

Deformation and Buckling of Cylindrical Shells Subjected to Heating*

By

Megumi SUNAKAWA

Summary. The problems of thermal deformation and buckling of cylindrical shells arise in connection with air-frame bodies subjected to the aerodynamic heating at supersonic speeds and with combustion chambers and exhaust pipes exposed to extremely hot gases.

The authors analyzed previously the thermal deformations of flat plates and pointed out that it was impossible to disregard the effect of temperature gradient through the thickness on their deformations depending on the temperature distribution and the boundary conditions.

In this paper, the deformation and buckling of cylindrical shells subjected to heating are analyzed. Taking the finite deformation into account, the fundamental non-linear simultaneous partial differential equations for the thermoelastic problems of a cylindrical shell, including also the terms concerning the normal pressure as well as the effect of temperature gradient through the thickness, are derived from the variational principle and are analyzed for the following three extreme cases, that is, for the cases where

- i) the cylindrical shell, subjected to heating and normal pressure, is free from the constraints upon the thermal expansion and the axial displacement, but restrained laterally against the pressure at the edges,
 - ii) the cylindrical shell is unrestrained longitudinally but fully restrained laterally at the edges, and
 - iii) the cylindrical shell is clamped completely at the edges,
- and then some discussions on analytical and numerical results are given.

1. INTRODUCTION

The problems, on the structural components such as external plates of wing and fuselage, combustion chambers and exhaust pipes, resulting from the aerodynamic heating encountered in the supersonic flight and from the extremely high temperature induced by the internal combustion engine, have recently been of great importance in connection not only with the deteriorations of strength and rigidity of structures, but also with the influences imposed upon the aerodynamic and aeroelastic characteristics of flying bodies.

The thermal deformation behaviours of structural components are considered to be broadly classified into two kinds, that is, the Euler buckling type (rapidly changing deformation type) and the gradually changing deformation type. When the structural components are heated, they exhibit either of deformation type,

* Read at the Symposium on Space Science and Technology of the Japan Society for Aeronautical and Space Sciences, February 25, 1961.

depending on the temperature distributions and the boundary conditions. That is to say, the flat plates will generally take the Euler type of deformation when the Biot Number is small, while they will take the gradually changing type of deformation for the case of large Biot Number, and the structural components having finite curvatures such as cylindrical and spherical shells will naturally take the gradually changing type of deformation regardless of the heating and boundary conditions. The larger the Biot Number is, the steeper the temperature gradient through the thickness becomes under the transient heating condition. That is to say, when the structural components are thicker, their thermal conductivity is smaller, and the heat transfer is larger, then the Biot Number will be larger; when the heating conditions are much severe as seen in the up-to-date supersonic flying bodies, the temperature gradient through the thickness will become steep. As described below, the effect of temperature gradient through the thickness is equivalent to that of normal pressure depending on its distribution over the plane and on the boundary conditions, and it is therefore considered that this effect will have very important influences upon the deformation behaviours.

A number of theoretical and experimental studies have been pursued on the thermal deformations and stresses which are of practical importance in the field of the aeronautical and space sciences as well as in many other fields of engineering. The purports of the problems have been gradually clarified with the accumulation of such studies. The thermoelastic equations taking the finite deformation into account are non-linear which make it extremely difficult to solve completely satisfying all the boundary conditions. Accordingly, it seems that the previous efforts have been concentrated to the attainment of the practical solutions as briefly as possible without losing sight of their purports. The studies hitherto carried out in these fields have been limited mainly to rectangular plates for the cases of two-dimensional problems without taking into consideration the effect of temperature gradient through the thickness; while as far as cylindrical shells are concerned there seems to be few analyses, except the studies by Zuk and by Hoff for the special case where there is no constraint upon the axial displacement. Zuk [1], assuming $\sigma_{11} = \sigma_{12} = 0$ and $\sigma_{22} = E\alpha T(1 + \cos \pi x/L)/2$, approximating also the normal displacement with only the first term of infinite biharmonic series, applied the Galerkin method to the shell equation by Donnell and gave approximately the critical temperature of the clamped cylindrical shell with the parameter of the circumferential length of buckled waves for the above special mode of σ_{22} . After completing the original manuscript, the author noticed that Hoff [2], in his study, assumed $\sigma_{11} = \sigma_{12} = 0$ and $\sigma_{22} = KE \sum_{m=0}^{\infty} s_m \cos(m\pi x/R\lambda)$ where $\lambda = 2L/R$ and K is a multiplying factor (a number), and expressed the displacements in the three directions by the infinite series with the parameters of the circumferential number of buckled waves and $2L/R$ so as to satisfy the boundary conditions, and applied the Rayleigh-Ritz method to the shell equation by Donnell, then obtained in the general form the critical temperature of a simply supported cylindrical shell where the hoop stress, σ_{22} varies

in the axial direction. In his example of analysis where the temperature rise is assumed as uniform over the shell, the value of σ_{22} approximated with the first eight terms has some error, for example, at the edges, where the series gives 0.907 instead of unity, and the uncertainty is suspended in a practical use of his results for such a problem where the stress state in the vicinity of the edge has a much important effect, and the process of solving the coefficients determinant seems to be rather complicated.

All these researches have dealt with the cases, where the Biot Number is comparatively small, resulting in the uniform temperature through the thickness, and have put forth the analyses for the Euler type buckling. That is to say, most of thermal deformation problems have been treated as the buckling problems, paying no attention to the temperature gradient through the thickness.

However, the heating to which the structural components of supersonic airplanes and missiles are subjected has recently become more and more intense and the structural components themselves tend to become thicker; and so it seems necessary to deal with the cases of large Biot Number. It does not matter to consider that the temperature is uniform through the thickness for the cases where the Biot Number is small. When the Number becomes large, however, the temperature gradient through the thickness cannot be ignored; and consequently it can be presumed that the deformation behaviours of flat plates do not necessarily present only the buckling phenomenon of Euler-type, and that the cylindrical shells undoubtedly start to deform from the beginning of heating and may buckle after that in some special cases.

From the above-mentioned viewpoints, paying special attention to the effect of temperature gradient through the thickness, the authors [3]~[7] analyzed previously the thermal deformations of flat plates and pointed out that it was impossible to disregard the effect of temperature gradient through the thickness on their deformations depending on the temperature distribution and the boundary conditions.

In this paper, the thermal deformation and buckling of cylindrical shells are analyzed. Taking the finite deformation into account, the fundamental equations for the thermoelastic problems of a cylindrical shell, including also the terms concerning the normal pressure as well as the effect of temperature gradient through the thickness, are derived from the variational principle first of all with the aid of the theorem of the stationary potential energy. Checking the terms in the fundamental equations critically, the deformation behaviours are studied, paying special attention to the effect of temperature gradient through the thickness which played an important role for the cases of flat plates. Then, the problems are analyzed for some typical cases, that is, for the cases where i) the cylindrical shell, subjected to heating and normal pressure, is free from the constraints upon the thermal expansion and the axial displacement, but restrained laterally against the pressure at the edges, ii) the cylindrical shell is unrestrained longitudinally but fully restrained laterally at the edges, and iii) the cylindrical shell is clamped at the edges, and some discussions on the analytical and numerical results are given.

Nomenclatures and Symbols

d	thickness of shell.
h	heat transfer coefficient.
l, n	half wave length in the axial direction and number of the buckled waves in the circumferential direction of cylindrical shell.
p	normal pressure (positive for the positive direction of z -axis).
t	time.
u, v, w	displacement components in the median surface in x -, y -, z -directions.
x, y, z	rectangular coordinates, $y = R\theta$.
A, C	integral constants.
D	flexural modulus of rigidity, $D \equiv Ed^3/12(1-\nu^2)$.
E	modulus of elasticity.
F	free energy per unit volume.
L, R	half length and radius of cylindrical shell.
T	temperature rise above the unstrained state.
T_E	equilibrium wall surface temperature.
\bar{T}, \tilde{T}	Eq. (2.10).
α, ρ, c_p, k	coefficient of thermal expansion, density, specific heat and thermal conductivity of the material.
β	Eq. (3.10).
δ	deflection.
$\epsilon_{11}, \epsilon_{22}$	extensional strains in x -, y -directions.
ϵ_{12}	shearing strain in xy -plane.
σ_{11}, σ_{22}	extensional stresses in x -, y -directions.
σ_{12}	shearing stress in xy -plane.
κ_1, κ_2	changes of curvature of the median surface about x -, y -axes.
κ_{12}	change of twist of the median surface.
ν	Poisson's ratio.
ξ, η	Eq. (3.12), Eq. (4.15), Eq. (5.11).
χ	stress function.
Π	total potential energy.
$\bar{\Phi}, \tilde{\Phi}, \Psi$	non-dimensional variables referred to the mean temperature, temperature gradient through the thickness and normal pressure, respectively, Eq. (2.19).
∇^2, ∇^4	operators,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

Subscripts " x " and " y " denote the partial differentiation with respect to x and y . Bar over letter refers to the median surface.

