SIMPLY SUPPORTED AT FOUR EDGES

A rectangular plate which is simply supported at four edges and subjected only to an arbitrary symmetrical temperature due to the heating at its upper face, that is $p(x, y)=0$ in Eq. (21), is considered.

The temperature distribution over the plate is assumed to be symmetrical about the midpoint of the plate (Fig. 1), and then $\bar{T}(x, y)$ and $\tilde{T}(x, y)$ can be expressed as

$$\frac{\bar{T}}{d}=\sum_{i,j:\,\text{even}}\sum \bar{T}_{ij}\cos\frac{i\pi x}{2a}\cos\frac{j\pi y}{2b}, \tag{25}$$

$$\frac{\tilde{T}}{d^2}=\tilde{T}_e+\sum_{p,q:\,\text{odd}}\sum \tilde{T}_{pq}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}. \tag{26}$$

The symmetrical deflection mode corresponding to the symmetrical temperature distribution is assumed as follows in the first approximation:

$$w(x, y)=\delta\sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}, \tag{27}$$

where δ is the maximum deflection at the midpoint of the plate. Substituting Eqs. (25) and (27) into Eq. (22), we obtain the following expressions for the compatibility equation:

$$\begin{aligned}\nabla^4\chi=&\frac{\pi^4E\delta^2}{32a^2b^2}\left(\cos\frac{\pi x}{a}+\cos\frac{\pi y}{b}\right)+E\alpha\left\{\sum_{i=2}^{\infty}\left(\frac{i\pi}{2a}\right)^2\bar{T}_{i0}\cos\frac{i\pi x}{2a}\right.\\&\left.+\sum_{j=2}^{\infty}\left(\frac{j\pi}{2b}\right)^2\bar{T}_{0j}\cos\frac{j\pi y}{2b}+\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left[\left(\frac{i\pi}{2a}\right)^2+\left(\frac{j\pi}{2b}\right)^2\right]\bar{T}_{ij}\cos\frac{i\pi x}{2a}\cos\frac{j\pi y}{2b}\right\},\end{aligned} \tag{28}$$

The solution of χ obtained by integrating Eq. (28) is

$$\begin{aligned}\chi=&\frac{C_1}{2}x^2+\frac{C_2}{2}y^2+\frac{E\delta^2}{32}\left[\left(\frac{a}{b}\right)^2\cos\frac{\pi x}{a}+\left(\frac{b}{a}\right)^2\cos\frac{\pi y}{b}\right]\\&+E\alpha\left\{\sum_{i=2}^{\infty}\frac{\bar{T}_{i0}}{(i\pi/2a)^2}\cos\frac{i\pi x}{2a}+\sum_{j=2}^{\infty}\frac{\bar{T}_{0j}}{(j\pi/2b)^2}\cos\frac{j\pi y}{2b}\right.\\&\left.+\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{\bar{T}_{ij}}{[(i\pi/2a)^2+(j\pi/2b)^2]}\cos\frac{i\pi x}{2a}\cos\frac{j\pi y}{2b}\right\}.\end{aligned} \tag{29}$$

The integral constants C_1 and C_2 can be determined so as to satisfy the boundary conditions of Eqs. (23–1) and (23–6), that is $u=0$ at $x=0$ and $2a$ and $v=0$ at $y=0$ and $2b$, as

$$\left.\begin{aligned}C_1&=-\frac{E\alpha\bar{T}_{00}}{(1-\nu)}+\frac{\pi^2E\delta^2}{32(1-\nu^2)}\left(\frac{\nu}{a^2}+\frac{1}{b^2}\right),\\C_2&=-\frac{E\alpha\bar{T}_{00}}{(1-\nu)}+\frac{\pi^2E\delta^2}{32(1-\nu^2)}\left(\frac{1}{a^2}+\frac{\nu}{b^2}\right).\end{aligned}\right\} \tag{30}$$

Then, the displacement components $u(x, y)$ and $v(x, y)$ in a point on the median surface in the x and y directions, respectively, are given by the following expressions:

$$u(x, y)=\frac{\pi^2\delta^2}{32}\left[\frac{\nu}{b^2}-\frac{1}{a^2}\left(1-\cos\frac{\pi y}{b}\right)\right]\frac{a}{\pi}\sin\frac{\pi x}{a}$$

$$+\alpha(1+\nu)\left[\sum_{i=2}^{\infty}\frac{\bar{T}_{i0}}{(i\pi/2a)}\sin\frac{i\pi x}{2a}+\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{(i\pi/2a)\bar{T}_{ij}}{[(i\pi/2a)^2+(j\pi/2b)^2]}\sin\frac{i\pi x}{2a}\cos\frac{j\pi y}{2b}\right], \tag{31}$$

$$v(x,y)=\frac{\pi^2\delta^2}{32}\left[\frac{\nu}{a^2}-\frac{1}{b^2}\left(1-\cos\frac{\pi x}{a}\right)\right]\frac{b}{\pi}\sin\frac{\pi y}{b}$$
$$+\alpha(1+\nu)\left[\sum_{j=2}^{\infty}\frac{\bar{T}_{0j}}{(j\pi/2b)}\sin\frac{j\pi y}{2b}+\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{(j\pi/2b)\bar{T}_{ij}}{[(i\pi/2a)^2+(j\pi/2b)^2]}\cos\frac{i\pi x}{2a}\sin\frac{j\pi y}{2b}\right]. \tag{32}$$

where, v and u are not zero at $x=0$, $2a$ and $y=0$, $2b$, respectively; however, the condition of $u_y+v_x+w_xw_y=0$ is satisfied at all edges and so the condition of $\delta\Pi=0$ (the minimum potential energy) is exactly satisfied [(Eqs. (23–2) and (23–5)].