2. DERIVATION OF FUNDAMENTAL EQUATIONS

A cylindrical shell as shown in Fig. 1 is considered, and it is assumed that the shell is heated at the outer face and there exists a temperature gradient through the thickness in the initial transient stage of heating.

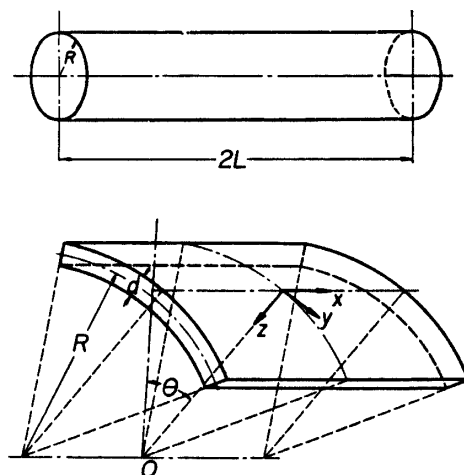


FIGURE 1. Cylindrical shell.

The two-dimensional stress-strain law can be given by

$$\left. \begin{aligned} E\epsilon_{11} &= \sigma_{11} - \nu\sigma_{22} + E\alpha T, \\ E\epsilon_{22} &= \sigma_{22} - \nu\sigma_{11} + E\alpha T, \\ G\epsilon_{12} &= \frac{E}{2(1+\nu)}\epsilon_{12} = \sigma_{12}. \end{aligned} \right\} \quad (2.1)$$

where the strain components can be given as follows:

$$\left. \begin{aligned} \epsilon_{11} &= \bar{\epsilon}_{11} - z\kappa_1, \\ \epsilon_{22} &= \bar{\epsilon}_{22} - z\kappa_2, \\ \epsilon_{12} &= \bar{\epsilon}_{12} - 2z\kappa_{12}. \end{aligned} \right\} \quad (2.2)$$

The strain components and curvature changes of the median surface can be expressed, taking account of the terms of the first order only in the finite deformation as

$$\left. \begin{aligned} \bar{\epsilon}_{11} &= u_x + \frac{1}{2}w_x^2, \\ \bar{\epsilon}_{22} &= v_y - \frac{w}{R} + \frac{1}{2}w_y^2, \\ \bar{\epsilon}_{12} &= u_y + v_x + w_x w_y, \end{aligned} \right\} \quad (2.3)$$

$$\kappa_1 = w_{xx}, \quad \kappa_2 = w_{yy} + \frac{w}{R^2}, \quad \kappa_{12} = w_{xy}, \quad (2.4)$$

and so Eqs. (2.2) reduce to the following expressions.

$$\left. \begin{aligned} \varepsilon_{11} &= u_x + \frac{1}{2} w_x^2 - z w_{xx}, \\ \varepsilon_{22} &= v_y - \frac{w}{R} + \frac{1}{2} w_y^2 - z \left(w_{yy} + \frac{w}{R^2} \right), \\ \varepsilon_{12} &= u_y + v_x + w_x w_y - 2z w_{xy}. \end{aligned} \right\} \quad (2.5)$$

The equilibrium equation can be derived by the variational method with the aid of the well-known theorem of the stationary potential energy. The detail of the process is the same as that in the case of rectangular plate [4], [5], and so it will be briefly described here.

Now, a cylindrical shell is considered to be subjected to the normal pressure only as the external force in addition to the heating. Then, the total potential energy, Π of a body which is bounded by the planes which are normal to the x - and y -axes at $x=x_1$ and x_2 and $y=y_1$ and y_2 , respectively, and by the outer and inner faces can be expressed by

$$\Pi = \int_V F dV - \int_{y_1}^{y_2} \int_{x_1}^{x_2} p(x, y) w(x, y) dx dy. \quad (2.6)$$

By the variational operation on u , v and w with the aid of the usual theorem of the stationary potential energy, that is, from $\delta\Pi=0$, the following equilibrium equations in the three directions can be obtained.

$$\begin{aligned} & \frac{Ed}{2(1-\nu^2)} \left[2 \frac{\partial}{\partial x} \left\{ \left(u_x + \frac{1}{2} w_x^2 \right) + \nu \left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) \right\} + (1-\nu) \frac{\partial}{\partial y} (u_y + v_x + w_x w_y) \right] \\ &= \frac{Ed\alpha}{(1-\nu)} \bar{T}_x, \end{aligned} \quad (2.7)$$

$$\begin{aligned} & \frac{Ed}{2(1-\nu^2)} \left[2 \frac{\partial}{\partial y} \left\{ \left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right\} + (1-\nu) \frac{\partial}{\partial x} (u_y + v_x + w_x w_y) \right] \\ &= \frac{Ed\alpha}{(1-\nu)} \bar{T}_y, \end{aligned} \quad (2.8)$$

$$\begin{aligned} & \frac{Ed}{2(1-\nu^2)} \left[2 \frac{\partial}{\partial x} \left\{ \left[\left(u_x + \frac{1}{2} w_x^2 \right) + \nu \left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) \right] w_x \right\} + 2 \frac{\partial}{\partial y} \left\{ \left[\left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right] w_y \right\} \right. \\ & \quad \left. + (1-\nu) \frac{\partial}{\partial y} [(u_y + v_x + w_x w_y) w_x] \right] - \frac{Ed^3}{12(1-\nu^2)} [w_{xxxx} + 2w_{xxyy} + w_{yyyy}] \\ & \quad + \frac{2}{R^2} (\nu w_{xx} + w_{yy}) + \frac{w}{R^4} + \frac{Ed}{(1-\nu^2)} \left[\left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right] \left(\frac{1}{R} \right) \end{aligned}$$

$$= \frac{E\alpha}{(1-\nu)} \left\{ d \left[\frac{\partial}{\partial x} (\bar{T} w_x) + \frac{\partial}{\partial y} (\bar{T} w_y) + \frac{\bar{T}}{R} \right] + d^2 \left(\tilde{T}_{xx} + \tilde{T}_{yy} + \frac{\tilde{T}}{R^2} \right) \right\} - p, \quad (2.9)$$

where

$$\bar{T} = \frac{1}{d} \int_{-d/2}^{d/2} T(x, y, z) dz, \quad \tilde{T} = \frac{1}{d^2} \int_{-d/2}^{d/2} z T(x, y, z) dz. \quad (2.10)$$

If the extensional stresses $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ and the shearing stress $\bar{\sigma}_{12}$ in the median surface corresponding to $\bar{\epsilon}_{11}$, $\bar{\epsilon}_{22}$ and $\bar{\epsilon}_{12}$ respectively are introduced with the following relation:

$$\left. \begin{aligned} \bar{\epsilon}_{11} &= \frac{1}{E} (\bar{\sigma}_{11} - \nu \bar{\sigma}_{22}) + \alpha \bar{T}, \\ \bar{\epsilon}_{22} &= \frac{1}{E} (\bar{\sigma}_{22} - \nu \bar{\sigma}_{11}) + \alpha \bar{T}, \\ \bar{\epsilon}_{12} &= \frac{2(1+\nu)}{E} \bar{\sigma}_{12}, \end{aligned} \right\} \quad (2.11)$$

then Eqs. (2.7) and (2.8) are reduced to

$$\left. \begin{aligned} \frac{\partial \bar{\sigma}_{11}}{\partial x} + \frac{\partial \bar{\sigma}_{12}}{\partial y} &= 0, \\ \frac{\partial \bar{\sigma}_{12}}{\partial x} + \frac{\partial \bar{\sigma}_{22}}{\partial y} &= 0. \end{aligned} \right\} \quad (2.12)$$

This pair of equations can be satisfied by introducing the stress function χ , defined by the relations

$$\bar{\sigma}_{11} = \chi_{yy}, \quad \bar{\sigma}_{22} = \chi_{xx}, \quad \bar{\sigma}_{12} = -\chi_{xy}. \quad (2.13)$$

Then, the equilibrium equation of Eq. (2.9) in the z -direction is expressed by

$$\begin{aligned} D \left[\nabla^4 w + \frac{2}{R^2} (w_{yy} + \nu w_{xx}) + \frac{w}{R^4} \right] \\ = d(\chi_{yy} w_{xx} - 2\chi_{xy} w_{xy} + \chi_{xx} w_{yy}) + \frac{d}{R} \chi_{xx} - \frac{Ed^2\alpha}{(1-\nu)} \left(\nabla^2 \tilde{T} + \frac{\tilde{T}}{R^2} \right) + p. \end{aligned} \quad (2.14)$$

Eliminating u and v from the three of Eqs. (2.3) and using Eqs. (2.11) and (2.13), the following compatibility equation can be obtained.

$$\nabla^4 \chi = E \left[w_{xy}^2 - w_{xx} w_{yy} - \frac{w_{xx}}{R} \right] - E\alpha \nabla^2 \bar{T}. \quad (2.15)$$

All the natural boundary conditions obtained from the operation of $\delta \Pi = 0$ are summarized in Eqs. (2.16).

at $x = x_1$ and x_2 ,

$$\left. \begin{aligned}
& \frac{Ed}{(1-\nu^2)} \left\{ \left(u_x + \frac{1}{2} w_x^2 \right) + \nu \left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) \right\} - \frac{Ed\alpha}{(1-\nu)} \bar{T} = 0, \\
& u_y + v_x + w_x w_y = 0, \\
& \frac{Ed}{(1-\nu^2)} \left\{ \left[\left(u_x + \frac{1}{2} w_x^2 \right) + \nu \left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) \right] w_x + \frac{(1-\nu)}{2} (u_y + v_x + w_x w_y) w_y \right. \\
& \quad \left. - \frac{d^2}{12} \left[w_{xxx} + \nu \left(w_{xyy} + \frac{w_x}{R^2} \right) + 2(1-\nu) w_{xyy} \right] \right\} - \frac{Ed\alpha}{(1-\nu)} (\bar{T} w_x + d \tilde{T}_x) = 0, \\
& D \left[w_{xx} + \nu \left(w_{yy} + \frac{w}{R^2} \right) \right] + \frac{Ed^2\alpha}{(1-\nu)} \tilde{T} = 0,
\end{aligned} \right\} \quad (2.16-1)$$

at $y=y_1$ and y_2 ,

$$\left. \begin{aligned}
& u_y + v_x + w_x w_y = 0, \\
& \frac{Ed}{(1-\nu^2)} \left\{ \left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right\} - \frac{Ed\alpha}{(1-\nu)} \bar{T} = 0, \\
& \frac{Ed}{(1-\nu^2)} \left\{ \left[\left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right] w_y + \frac{(1-\nu)}{2} (u_y + v_x + w_x w_y) w_x \right. \\
& \quad \left. - \frac{d^2}{12} \left[\left(w_{xyy} + \frac{w_y}{R^2} \right) + \nu w_{xxx} + 2(1-\nu) w_{xyy} \right] \right\} - \frac{Ed\alpha}{(1-\nu)} (\bar{T} w_y + d \tilde{T}_y) = 0, \\
& D \left[\left(w_{yy} + \frac{w}{R^2} \right) + \nu w_{xx} \right] + \frac{Ed^2\alpha}{(1-\nu)} \tilde{T} = 0,
\end{aligned} \right\} \quad (2.16-2)$$

at $x=x_1$ and x_2 , and $y=y_1$ and y_2 ,

$$w_{xy} = 0. \quad (2.16-3)$$

Equations (2.14) and (2.15) are the fundamental equations for the thermoelastic problems of a heated cylindrical shell. The problem will then be how to solve the non-linear simultaneous partial differential equations, Eqs. (2.14) and (2.15) under the reasonable boundary conditions and the given distributions of \bar{T} , \tilde{T} and p .

It is too difficult to obtain the exact solution of the non-linear simultaneous partial differential equations, Eqs. (2.14) and (2.15), and so it has been usual to use the energy method or the successive approximation method in solving them. In those cases, the compatibility equation, Eq. (2.15) can be integrated by using the first approximate deflection function under the given temperature distribution, T , and the term concerning the mean temperature, \bar{T} appears in the solution of the stress function, χ through the process where the boundary conditions on the displacement in the shell plane are satisfied. Therefore, the influence of the mean temperature is introduced through the stress function, χ in the right hand side of the equilibrium equation, Eq. (2.14). It will also be seen in the equilibrium equation that the second and third terms, which are due to the cir-

cumferential stress component and the temperature gradient through the thickness, respectively, have the effect equivalent to the normal pressure, and so the cylindrical shell can be predicted to start to deform from the beginning of heating. Further it may have the critical temperature, where the deformation mode changes abruptly, under some heating and boundary conditions.

Equations (2.14) and (2.15) coincide naturally with the fundamental equations of a rectangular plate, [4], [5] by putting the radius of curvature, R of a cylindrical shell infinitely large.

Next, the effect of temperature gradient through the thickness on the thermal deformation of a cylindrical shell is critically studied. The terms which have the effect equivalent to that of normal pressure in Eq. (2.14) are presented as follows:

$$\left(\frac{d}{R}\right)\chi_{xx} - \left(\frac{d}{R}\right)^2 \frac{E\alpha}{(1-\nu)}(R^2\nabla^2\tilde{T} + \tilde{T}) + p. \quad (2.17)$$

The term of $E\alpha\tilde{T}$ is included in the first term, $\chi_{xx} = \bar{\sigma}_{22}$ as already mentioned. The magnitude of \tilde{T} in the second term is usually less than one-tenth of that of \bar{T} [4] and d/R takes practically a value less than 10^{-2} , so it may be considered that the second term can be neglected compared with the first term except the case where the value of $R^2\nabla^2\tilde{T}$ is particularly large. However, in practice, the cases where the heating is applied not alone but together with the normal pressure are necessary to be frequently analyzed, and for this purpose it will be important to check the degree of the effect of heating compared with that of normal pressure. Then, putting

$$\left. \begin{aligned} \bar{T}(x, y) &\equiv \bar{T}_0, \\ \tilde{T}(x, y) &\equiv \tilde{T}_0, \\ p(x, y) &\equiv p_0, \end{aligned} \right\} \quad (2.18)$$

$$\left. \begin{aligned} \bar{\Phi} &= \left(\frac{d}{R}\right)\alpha\bar{T}_0, \\ \tilde{\Phi} &= \left(\frac{d}{R}\right)^2 \frac{\alpha\tilde{T}_0}{(1-\nu)}, \\ \psi &= \frac{p_0}{E}, \end{aligned} \right\} \quad (2.19)$$

$$\frac{\bar{\Phi}}{\psi} = \left(\frac{d}{R}\right)\alpha\bar{T}_0 / \left(\frac{p_0}{E}\right), \quad (2.20)$$

$$\frac{\tilde{\Phi}}{\psi} = \left(\frac{d}{R}\right)^2 \frac{\alpha\tilde{T}_0}{(1-\nu)} / \left(\frac{p_0}{E}\right), \quad (2.21)$$

the orders of ratios, Eq. (2.20) and (2.21) are examined.

As an example, the case is considered where the outer face of a cylindrical shell is instantaneously exposed to the equilibrium wall surface temperature, T_E .

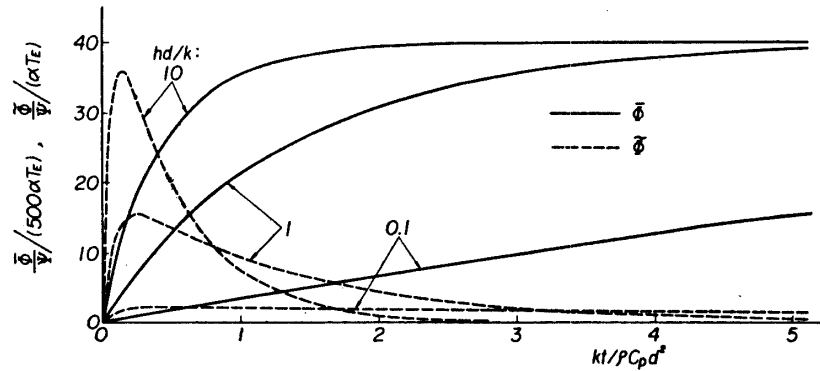


FIGURE 2. Variation of the ratios of mean temperature and temperature gradient to normal pressure with the time, $d/R=2 \times 10^{-2}$, $p_0/E=10^{-6}$.