Substituting the stress function χ just obtained as in Eq. (29) and the deflection mode w of the first approximation in Eq. (27) into Eq. (21), we obtain the following expression for the equilibrium equation:

$$\frac{Ed^2}{12(1-\nu^2)}\nabla^4 w$$
$$=-\left\{\left[C_2\left(\frac{\pi}{2a}\right)^2+C_1\left(\frac{\pi}{2b}\right)^2\right]\delta+\frac{E\delta^3}{16}\left[\left(\frac{\pi}{2a}\right)^4+\left(\frac{\pi}{2b}\right)^4\right]\right\}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}$$
$$+\frac{E\delta^3}{16}\left[\left(\frac{\pi}{2a}\right)^4\sin\frac{\pi x}{2a}\sin\frac{3\pi y}{2b}+\left(\frac{\pi}{2b}\right)^4\sin\frac{3\pi x}{2a}\sin\frac{\pi y}{2b}\right]$$
$$+\frac{E\alpha\delta}{2}\left\{\sum_{j=2}^{\infty}\left(\frac{\pi}{2a}\right)^2\bar{T}_{0j}\left[-\sin\frac{\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}+\sin\frac{\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}\right]\right.$$
$$\left.+\sum_{i=2}^{\infty}\left(\frac{\pi}{2b}\right)^2\bar{T}_{i0}\left[-\sin\frac{(i-1)\pi x}{2a}\sin\frac{\pi y}{2b}+\sin\frac{(i+1)\pi x}{2a}\sin\frac{\pi y}{2b}\right]\right\}$$
$$+E\alpha\delta\left(\frac{\pi}{4ab}\right)^2\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{\bar{T}_{ij}}{[(i/a)^2+(j/b)^2]}\left\{(i-j)^2\left[\sin\frac{(i-1)\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}\right.\right.$$
$$\left.+\sin\frac{(i+1)\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}\right]-(i+j)^2\left[\sin\frac{(i-1)\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}\right.$$
$$\left.\left.+\sin\frac{(i+1)\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}\right]\right\}$$
$$+\frac{E\alpha d}{(1-\nu)}\sum_{p,q:\,\text{odd}}\sum\left[\left(\frac{p\pi}{2a}\right)^2+\left(\frac{q\pi}{2b}\right)^2\right]\tilde{T}_{pq}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}. \tag{33}$$

The solution of w obtained by integrating Eq. (33) is given as follows:

$$\frac{Ed^2}{12(1-\nu^2)}w$$
$$=-\left\{\left[C_2\left(\frac{\pi}{2a}\right)^2+C_1\left(\frac{\pi}{2b}\right)^2\right]\delta\right.$$
$$\left.+\frac{E\delta^3}{16}\left[\left(\frac{\pi}{2a}\right)^4+\left(\frac{\pi}{2b}\right)^4\right]\right\}\frac{1}{[(\pi/2a)^2+(\pi/2b)^2]^2}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}$$
$$+\frac{E\delta^3}{16}\left\{\frac{(\pi/2a)^4}{[(\pi/2a)^2+(3\pi/2b)^2]^2}\sin\frac{\pi x}{2a}\sin\frac{3\pi y}{2b}+\frac{(\pi/2b)^4}{[(3\pi/2a)^2+(\pi/2b)^2]^2}\sin\frac{3\pi x}{2a}\sin\frac{\pi y}{2b}\right\}$$

$$+\frac{E\alpha\delta}{2}\Bigg\{\sum_{j=2}^{\infty}\bar{T}_{0j}\Bigg[\frac{-(\pi/2a)^2}{[(\pi/2a)^2+([j-1]\pi/2b)^2]^2}\sin\frac{\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}$$
$$+\frac{(\pi/2a)^2}{[(\pi/2a)^2+([j+1]\pi/2b)^2]^2}\sin\frac{\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}\Bigg]$$
$$+\sum_{i=2}^{\infty}\bar{T}_{i0}\Bigg[\frac{-(\pi/2b)^2}{[([i-1]\pi/2a)^2+(\pi/2b)^2]^2}\sin\frac{(i-1)\pi x}{2a}\sin\frac{\pi y}{2b}$$
$$+\frac{(\pi/2b)^2}{[([i+1]\pi/2a)^2+(\pi/2b)^2]^2}\sin\frac{(i+1)\pi x}{2a}\sin\frac{\pi y}{2b}\Bigg]\Bigg\}$$
$$+E\alpha\delta\left(\frac{\pi}{4ab}\right)^2\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{\bar{T}_{ij}}{[(i/a)^2+(j/b)^2]}$$
$$\times\Bigg\{(i-j)^2\Bigg[\frac{1}{\{([i-1]\pi/2a)^2+([j-1]\pi/2b)^2\}^2}\sin\frac{(i-1)\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}$$
$$+\frac{1}{\{([i+1]\pi/2a)^2+([j+1]\pi/2b)^2\}^2}\sin\frac{(i+1)\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}\Bigg]$$
$$-(i+j)^2\Bigg[\frac{1}{\{([i-1]\pi/2a)^2+([j+1]\pi/2b)^2\}^2}\sin\frac{(i-1)\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}$$
$$+\frac{1}{\{([i+1]\pi/2a)^2+([j-1]\pi/2b)^2\}^2}\sin\frac{(i+1)\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}\Bigg]\Bigg\}$$
$$+\frac{E\alpha d}{(1-\nu)}\sum_{p,q:\,\mathrm{odd}}\sum\frac{\widetilde{T}_{pq}}{[(p\pi/2a)^2+(q\pi/2b)^2]}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}$$
$$+\sum_{m:\,\mathrm{odd}}\Bigg\{\Bigg[A_{1m}\cosh\frac{m\pi(y-b)}{2a}+A_{2m}\frac{m\pi(y-b)}{2a}\sinh\frac{m\pi(y-b)}{2a}\Bigg]\sin\frac{m\pi x}{2a}$$
$$+\Bigg[B_{1m}\cosh\frac{m\pi(x-a)}{2b}+B_{2m}\frac{m\pi(x-a)}{2b}\sinh\frac{m\pi(x-a)}{2b}\Bigg]\sin\frac{m\pi y}{2b}\Bigg\}. \quad (34)$$