The variations of Eqs. (2.20) and (2.21) with the time are shown in Fig. 2 putting $d/R=2 \times 10^{-2}$, $p_0/E=10^{-6}$ and $\nu=1/3$, where the heat conduction along the shell plane and the heat transfer from the inner face are neglected. If αT_E is specified as equal to 2×10^{-3} , the maximum values of $\bar{\Phi}$ and $\tilde{\Phi}$ are nearly 40Ψ and 0.03Ψ , respectively, when $hd/k=1$. This means that $T_E=100^\circ\text{C}$ and $p_0 \doteq 2\text{ atm.}$, if $\alpha=2 \times 10^{-5}$ and $E=2 \times 10^4 \text{ Kg/mm}^2$. Therefore, the value of $\bar{\Phi}_{\text{max}}$ corresponds to about 80 atm. and its effect will play an important role, while the effect of $\tilde{\Phi}$ is only 3 per cent of that of p_0 . However, in the practical design, some attention is paid to remove the constraint upon the thermal expansion, and hence χ_{xx} will become much smaller. On the other hand, as a consequence of severe heating, the temperature gradient through the thickness is inevitably generated, which cannot be ignored. In the above-mentioned example, the effect of $\tilde{\Phi}$ is only 3 per cent of that of p_0 , but its effect becomes larger with $E\alpha T_E$, because $\tilde{\Phi}/\Psi$ is proportional to $E\alpha T_E$. This is the reason why the effect of $\tilde{\Phi}$ is studied here.

Next, the relation between $\tilde{\Phi}/\Psi$ and d/R is calculated and shown in Fig. 3, where $p_0/E=10^{-6}$. For example, $\tilde{\Phi}/\Psi=0.6$ for the case of $d/R=2 \times 10^{-2}$ and $\alpha \tilde{T}_0=10^{-3}$. This ratio is proportional to $(d/R)^2$, and so the effect of temperature

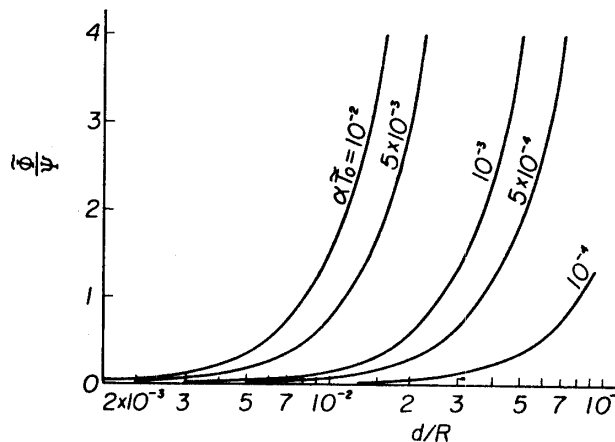


FIGURE 3. Relation between the ratio of temperature gradient to normal pressure and the thickness, $p_0/E=10^{-6}$.

gradient through the thickness becomes relatively more important with the increase of d/R .

From the above discussions, it may be concluded that the effect of temperature gradient through the thickness on the deformation is remarkable in comparison with the normal pressure for the following cases, that is, i) where the value of d/R is large, and ii) where the values of material constants α and E are large, and iii) in the early stage of heating where the Biot Number hd/k and T_E are large, and then this effect is supposed to become larger than that of normal pressure in some special cases.

3. THERMAL BUCKLING OF A CYLINDRICAL SHELL (I). *The case where the cylindrical shell is free from the constraints upon the thermal expansion and the axial displacement, but restrained laterally against the pressure at the edges.*

The cylindrical shell which is free from the constraints upon the thermal expansion and the axial displacement may buckle at the critical temperature by the existence of temperature gradient through the thickness only. In this section, the critical buckling value of a cylindrical shell in the above-mentioned case will be discussed. That is, the temperature gradient through the thickness due to the rapid heating on the inner face has the effect equivalent to that of external pressure as mentioned in the preceding section, and so it may cause the buckling phenomenon, depending on the temperature distribution and also on the boundary conditions. The temperature rise, temperature gradient through the thickness and external pressure are assumed to take symmetrical distribution over the shell surface with respect to the centre along the axial length and to be uniform in the circumferential direction and are given in the following expressions.

$$\bar{T} = \sum_i \bar{T}_i \cos \frac{i\pi x}{2L}, \quad (i=0, 2, 4, \dots \text{even}), \quad (3.1)$$

$$\tilde{T} = \sum_j \tilde{T}_j \cos \frac{j\pi x}{2L}, \quad (j=0, 2, 4, \dots \text{even}), \quad (3.2)$$

$$p = \sum_k p_k \cos \frac{k\pi x}{2L}, \quad (k=0, 2, 4, \dots \text{even}). \quad (3.3)$$

3.1. *The deformation and stress state before buckling.*

The deformation mode for this case takes naturally an axisymmetrical one, and so all the terms concerning v and $\partial/\partial y$ vanish. Then, the equilibrium equation, Eq. (2.14) reduces to

$$D \left[\frac{d^4 w}{dx^4} + \frac{2\nu}{R^2} \frac{d^2 w}{dx^2} + \frac{w}{R^4} \right] - d\chi_{\nu\nu} w_{xx} - \frac{d}{R} \chi_{xx} = - \frac{E\alpha}{(1-\nu)} \left(\frac{d}{R} \right)^2 \left(\tilde{T} + R^2 \frac{d^2 \tilde{T}}{dx^2} \right) + p. \quad (3.4)$$

Now, the extensional stress in the circumferential direction can be expressed as

$$\frac{\chi_{xx}}{E} = \frac{1}{(1-\nu^2)} \left[\left(v_y - \frac{w}{R} + \frac{1}{2} w_y^2 \right) + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right] - \frac{\alpha \bar{T}}{(1-\nu)}. \quad (3.5)$$

Considering that the deformation is axisymmetrical, Eq. (3.5) is reduced to

$$\frac{\chi_{xx}}{E} = \frac{1}{(1-\nu^2)} \left[-\frac{w}{R} + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right] - \frac{\alpha \bar{T}}{(1-\nu)}. \quad (3.5a)$$

From the same consideration, the following equation for χ_{yy}/E can be obtained with the condition that the shell is longitudinally unrestrained.

$$\frac{\chi_{yy}}{E} = \frac{1}{(1-\nu^2)} \left[\left(u_x + \frac{1}{2} w_x^2 \right) - \nu \frac{w}{R} \right] - \frac{\alpha \bar{T}}{(1-\nu)} = 0. \quad (3.6)$$

By using Eq. (3.6), Eq. (3.5a) is reduced to

$$\frac{\chi_{xx}}{E} = -\frac{w}{R} - \alpha \bar{T}. \quad (3.7)$$

Accordingly, the equilibrium equation, Eq. (3.4) can be expressed as

$$\begin{aligned} \frac{d^4 w}{dx^4} + \frac{2\nu}{R^2} \frac{d^2 w}{dx^2} + \left[\frac{1}{R^4} + \frac{12(1-\nu^2)}{d^2 R^2} \right] w \\ = \frac{12(1-\nu^2)}{d^2 R} \left[-\alpha \bar{T} - \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \left(\tilde{T} + R^2 \frac{d^2 \tilde{T}}{dx^2} \right) + \left(\frac{R}{d} \right) \frac{p}{E} \right], \end{aligned} \quad (3.8)$$

or

$$\begin{aligned} \frac{d^4 w}{dx^4} + \frac{2\nu}{R^2} \frac{d^2 w}{dx^2} + 4\beta^4 w \\ = 4R\beta^4 \left[-\alpha \bar{T}_0 - \frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) + \left(\frac{R}{d} \right) \frac{p_0}{E} \right] \\ + 4R\beta^4 \left\{ -\alpha \sum_{i=2}^{\infty} \bar{T}_i \cos \frac{i\pi x}{2L} - \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \left[1 - \frac{j^2}{4} \left(\frac{\pi R}{L} \right)^2 \right] \tilde{T}_j \cos \frac{j\pi x}{2L} \right. \\ \left. + \frac{1}{E} \left(\frac{R}{d} \right) \sum_{k=2}^{\infty} p_k \cos \frac{k\pi x}{2L} \right\}, \end{aligned} \quad (3.9)$$

where

$$\beta^4 = \frac{Ed}{4DR^2} = \frac{3(1-\nu^2)}{d^2 R^2}. \quad (3.10)$$

The general solution of w obtained after integrating Eq. (3.9) is

$$\begin{aligned} -\frac{w}{R} = & \left[\alpha \bar{T}_0 + \frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) - \left(\frac{R}{d} \right) \frac{p_0}{E} \right] \\ & + 16\xi^4 \left[4\alpha \sum_{i=2}^{\infty} \frac{\bar{T}_i}{(i^4 \eta^4 - 8\nu i^2 \eta^2 + 64\xi^4)} \cos \frac{i\pi x}{2L} \right. \\ & + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{(4 - j^2 \eta^2) \tilde{T}_j}{(j^4 \eta^4 - 8\nu j^2 \eta^2 + 64\xi^4)} \cos \frac{j\pi x}{2L} \\ & \left. - \frac{4}{E} \left(\frac{R}{d} \right) \sum_{k=2}^{\infty} \frac{p_k}{(k^4 \eta^4 - 8\nu k^2 \eta^2 + 64\xi^4)} \cos \frac{k\pi x}{2L} \right] + A(x), \end{aligned} \quad (3.11)$$

where

$$\xi = R\beta, \quad \eta = \frac{\pi R}{L}, \quad (3.12)$$

and the complementary solution $A(x)$ can be given as

$$\begin{aligned} A(x) = & A_1 e^{-\sqrt{1-\frac{\nu}{2\xi^2}}\beta x} \cos \sqrt{1+\frac{\nu}{2\xi^2}}\beta x + A_2 e^{-\sqrt{1-\frac{\nu}{2\xi^2}}\beta x} \sin \sqrt{1+\frac{\nu}{2\xi^2}}\beta x \\ & + A_3' e^{\sqrt{1-\frac{\nu}{2\xi^2}}\beta x} \cos \sqrt{1+\frac{\nu}{2\xi^2}}\beta x + A_4' e^{\sqrt{1-\frac{\nu}{2\xi^2}}\beta x} \sin \sqrt{1+\frac{\nu}{2\xi^2}}\beta x. \end{aligned} \quad (3.13)$$

where the following condition is allowed to hold,

$$\frac{\nu}{2\xi^2} = \frac{\nu}{2\sqrt{3(1-\nu^2)}} \left(\frac{d}{R} \right) \ll 1, \quad (3.14)$$

and then Eq. (3.13) is reduced to

$$\begin{aligned} A(x) = & A_1 e^{-\beta x} \cos \beta x + A_2 e^{-\beta x} \sin \beta x + A_3' e^{\beta x} \cos \beta x + A_4' e^{\beta x} \sin \beta x \\ = & A_1 e^{-\beta x} \cos \beta x + A_2 e^{-\beta x} \sin \beta x \\ & + A_3 e^{\beta(x-2L)} \cos \beta(x-2L) + A_4 e^{\beta(x-2L)} \sin \beta(x-2L). \end{aligned} \quad (3.13a)$$

The integral constants A_i 's in the complementary solution can be determined so as to satisfy the boundary conditions on w , that is, at $x=0$ and $2L$

$$\left. \begin{aligned} w = & -R\alpha \sum_{i=0}^{\infty} \bar{T}_i, \\ w_x = & 0, \end{aligned} \right\} \quad (3.15)$$

as

$$\left. \begin{aligned} A_1 = A_3 = & -\frac{1-e^{-2\beta L}(\cos 2\beta L - \sin 2\beta L)}{1-e^{-4\beta L}+2e^{-2\beta L}\sin 2\beta L} A_0, \\ A_2 = -A_4 = & -\frac{1-e^{-2\beta L}(\cos 2\beta L + \sin 2\beta L)}{1-e^{-4\beta L}+2e^{-2\beta L}\sin 2\beta L} A_0, \end{aligned} \right\} \quad (3.16)$$

where

$$\begin{aligned} -A_0 = & \left[-\frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) + \left(\frac{R}{d} \right) \frac{p_0}{E} \right] + \alpha \sum_{i=2}^{\infty} \frac{(i^4 \eta^4 - 8\nu i^2 \eta^2) \bar{T}_i}{(i^4 \eta^4 - 8\nu i^2 \eta^2 + 64\xi^4)} \\ & + 16\xi^4 \left[-\frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{(4-j^2 \eta^2) \tilde{T}_j}{(j^4 \eta^4 - 8\nu j^2 \eta^2 + 64\xi^4)} \right. \\ & \left. + \frac{4}{E} \left(\frac{R}{d} \right) \sum_{k=2}^{\infty} \frac{p_k}{(k^4 \eta^4 - 8\nu k^2 \eta^2 + 64\xi^4)} \right]. \end{aligned} \quad (3.17)$$

Next, the stress state corresponding to the above deformation mode will be analyzed. Integrating the compatibility equation, Eq. (2.15) by the use of Eq. (3.11), the following expression for the stress function, χ can be obtained.

$$\begin{aligned} \frac{\chi}{E} = & -\frac{C_1}{2}y^2 - \frac{C_2}{2}x^2 + 16\xi^4 \iint \left[4\alpha \sum_{i=2}^{\infty} \frac{\bar{T}_i}{(i^4\eta^4 - 8\nu i^2\eta^2 + 64\xi^4)} \cos \frac{i\pi x}{2L} \right. \\ & + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{(4-j^2\eta^2)\tilde{T}_j}{(j^4\eta^4 - 8\nu j^2\eta^2 + 64\xi^4)} \cos \frac{j\pi x}{2L} \\ & - \frac{4}{E} \left(\frac{R}{d} \right) \sum_{k=2}^{\infty} \frac{p_k}{(k^4\eta^4 - 8\nu k^2\eta^2 + 64\xi^4)} \cos \frac{k\pi x}{2L} \left. \right] dx dx \\ & + \iint A(x) dx dx - \alpha \iint \sum_{i=2}^{\infty} \bar{T}_i \cos \frac{i\pi x}{2L} dx dx. \end{aligned} \quad (3.18)$$

The integral constants C_1 and C_2 can be determined so as to satisfy the conditions of $\bar{\sigma}_{11}=0$ and $v_y=0$ as

$$\left. \begin{aligned} C_1 &= 0, \\ C_2 &= -\frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) + \left(\frac{R}{d} \right) \frac{p_0}{E}. \end{aligned} \right\} \quad (3.19)$$

Finally, the stresses of the cylindrical shell before buckling can be expressed as follows:

$$\begin{aligned} \frac{\bar{\sigma}_{11}}{E} &= \frac{\chi_{yy}}{E} = 0, \\ \frac{\bar{\sigma}_{22}}{E} &= \frac{\chi_{xx}}{E} = - \left[-\frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) + \left(\frac{R}{d} \right) \frac{p_0}{E} \right] - \alpha \sum_{i=2}^{\infty} \frac{(i^4\eta^4 - 8\nu i^2\eta^2)\bar{T}_i}{(i^4\eta^4 - 8\nu i^2\eta^2 + 64\xi^4)} \cos \frac{i\pi x}{2L} \\ &\quad - 16\xi^4 \left[-\frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{(4-j^2\eta^2)\tilde{T}_j}{(j^4\eta^4 - 8\nu j^2\eta^2 + 64\xi^4)} \cos \frac{j\pi x}{2L} \right. \\ &\quad + \frac{4}{E} \left(\frac{R}{d} \right) \sum_{k=2}^{\infty} \frac{p_k}{(k^4\eta^4 - 8\nu k^2\eta^2 + 64\xi^4)} \cos \frac{k\pi x}{2L} \left. \right] \\ &\quad + A_1 [e^{-\beta x} \cos \beta x + e^{\beta(x-2L)} \cos \beta(x-2L)] \\ &\quad + A_2 [e^{-\beta x} \sin \beta x - e^{\beta(x-2L)} \sin \beta(x-2L)], \\ \frac{\bar{\sigma}_{12}}{E} &= -\frac{\chi_{xy}}{E} = 0. \end{aligned} \quad (3.20)$$

3.2. The critical buckling value.

The buckling mode, which is an additional deflection to the axisymmetrical deflection state, is assumed as follows so as to satisfy the clamped edge conditions:

$$\begin{aligned} w' &= \sum_m \delta_m \sin^2 \frac{m\pi x}{2L} \cos \frac{ny}{R} \\ &\doteq \delta \left(1 - \cos \frac{\pi x}{L} \right) \cos \frac{ny}{R}, \end{aligned} \quad (3.21)$$

where, 2δ is the maximum deflection at the midpoint of shell and n is the number of buckled waves in the circumferential direction.