where A_{1m}, A_{2m}, B_{1m} and B_{2m} are the integral constants and are to be determined from the boundary conditions on w. That is, from the condition of $w=0$ at $x=0$ and $2a$, B_{1m} can be expressed as follows:

$$B_{1m}=-B_{2m}\mu_{2m}\tanh\mu_{2m}, \quad (35\text{-}1)$$

where

$$\mu_{2m}=\frac{m\pi a}{2b} \quad (36\text{-}1)$$

Similarly, from the condition of $w=0$ at $y=0$ and $2b$,

$$A_{1m}=-A_{2m}\mu_{1m}\tanh\mu_{1m}, \quad (35\text{-}2)$$

where

$$\mu_{1m}=\frac{m\pi b}{2a}. \quad (36\text{-}2)$$

Furthermore, with the aid of Eqs. (23–4) and (23–8), A_{2m} and B_{2m} can be determined as follows:

$$2d\sum_{m:\,\mathrm{odd}}B_{2m}\left(\frac{m\pi}{2b}\right)^2\cosh\mu_{2m}\sin\frac{m\pi y}{2b}=-\frac{d^2E\alpha}{(1-\nu)}\widetilde{T}_e=-\frac{4d^2E\alpha\widetilde{T}_e}{(1-\nu)\pi}\sum_{m:\,\mathrm{odd}}\frac{1}{m}\sin\frac{m\pi y}{2b},$$

$$2d\sum_{m:\,\mathrm{odd}} A_{2m}\left(\frac{m\pi}{2a}\right)^2 \cosh\mu_{1m}\sin\frac{m\pi x}{2a}=-\frac{d^2E\alpha}{(1-\nu)}\widetilde{T}_e=-\frac{4d^2E\alpha\widetilde{T}_e}{(1-\nu)\pi}\sum_{m:\,\mathrm{odd}}\frac{1}{m}\sin\frac{m\pi x}{2a},$$

$$\therefore\quad B_{2m}=-\frac{8b^2dE\alpha\widetilde{T}_e}{(1-\nu)(m\pi)^3\cosh\mu_{2m}}, \tag{37–1}$$

$$A_{2m}=-\frac{8a^2dE\alpha\widetilde{T}_e}{(1-\nu)(m\pi)^3\cosh\mu_{1m}}. \tag{37–2}$$

Substituting Eqs. (30), (35) and (37) into Eq. (34), we finally obtain the deflection mode of $w(x, y)$:

$$\begin{aligned}
&\frac{1}{12(1-\nu^2)}\left(\frac{d}{b}\right)^2\left(\frac{w}{d}\right)\\
&=\left\{\frac{4\alpha\overline{T}_{00}}{(1-\nu)\pi^2}\frac{\lambda^2}{(1+\lambda^2)}\left(\frac{\delta}{d}\right)\right.\\
&\quad\left.-\left[\frac{1}{8(1-\nu^2)}\frac{(1+2\nu\lambda^2+\lambda^4)}{(1+\lambda^2)^2}+\frac{1}{16}\frac{(1+\lambda^4)}{(1+\lambda^2)^2}\right]\left(\frac{d}{b}\right)^2\left(\frac{\delta}{d}\right)^3\right\}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}\\
&\quad+\frac{1}{16}\left(\frac{d}{b}\right)^2\left(\frac{\delta}{d}\right)^3\left\{\frac{1}{(1+9\lambda^2)^2}\sin\frac{\pi x}{2a}\sin\frac{3\pi y}{2b}+\frac{\lambda^4}{(9+\lambda^2)^2}\sin\frac{3\pi x}{2a}\sin\frac{\pi y}{2b}\right\}\\
&\quad+\frac{\alpha}{\pi^2}\left(2\left\{\sum_{j=2}^{\infty}\overline{T}_{0j}\left[\frac{-\lambda^2}{\{1+(j-1)^2\lambda^2\}^2}\sin\frac{\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}\right.\right.\right.\\
&\qquad\qquad\left.+\frac{\lambda^2}{\{1+(j+1)^2\lambda^2\}^2}\sin\frac{\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}\right]\\
&\qquad+\sum_{i=2}^{\infty}\overline{T}_{i0}\left[\frac{-\lambda^4}{\{(i-1)^2+\lambda^2\}^2}\sin\frac{(i-1)\pi x}{2a}\sin\frac{\pi y}{2b}\right.\\
&\qquad\qquad\left.\left.+\frac{\lambda^4}{\{(i+1)^2+\lambda^2\}^2}\sin\frac{(i+1)\pi x}{2a}\sin\frac{\pi y}{2b}\right]\right\}\\
&\qquad+\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{\overline{T}_{ij}}{[i^2+\lambda^2j^2]}\left\{(i-j)^2\left[\frac{\lambda^4}{\{(i-1)^2+(j-1)^2\lambda^2\}^2}\sin\frac{(i-1)\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}\right.\right.\\
&\qquad\qquad\left.+\frac{\lambda^4}{\{(i+1)^2+(j+1)^2\lambda^2\}^2}\sin\frac{(i+1)\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}\right]\\
&\qquad\quad-(i+j)^2\left[\frac{\lambda^4}{\{(i-1)^2+(j+1)^2\lambda^2\}^2}\sin\frac{(i-1)\pi x}{2a}\sin\frac{(j+1)\pi y}{2b}\right.\\
&\qquad\qquad\left.\left.\left.+\frac{\lambda^4}{\{(i+1)^2+(j-1)^2\lambda^2\}^2}\sin\frac{(i+1)\pi x}{2a}\sin\frac{(j-1)\pi y}{2b}\right]\right\}\right)\left(\frac{\delta}{d}\right)\\
&\quad+\frac{4\alpha}{(1-\nu)\pi^2}\sum\sum_{p,q:\,\mathrm{odd}}\frac{\lambda^2\widetilde{T}_{pq}}{[p^2+\lambda^2q^2]}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}\\
&\quad+\frac{8\alpha\widetilde{T}_e}{(1-\nu)\pi^3}\sum_{m:\,\mathrm{odd}}\frac{1}{m^3}\left\{\frac{\lambda^2}{\cosh\mu_{1m}}\left[\mu_{1m}\tanh\mu_{1m}\cosh\frac{m\pi(y-b)}{2a}\right.\right.\\
&\qquad\qquad\left.-\frac{m\pi(y-b)}{2a}\sinh\frac{m\pi(y-b)}{2a}\right]\sin\frac{m\pi x}{2a}\\
&\qquad\qquad+\frac{1}{\cosh\mu_{2m}}\left[\mu_{2m}\tanh\mu_{2m}\cosh\frac{m\pi(x-a)}{2b}\right.
\end{aligned}$$

$$-\frac{m\pi(x-a)}{2b}\sinh\frac{m\pi(x-a)}{2b}\Big]\sin\frac{m\pi y}{2b}\Big\}, \tag{38}$$

where

$$\lambda=\frac{a}{b}. \tag{39-1}$$

Putting $w=\delta$ at $x=a$ and $y=b$ in Eq. (38) we finally obtain the following expression:

$$\frac{1}{12(1-\nu^2)}\left(\frac{d}{b}\right)^2\left(\frac{\delta}{d}\right)+\Big\{\frac{1}{8(1-\nu^2)}\frac{(1+2\nu\lambda^2+\lambda^4)}{(1+\lambda^2)^2}+\frac{1}{16}\Big[\frac{1}{(1+9\lambda^2)^2}+\frac{\lambda^4}{(9+\lambda^2)^2}+\frac{(1+\lambda^4)}{(1+\lambda^2)^2}\Big]\Big\}\left(\frac{d}{b}\right)^2\left(\frac{\delta}{d}\right)^3$$

$$=\frac{4\alpha}{(1-\nu)\pi^2}\Big\{\lambda\tilde{T}_e\sum_{m:\text{odd}}\frac{(-1)^{\frac{m-1}{2}}}{m^2}\Big[\frac{\tanh\mu_m}{\cosh\mu_m}+\frac{\tanh(\lambda^2\mu_m)}{\cosh(\lambda^2\mu_m)}\Big]+\sum_{p,q:\text{odd}}\sum\frac{\lambda^2(-1)^{(\frac{p+q}{2}-1)}}{[p^2+\lambda^2q^2]}\tilde{T}_{pq}\Big\}$$

$$+\frac{\alpha}{\pi^2}\Big(\frac{4\bar{T}_{00}}{(1-\nu)}\frac{\lambda^2}{(1+\lambda^2)}+2\Big\{\sum_{j=2}^{\infty}(-1)^{\frac{j}{2}}\bar{T}_{0j}\Big[\frac{\lambda^2}{\{1+(j-1)^2\lambda^2\}^2}+\frac{\lambda^2}{\{1+(j+1)^2\lambda^2\}^2}\Big]+\sum_{i=2}^{\infty}(-1)^{\frac{i}{2}}\bar{T}_{i0}\Big[\frac{\lambda^4}{\{(i-1)^2+\lambda^2\}^2}+\frac{\lambda^4}{\{(i+1)^2+\lambda^2\}^2}\Big]\Big\}$$

$$+\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{(-1)^{\frac{i+j}{2}}\bar{T}_{ij}}{[i^2+\lambda^2j^2]}\Big\{(i-j)^2\Big[\frac{\lambda^4}{\{(i-1)^2+(j-1)^2\lambda^2\}^2}+\frac{\lambda^4}{\{(i+1)^2+(j+1)^2\lambda^2\}^2}\Big]+(i+j)^2\Big[\frac{\lambda^4}{\{(i-1)^2+(j+1)^2\lambda^2\}^2}+\frac{\lambda^4}{\{(i+1)^2+(j-1)^2\lambda^2\}^2}\Big]\Big\}\Big)\left(\frac{\delta}{d}\right), \tag{40}$$

where

$$\mu_m=\frac{m\pi}{2\lambda}, \tag{39-2}$$

and

$$\bar{T}_{00}=\frac{1}{4ab}\int_0^{2b}\int_0^{2a}\frac{\bar{T}}{d}dxdy, \tag{41-1}$$