Then, the Galerkin method is applied to the equilibrium equation, Eq. (2.14) corresponding to the minimum principle of the potential energy. By expressing each of the variables in Eq. (2.14) by the sum of two components, that is, the one corresponding to the axisymmetrical deflection state before buckling and the other corresponding to the buckled state which is referred with the superscript ", ", and then by eliminating the former part for the axisymmetrical deflection state before buckling, the following equation is finally obtained after applying the Galerkin method.

$$\int_0^{2\pi} \int_0^L \left\{ D \left[\nabla^4 w' + \frac{2}{R^2} (w'_{yy} + \nu w'_{xx}) + \frac{w'}{R^4} \right] - d \chi_{xx} \left(w'_{yy} + \frac{w'}{R^2} \right) - \frac{d}{R} \chi'_{xx} \right\} \{w'\} dx dy = 0. \quad (3.22)$$

By integrating Eq. (3.22) with Eqs. (3.20) and (3.21) and χ' obtained after integrating the compatibility equation, Eq. (2.15) with Eqs. (3.11) and (3.21) [where, after expressing each of variables in Eq. (2.15) by the sum of two components, that is, the one corresponding to the axisymmetrical deflection state before buckling and the other corresponding to the buckled state, and after eliminating the former part for the axisymmetrical deflection state before buckling, the resulting equation is integrated], and by neglecting the higher order of infinitesimals, the following expression for the critical buckling value is obtained.

$$\begin{aligned} \theta_{cr} &= \frac{\left(\frac{d}{R}\right)^2}{36(1-\nu^2)} \left\{ 3(n^2-1) + \eta^2 \left[2 + \frac{\eta^2 + 2(1-\nu)}{(n^2-1)} \right] \right\} \\ &\quad + \frac{\eta^4}{3(n^2-1)(n^2+\eta^2)^2} \left[1 + \operatorname{sech}^2 \frac{n\pi}{\eta} - \frac{\eta(3n^2-\eta^2)}{n\pi(n^2+\eta^2)} \tanh \frac{n\pi}{\eta} \right] \\ &\doteq \frac{\left(\frac{d}{R}\right)^2}{36(1-\nu^2)} \left\{ 3(n^2-1) + \eta^2 \left[2 + \frac{\eta^2 + 2(1-\nu)}{(n^2-1)} \right] \right\} \\ &\quad + \frac{\eta^4}{3(n^2-1)(n^2+\eta^2)^2}, \end{aligned} \quad (3.23)$$

where

$$\begin{aligned} \theta &= \left[-\frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R}\right) + \left(\frac{R}{d}\right) \frac{p_0}{E} \right] \\ &\quad - \frac{1}{3} \left\{ \left[\frac{2(\eta^2-2\nu)\eta^2\alpha\tilde{T}_2}{(4\xi^4-2\nu\eta^2+\eta^4)} - \frac{(2\eta^2-\nu)\eta^2\alpha\tilde{T}_4}{(\xi^4-2\nu\eta^2+4\eta^4)} \right] \right. \\ &\quad \left. - \frac{1}{(1-\nu)} \left(\frac{d}{R}\right) \left[\frac{8(1-\eta^2)\xi^4\alpha\tilde{T}_2}{(4\xi^4-2\nu\eta^2+\eta^4)} - \frac{(1-4\eta^2)\xi^4\alpha\tilde{T}_4}{2(\xi^4-2\nu\eta^2+4\eta^4)} \right] \right. \\ &\quad \left. + \frac{1}{E} \left(\frac{R}{d}\right) \left[\frac{8\xi^4 p_2}{(4\xi^4-2\nu\eta^2+\eta^4)} - \frac{\xi^4 p_4}{2(\xi^4-2\nu\eta^2+4\eta^4)} \right] \right\} \end{aligned}$$

$$-\frac{4(\xi^4 - \eta^4)\eta^5}{\pi\xi(\xi^4 + 4\eta^4)(4\xi^4 + \eta^4)} \left[\frac{1 + e^{-\frac{4\pi\xi}{\eta}} - 2e^{-\frac{2\pi\xi}{\eta}} \cos \frac{2\pi\xi}{\eta}}{1 - e^{-\frac{4\pi\xi}{\eta}} - 2e^{-\frac{2\pi\xi}{\eta}} \sin \frac{2\pi\xi}{\eta}} \right] A_0. \quad (3.24)$$

The value of n which minimizes the critical buckling value can be obtained, from $\partial\theta_{cr}/\partial n=0$, as

$$n \doteq \left[1 + 5.677 \left(\frac{R}{L} \right) \left(\frac{R}{d} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}. \quad (3.25)$$

Substituting Eq. (3.25) into Eq. (3.23), the critical buckling value is finally reduced to

$$\theta_{cr} \doteq \left[0.710 + 0.617 \left(\frac{R}{L} \right) \left(\frac{d}{R} \right)^{\frac{1}{2}} + 0.536 \left(\frac{R}{L} \right)^2 \left(\frac{d}{R} \right) \right] \left(\frac{R}{L} \right) \left(\frac{d}{R} \right)^{\frac{3}{2}}, \quad \left(\nu = \frac{1}{3} \right). \quad (3.26)$$

And, the critical buckling value for the infinitely long cylinder can be obtained by putting $\eta=0$ in Eqs. (3.23) and (3.24) as

$$\theta_{cr} = \frac{1}{4(1-\nu^2)} \left(\frac{d}{R} \right)^2, \quad (n=2), \quad (3.27)$$

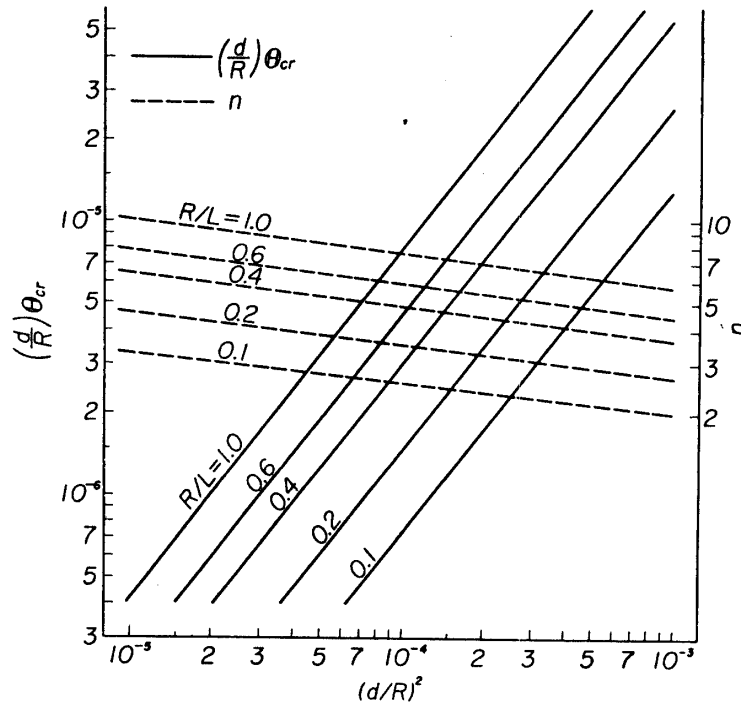


FIGURE 4. Critical value and number of the circumferential waves of the cylindrical shell subjected to heating and normal pressure, free from the constraints upon the thermal expansion and the axial displacement but restrained laterally against the pressure at the edges.

where

$$\Theta = -\frac{\alpha}{(1-\nu)}\left(\frac{d}{R}\right)\left(\tilde{T}_0 - \frac{2}{3}\tilde{T}_2 + \frac{1}{6}\tilde{T}_4\right) + \frac{1}{E}\left(\frac{R}{d}\right)\left(p_0 - \frac{2}{3}p_2 + \frac{1}{6}p_4\right). \quad (3.28)$$

Eqs. (3.25) and (3.26) are shown in Fig. 4. Eqs. (3.26) and (3.27), particularly putting $\bar{T} = \tilde{T} = 0$ and $p \equiv p_0$, are the solutions for a cylindrical shell subjected only to the uniform external pressure, and these results agree satisfactorily with those [8] obtained in other manners.

In the above, the effect of temperature gradient through the thickness on the thermal deformation of a cylindrical shell has been made clear to a certain extent. Then, the analyses for the cases where the thermal expansion is restrained will be shown in the following sections. For these cases, the effect of temperature gradient through the thickness is much less than that of the thermal circumferential stress due to the mean temperature rise as mentioned before, and the deformation mode is greatly influenced by the condition of constraint upon the deformation, that is, by the boundary conditions, and so the following two typical cases will be analyzed separately, where i) the cylindrical shell is unrestrained longitudinally but fully restrained laterally at the edges, and ii) the cylindrical shell is clamped at the edges, which will be shown later in 4. and 5., respectively.

4. THERMAL BUCKLING OF A CYLINDRICAL SHELL (II). *The case where the cylindrical shell is unrestrained longitudinally but fully restrained laterally at the edges.*

When the cylindrical shell, which is unrestrained longitudinally but fully restrained laterally at the edges, is heated, the axial stress is not induced, while the circumferential stress is considered to be generated convergently in the narrow domain near the clamped edges, where the lateral displacement is fully restrained, and so the local buckling is expected to be generated there.

There are the previous analyses by Zuk [1] and Hoff [2] on such a problem as mentioned in the Introduction. In this section, however, the analysis will be carried out rather practically as the local buckling problem paying special attention to the effect of clamping the edges. That is, the deflection mode of the axisymmetrical expansion state will be obtained after integrating the equilibrium equation first, and then the critical temperature for the circumferential buckling from the former state will be discussed.