$$\bar{T}_{i0}=\frac{1}{2ab}\int_0^{2b}\int_0^{2a}\frac{\bar{T}}{d}\cos\frac{i\pi x}{2a}dxdy,\quad (i=2,4,\cdots\text{ even}), \tag{41-2}$$

$$\bar{T}_{0j}=\frac{1}{2ab}\int_0^{2b}\int_0^{2a}\frac{\bar{T}}{d}\cos\frac{j\pi y}{2b}dxdy,\quad (j=2,4,\cdots\text{ even}), \tag{41-3}$$

$$\bar{T}_{ij}=\frac{1}{ab}\int_0^{2b}\int_0^{2a}\frac{\bar{T}}{d}\cos\frac{i\pi x}{2a}\cos\frac{j\pi y}{2b}dxdy,\quad (i,j=2,4,\cdots\text{ even}), \tag{41-4}$$

$$\tilde{T}_{pq}=\frac{1}{ab}\int_0^{2b}\int_0^{2a}\left(\frac{\tilde{T}}{d^2}-\tilde{T}_e\right)\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}dxdy,\quad (p,q=1,3,\cdots\text{ odd}). \tag{41-5}$$

This Eq. (40) gives the relation in question between the maximum deflection of δ at the midpoint of the plate and the given temperature distribution of T.

4. Numerical Examples

4.1. *The case where the temperature distribution through the thickness is specified as linear.*

The temperature distribution in a body should be given by analyzing the equation of heat conduction, but will be here assumed to be given in order to present the fundamental purport of the above analysis under transient heating conditions.

It is assumed that the temperature distribution on the xy-plane of the plate takes a parabolic form and the temperature distribution through the thickness is

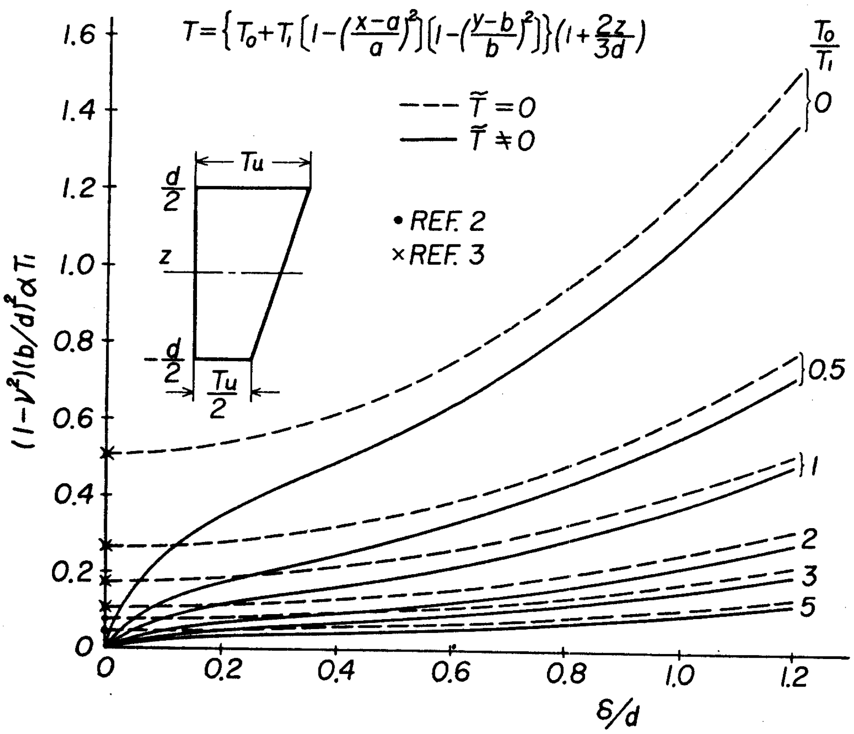

FIGURE 2. Relations between the temperature rise and the deflection, $\lambda=1$.

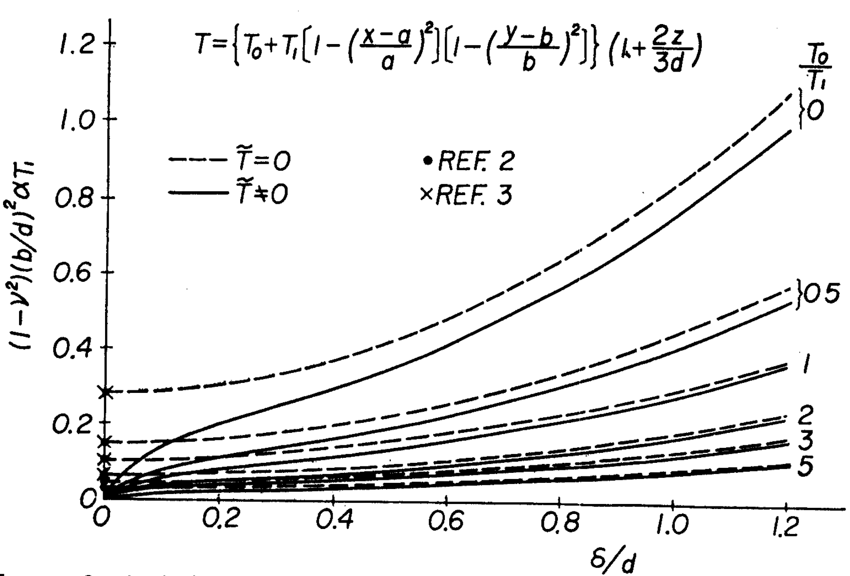

FIGURE 3. Relations between the temperature rise and the deflection, $\lambda=3$.

linear and the temperature on the lower face is one-half of that on the upper face, that is, the temperature distribution is expressed as follows:

$$T=\left\{T_0+T_1\left[1-\left(\frac{x-a}{a}\right)^2\right]\left[1-\left(\frac{y-b}{b}\right)^2\right]\right\}\left(1+\frac{2z}{3d}\right). \tag{42}$$

The relations between $(1-\nu^2)(b/d)^2\alpha T_1$ and δ/d can be obtained from Eq. (40) and are shown in Fig. 2 ($\lambda=1$) and Fig. 3 ($\lambda=3$) with the parameter of T_0/T_1. It can be seen from these figures that for the case where the temperature gradient through the thickness is taken into account the plate starts to deflect from the beginning of heating without exhibiting Euler buckling phenomenon at the critical temperature.

To put $\tilde{T}$ equal to zero in Eq. (21) means the absence of the temperature gradient through the thickness and the results for this case can be obtained easily by putting $\tilde{T}=0$ in Eq. (40). They are shown in Figs. 2 and 3 with dashed lines which show the deflections after buckling. The critical buckling temperatures which were obtained with the energy method [2][3] correspond to the values of the ordinate at $\delta/d=0$ in the dashed curves and the agreement between them is found to be good. The variation of these critical values of temperature with the aspect ratio of λ is shown in Fig. 4.

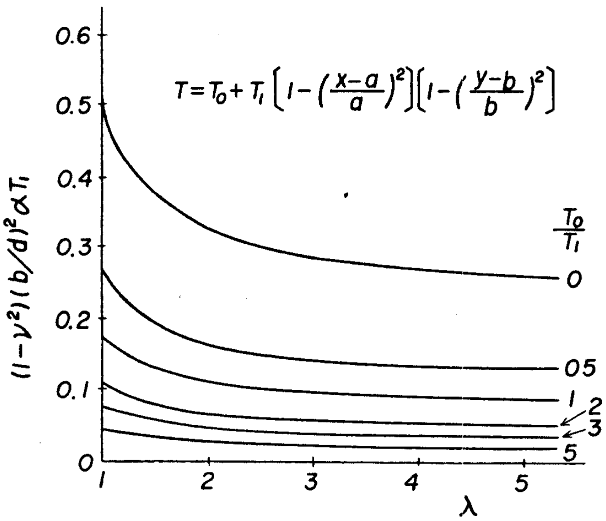

FIGURE 4. Variation of thermal buckling coefficient with aspect ratio.

4.2. *The case of the instantaneous heating.*

For this case of the instantaneous aerodynamic heating due to supersonic flight, it is necessary to take account of the variation of T, $\bar{T}$ and $\tilde{T}$ with time. It is assumed that the upper face of the plate is instantaneously exposed to the constant uniform adiabatic wall temperature of T_E. Neglecting the heat conduction along the plate plane, we consider the one-dimensional heat flow through the thickness of the plate only for the sake of simplicity. Then the temperature distribution through the thickness under the instantaneous heating condition can be given by [12]

$$\frac{T(z)}{T_E}=1-2\sum_{n=0}^{\infty}\frac{\sin P_n \exp(-P_n^2 kt/\rho c_p d^2)\cos[P_n(z+d/2)/d]}{P_n+\sin P_n \cos P_n}, \tag{43}$$

where, d, h, k, ρ and c_p are the thickness of the plate, heat transfer coefficient, heat conductivity, density and specific heat, respectively, and assumed to be constant. P_n is the n-th root of the following equation

$$P_n \tan P_n=\frac{hd}{k}, \tag{44}$$

and are given as shown in Tab. 1 [13] with the parameter of Biot Number hd/k.

TABLE 1

hd/k	P_0	P_1	P_2	P_3
1	0.8603	3.4256	6.4373	9.5293
5	1.3138	4.0336	6.9096	9.8928
10	1.4289	4.3058	7.2281	10.2003

The variation of the temperature distribution through the thickness with the time parameter of $kt/\rho c_p d^2$ is shown in Fig. 5 for the case of $hd/k=1$ as an example.

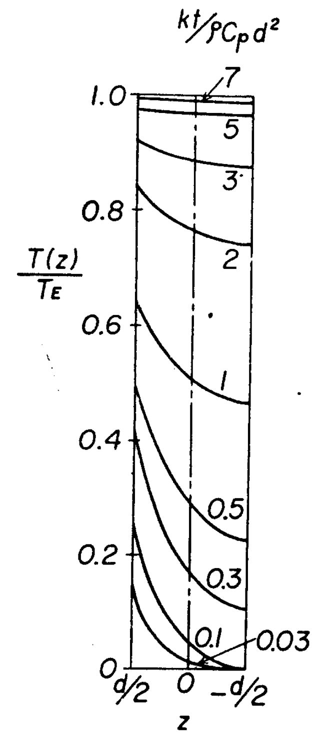

FIGURE 5. Variation of temperature distribution through the thickness with the time under aerodynamic heating, $hd/k=1$.