The temperature rise is assumed to be uniform over the shell, and the effect of temperature gradient through the thickness can be neglected for this case as stated above, that is,

$$\left. \begin{aligned} \bar{T} &\equiv \bar{T}_0, \\ \tilde{T} &= 0. \end{aligned} \right\} \quad (4.1)$$

4.1. The deformation and stress state before buckling.

The deformation mode for this case also takes naturally an axisymmetrical one, and so all the terms concerning v and $\partial/\partial y$ vanish. And, taking only the one side edge into account, because the edge effect is considered to be extremely local, and putting $\tilde{T}=p=0$, the equilibrium equation, Eq. (2.14) reduces to

$$D\left[\frac{d^4w}{dx^4} + \frac{2\nu}{R^2}\frac{d^2w}{dx^2} + \frac{w}{R^4}\right] - d\chi_{\nu\nu}w_{xx} - \frac{d}{R}\chi_{xx} = 0. \quad (4.2)$$

In the same manner as in 3.1., Eq. (4.2) is reduced to

$$\frac{d^4w}{dx^4} + \frac{2\nu}{R^2}\frac{d^2w}{dx^2} + 4\beta^4w = -4\beta^4R\alpha\bar{T}_0, \quad (4.3)$$

where

$$\beta^4 = \frac{Ed}{4DR^2} = \frac{3(1-\nu^2)}{d^2R^2}. \quad (4.4)$$

The general solution of w obtained after integrating Eq. (4.3) is

$$\begin{aligned} w = & -R\alpha\bar{T}_0 + e^{-\sqrt{1-\frac{\nu}{2R^2\beta^2}}\beta x} \left[A_1 \cos \sqrt{1+\frac{\nu}{2R^2\beta^2}}\beta x + A_2 \sin \sqrt{1+\frac{\nu}{2R^2\beta^2}}\beta x \right] \\ & + e^{\sqrt{1-\frac{\nu}{2R^2\beta^2}}\beta x} \left[A_3 \cos \sqrt{1+\frac{\nu}{2R^2\beta^2}}\beta x + A_4 \sin \sqrt{1+\frac{\nu}{2R^2\beta^2}}\beta x \right] \\ = & -R\alpha\bar{T}_0 + e^{-\beta x}(A_1 \cos \beta x + A_2 \sin \beta x) + e^{\beta x}(A_3 \cos \beta x + A_4 \sin \beta x). \end{aligned} \quad (4.5)$$

The integral constants A_i 's in the complementary solution can be determined so as to satisfy the boundary conditions on w , that is,

$$\left. \begin{aligned} \text{at } x=0 & \quad w=w_x=0, \\ \text{at } x=\infty & \quad w=-R\alpha\bar{T}_0, \end{aligned} \right\} \quad (4.6)$$

as

$$A_1=A_2=R\alpha\bar{T}_0, \quad A_3=A_4=0. \quad (4.7)$$

Then, w can be expressed with Eqs. (4.5) and (4.7) as

$$w = -R\alpha\bar{T}_0[1 - e^{-\beta x}(\cos \beta x + \sin \beta x)]. \quad (4.8)$$

Eq. (4.8) can be derived also from the analytical result [9] on the bending of a long cylindrical shell by a load uniformly distributed along a circular section.

Next, the stress state corresponding to the above deformation mode will be analyzed. Integrating the compatibility equation, Eq. (2.15) by the use of Eq. (4.8), the following expression for the stress function, χ can be obtained.

$$\frac{\chi}{E} = -\frac{1}{2\beta^2}\alpha\bar{T}_0e^{-\beta x}(\cos \beta x - \sin \beta x) + C_1x^2 + C_2y^2. \quad (4.9)$$

The integral constants C_1 and C_2 can be determined so as to satisfy the conditions of $\bar{\sigma}_{11}=0$ and $v_y=0$ as

$$C_1 = C_2 = 0. \quad (4.10)$$

Finally, the stress state of the cylindrical shell before buckling can be expressed as follows:

$$\left. \begin{aligned} \bar{\sigma}_{11} &= \chi_{yy} = 0, \\ \bar{\sigma}_{22} &= \chi_{xx} = -E\alpha \bar{T}_0 e^{-\beta x} (\cos \beta x + \sin \beta x), \\ \bar{\sigma}_{12} &= -\chi_{xy} = 0. \end{aligned} \right\} \quad (4.11)$$

4.2. The critical buckling temperature.

The buckling mode, which is an additional deflection to the axisymmetrical deflection before buckling, is assumed as follows so as to satisfy the clamped edge conditions:

$$w' = \delta e^{-\beta x} \left(1 - \cos \frac{\pi x}{l} \right) \cos \frac{ny}{R}, \quad (4.12)$$

where, δ denotes the deflection factor, and l and n are half the wave length in the axial direction and the number of the buckled waves in the circumferential direction, respectively.

Then, the Galerkin method is applied to the equilibrium equation, Eq. (2.14) corresponding to the minimum principle of the potential energy. By expressing each of the variables in Eq. (2.14) by the sum of two components, that is, the one corresponding to the axisymmetrical deflection state before buckling and the other corresponding to the buckled state which is referred with the superscript ", ", and then by eliminating the former part for the axisymmetrical deflection state before buckling, the following equation is finally obtained after applying the Galerkin method.

$$\int_0^{\pi R/n} \int_0^\infty \left\{ D \left[\nabla^4 w' + \frac{2}{R^2} (w'_{yy} + \nu w'_{xx}) + \frac{w'}{R^4} \right] - d\chi_{xx} \left(w'_{yy} + \frac{w'}{R^2} \right) - \frac{d}{R} \chi'_{xx} \right\} \{w'\} dx dy = 0. \quad (4.13)$$

By integrating Eq. (4.13) with Eqs. (4.11) and (4.12), the following expression for the critical buckling temperature is obtained.

$$\alpha \bar{T}_{0cr} = \frac{5(100\xi^4 + 16\xi^2\eta^2 + \eta^4)(25\xi^4 + 16\xi^2\eta^2 + 4\eta^4) \times [\xi^2(15\xi^2 + 4\eta^2) + 2(\xi^2 + \eta^2)(n^2 - \nu) + \eta^4 + 3(n^2 - 1)^2]}{144(1 - \nu^2)n^2(\xi^2 + \eta^2)(4\xi^2 + \eta^2)(76\xi^4 + 55\xi^2\eta^2 + 4\eta^4)} \left(\frac{d}{R} \right)^2, \quad (4.14)$$

where

$$\xi = R\beta, \quad \eta = \frac{\pi R}{l}. \quad (4.15)$$

The axial length of buckled wave is here assumed as

$$2l = \frac{\pi}{k\beta}, \quad (4.16)$$

then,

$$\eta = 2k\xi. \quad (4.17)$$

By using Eq. (4.17), Eq. (4.14) is reduced to

$$\alpha \bar{T}_{0cr} = \frac{5(25 + 16k^2 + 4k^4)(25 + 64k^2 + 64k^4) \times [\xi^4(15 + 16k^2 + 16k^4) + 2\xi^2(1 + 4k^2)(n^2 - \nu) + 3(n^2 - 1)^2]}{576(1 - \nu^2)n^2(1 + k^2)(1 + 4k^2)(19 + 55k^2 + 16k^4)} \left(\frac{d}{R}\right)^2. \quad (4.18)$$

The values of n and k which minimize the critical temperature, \bar{T}_{0cr} can be obtained by solving the following simultaneous equations:

$$\frac{\partial \bar{T}_{0cr}}{\partial n} = 0, \quad (4.19)$$

$$\frac{\partial \bar{T}_{0cr}}{\partial k} = 0, \quad (4.20)$$

as

$$k \doteq 0.99, \quad (4.21)$$

$$n \doteq 2.53 \left(\frac{R}{d}\right)^{\frac{1}{2}}. \quad (4.22)$$

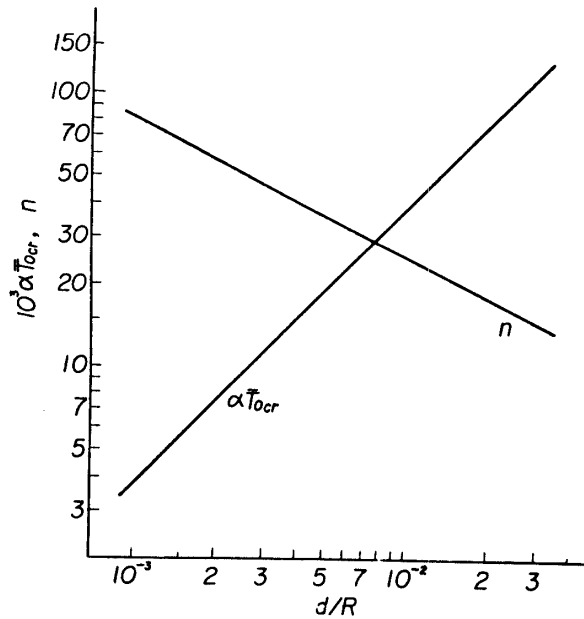


FIGURE 5. Local buckling temperature and number of the circumferential waves of the cylindrical shell, unrestrained longitudinally and fully restrained laterally at the edges.