The variation of the mean temperature $\bar{T}$ and the temperature gradient $\tilde{T}$ with the time under instantaneous aerodynamic heating can be obtained by substituting Eq. (43) into Eqs. (14) as follows:

$$\frac{\bar{T}}{d}=T_E\left[1-2\sum_{n=0}^{\infty}\frac{\sin^2 P_n}{P_n(P_n+\sin P_n \cos P_n)}\exp\left(-\frac{P_n^2 kt}{\rho c_p d^2}\right)\right], \tag{45}$$

$$\frac{\tilde{T}}{d^2}=-T_E\sum_{n=0}^{\infty}\frac{\sin P_n[P_n\sin P_n+2(\cos P_n-1)]}{P_n^2(P_n+\sin P_n\cos P_n)}\exp\left(-\frac{P_n^2kt}{\rho c_p d^2}\right), \qquad (46)$$

and are shown in Fig. 6 with the parameter of hd/k. From Fig. 6, we can see that the mean temperature $\bar{T}$ reaches the adiabatic wall temperature with the lapse of time, while the temperature gradient $\tilde{T}$ has a maximum and then decreases

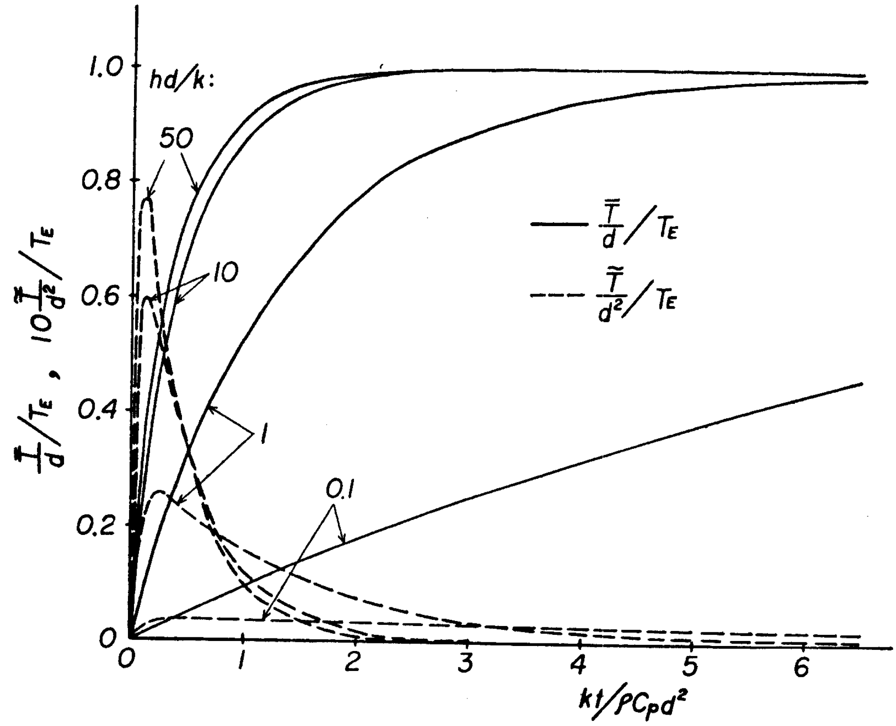

FIGURE 6. Variations of mean temperature and temperature gradient through the thickness with the time.

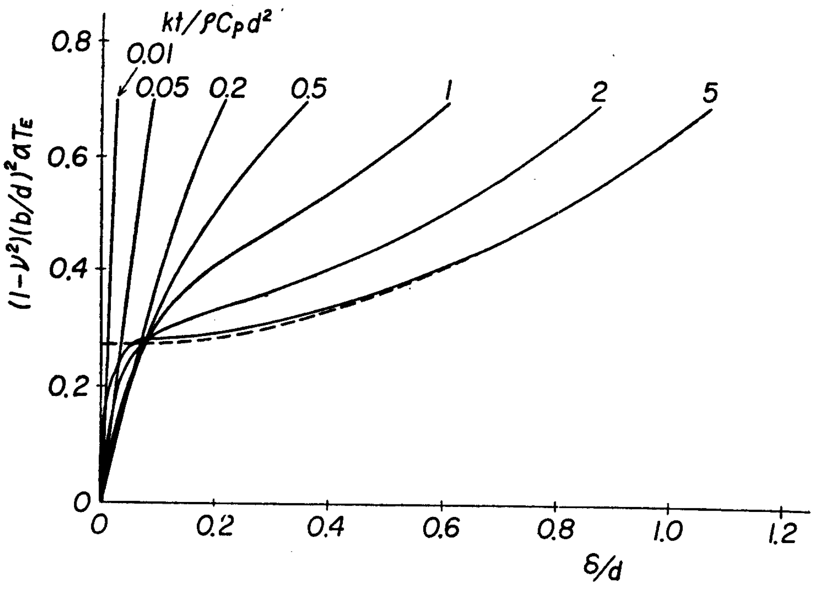

FIGURE 7. Relations between the temperature rise and the deflection with the parameter of time, $\lambda=1$, $hd/k=1$.

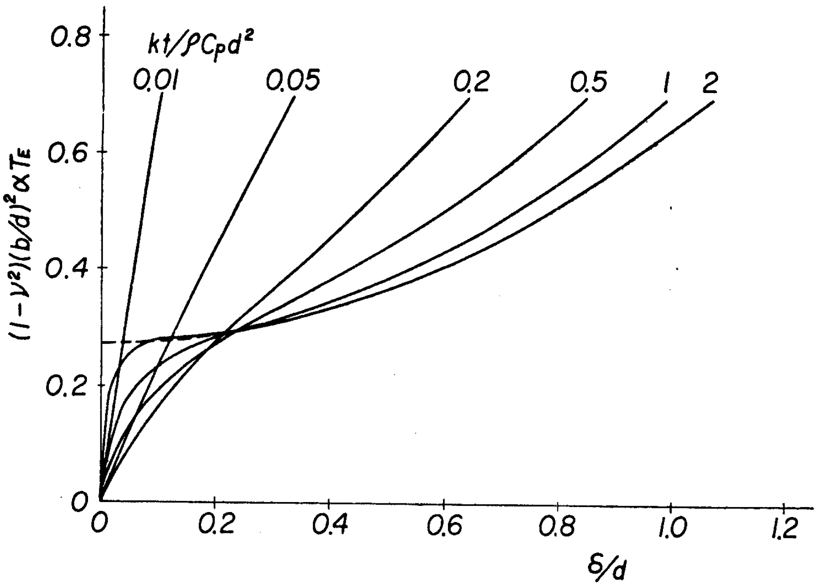

FIGURE 8. Relations between the temperature rise and the deflection with the parameter of time, $\lambda=1$, $hd/k=10$.

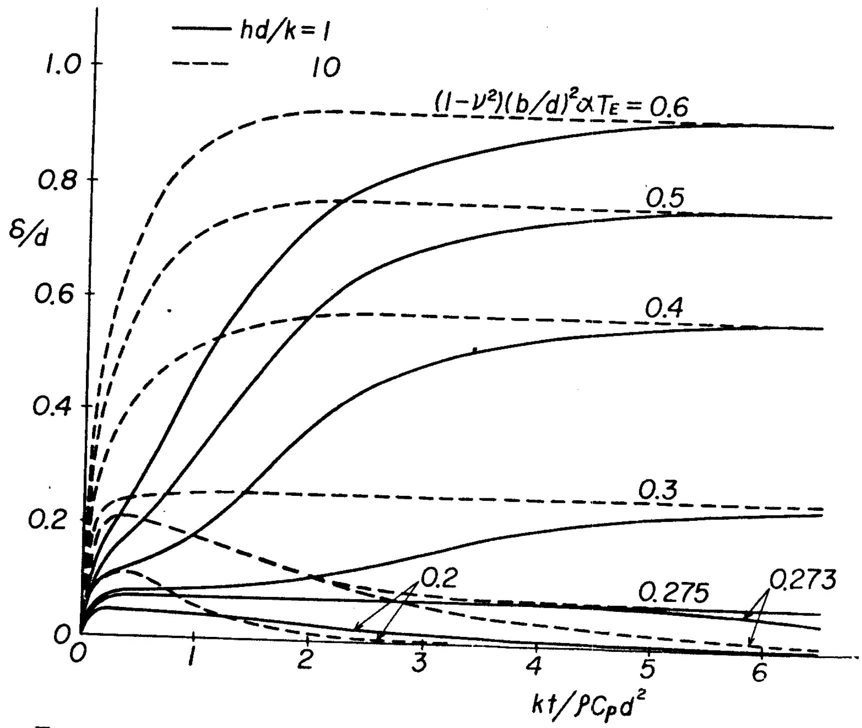

FIGURE 9. Variation of deflection at the midpoint of the plate with the time, $\lambda=1$.

with the variation of temperature distribution through the thickness as seen in Fig. 5.

The relations between the temperature rise and the deflection under such a transient heating condition are shown in Figs. 7 and 8 corresponding to Biot Numbers equal to 1 and 10, respectively, where the temperature distribution over the plate is assumed to be uniform. It can be seen from these figures that the relations between $(1-\nu^2)(b/d)^2\alpha T_E$ and δ/d change with the increase of nondimensional time $kt/\rho c_p d^2$ and finally become consistent with the curves for the case of $\widetilde{T}=0$. Next, the relations between $kt/\rho c_p d^2$ and δ/d under the specified adiabatic wall temperature are shown in Fig. 9. The maximum deflection δ/d at the midpoint of the plate increases with the lapse of time $kt/\rho c_p d^2$ approaching to a constant value when the adiabatic wall temperature of T_E is higher than the critical temperature of Euler buckling which is $(1-\nu^2)(b/d)^2\alpha(T_E)_{cr.}=0.274$ for the case in this figure. On the other hand, when T_E is lower than $(T_E)_{cr.}$, the plate deflects at the initial stage of heating, but then gradually comes back to the initial flat state. This latter phenomenon is interesting and is considered to be worthy of note.

POSTSCRIPT

The deformation and the thermal stress of a rectangular flat plate simply supported at four edges and subjected to an arbitrary symmetrical temperature were analyzed considering the effect of the temperature gradient through the thickness under transient heating condition. It was shown that the plate does not present the phenomenon of Euler buckling but starts to deflect from the beginning of heating according to the temperature distribution and the boundary conditions. In the supersonic airplanes and missiles, the deformation has much effect on the aerodynamic and aeroelastic problems, so such an analysis on the deformation and thermal stress under transient heating condition as treated in this paper will be as much important as the analysis of the critical buckling done previously.

However, it should be noticed that when the adiabatic wall temperature in aerodynamic heating is lower than the temperature of Euler buckling, the plate comes back to the initial flat state with the lapse of time even though it deflects at the beginning of heating.

Moreover, we must take account of the negative pressure over the surface which will make the deflection larger, although it has been neglected in the present paper.

The phenomenon which was indicated in this paper is more remarkable for the case where the Boit Number is larger; however, even though the Boit Number is small, the small initial deflection induced by the transient heating will have large effect on the total deflection of plate, especially when the external edge forces are applied in the plate plane.

The case for the clamped boundary condition has been also analyzed and will be published later.

We would like to express our sincere appreciations to Professors K. Ikeda and Y. Yoshimura of the Aeronautical Research Institute, University of Tokyo, for their valuable discussions on our analysis.

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October 19, 1960.

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