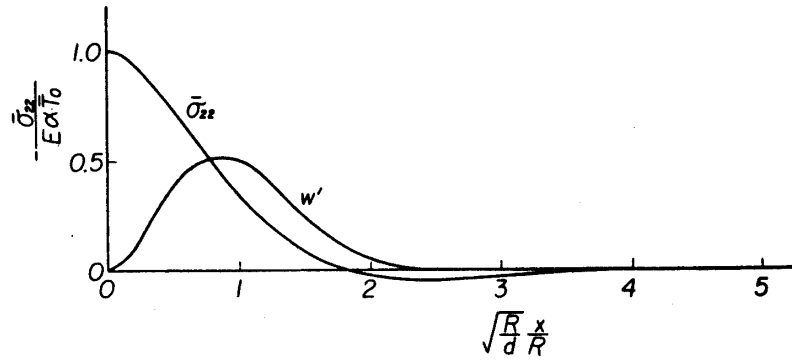


FIGURE 6. Longitudinal variations of the thermal circumferential stress before buckling and the buckling deflection mode, unrestrained longitudinally and fully restrained laterally at the edges.

Substituting Eqs. (4.4), (4.15), (4.21) and (4.22) into Eq. (4.18), the critical buckling temperature is finally given by

$$\alpha \bar{T}_{0cr} = 3.76 \left(\frac{d}{R} \right), \quad \left(\nu = \frac{1}{3} \right). \quad (4.23)$$

Eqs. (4.22) and (4.23) are shown in Fig. 5, and Eqs. (4.11) and (4.12) are shown in Fig. 6.

Thus, the critical local buckling temperature of the cylindrical shell has been obtained. It can be seen in Eq. (4.23) that this critical temperature is so high except for the very thin cylindrical shell. That is, $\bar{T}_{0cr} = 190, 950$ and 1900°C corresponding to $d/R = 10^{-3}, 5 \times 10^{-3}$ and 10^{-2} , respectively, where $\alpha = 2 \times 10^{-5}$, as an example. And so, generally speaking, the material property will be deteriorated and the plastic deformation will take place being caused by the thermal stress given by Eq. (4.11) before the buckling occurs, and then the local thermal buckling cannot be considered to occur. However, the attention should be paid to the case where the cylindrical shell is comparatively thin.

In this analysis, considering the comparatively long cylindrical shell, the clamping effect was assumed to be limited to the vicinity of the edge and the analysis on the one side edge only was carried out. Because, the domain where the thermal stress and the thermal buckling deformation have marked effects is limited to the one of $x < \pi/\beta \doteq 2.5\sqrt{d/R} R$, as easily can be seen from Eqs. (4.11), (4.12), (4.16) and (4.21) and Fig. 6, and, for example, it reduces to $x < 25$ mm for the case of $d/R = 10^{-2}$ and $R = 100$ mm. Therefore, such an analysis as in this section may be considered sufficiently accurate from the engineering point of view. But it will be necessary to re-examine the deflection mode and related terms for the case of extremely short cylindrical shell.

The numerical results by this analysis and by Hoff [2] are shown in Table 1, for reference, although the boundary conditions in both analyses are different from each other and thus the results cannot be compared directly. But it is interesting to note that the critical buckling temperature by this analysis is lower than that by Hoff, and it seems that this result presents an example of the fact

TABLE 1.

	Hoff	Author
Material	Steel ($E=2.03 \times 10^4 \text{ Kg/mm}^2$, $\alpha=1.32 \times 10^{-5}/^\circ\text{C}$)	
d/R	3.31×10^{-3} ($R=254 \text{ mm}$, $d=0.84 \text{ mm}$)	
n	32	44
\bar{T}_{0cr}	1300°C	950°C
Edge Cond.	Simple Support	Clamp

that the critical values of shells, having curvatures of finite magnitude such as cylindrical and spherical shells, are not necessarily higher for the case of clamped edge conditions than for that of simple support in some special modes of buckling.

5. THERMAL DEFORMATION OF A CYLINDRICAL SHELL.

The case where the cylindrical shell is clamped at the edges.

5.1. Solutions of the fundamental equations.

The cylindrical shell, clamped completely at the edges, starts to expand in the radial direction only inducing the axial compression which makes the radial deflection larger. Therefore, the cylindrical shell is considered to deform from the beginning of heating.

It is assumed that the temperature rise, axisymmetrical and symmetrical with respect to the centre of length, is only applied, that is, $p=0$ in Eq. (2.14), and that the deformation mode is axisymmetrical, too. Thus, all terms concerning v and $\partial/\partial y$ vanish, and the equilibrium equation, Eq. (2.14) and the compatibility equation, Eq. (2.15) are reduced to

$$\begin{aligned} \frac{d^4 w}{dx^4} + \frac{2\nu}{R^2} \frac{d^2 w}{dx^2} + \frac{w}{R^4} \\ = \frac{12(1-\nu^2)}{d^2} \left[\frac{\chi_{yy}}{E} w_{xx} + \frac{1}{R} \frac{\chi_{xx}}{E} - \frac{\alpha}{d(1-\nu)} \left(\frac{d}{R} \right)^2 \left(\bar{T} + R^2 \frac{d^2 \bar{T}}{dx^2} \right) \right], \end{aligned} \quad (5.1)$$

$$\nabla^4 \frac{\chi}{E} = -\frac{w_{xx}}{R} - \alpha \frac{d^2 \bar{T}}{dx^2}. \quad (5.2)$$

\bar{T} and \tilde{T} can be expressed according to the above assumption as

$$\bar{T} = \sum_i \bar{T}_i \cos \frac{i\pi x}{2L}, \quad (i=0, 2, 4, \dots \text{even}), \quad (5.3)$$

$$\tilde{T} = \sum_j \tilde{T}_j \cos \frac{j\pi x}{2L}, \quad (j=0, 2, 4, \dots \text{even}). \quad (5.4)$$

In the same manner as in 3.1., χ_{xx} and χ_{yy} reduce to

$$\frac{\chi_{xx}}{E} = \frac{1}{(1-\nu^2)} \left[-\frac{w}{R} + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right] - \frac{\alpha \bar{T}}{(1-\nu)}, \quad (5.5)$$

$$\frac{\chi_{yy}}{E} = \frac{1}{(1-\nu^2)} \left[\left(u_x + \frac{1}{2} w_x^2 \right) - \nu \frac{w}{R} \right] - \frac{\alpha \bar{T}}{(1-\nu)}, \quad (5.6)$$

where, χ_{yy} is independent of y because of the axisymmetrical deformation, and χ_{yy} is assumed to be constant in the x -direction depending on the temperature rise only. This assumption is considered to be reasonable in the first approximation, because it has already been shown in the large deflection analysis for a flat strip [3], [10] that the axial stress is constant in the x -direction during the deformation. Taking this analytical result into account, the following analysis is carried out as an Eigen value problem, by putting $\chi_{yy}/E = -C_1$ as

$$\frac{\chi_{yy}}{E} = \frac{1}{(1-\nu^2)} \left[\left(u_x + \frac{1}{2} w_x^2 \right) - \nu \frac{w}{R} \right] - \frac{\alpha \bar{T}}{(1-\nu)} = -C_1. \quad (5.6a)$$

Using Eqs. (5.5) and (5.6a), χ_{xx} can be expressed as

$$\frac{\chi_{xx}}{E} = -\frac{w}{R} - \alpha \bar{T} - \nu C_1. \quad (5.5a)$$

Then, the equilibrium equation, Eq. (5.1) is reduced to

$$\begin{aligned} \frac{d^4 w}{dx^4} + \left[\frac{2\nu}{R^2} + \frac{12(1-\nu^2)}{d^2} C_1 \right] \frac{d^2 w}{dx^2} + \left[\frac{1}{R^4} + \frac{12(1-\nu^2)}{d^2 R^2} \right] w \\ = -\frac{12(1-\nu^2)}{d^2 R} \left[\alpha \bar{T} + \nu C_1 + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \left(\tilde{T} + R^2 \frac{d^2 \tilde{T}}{dx^2} \right) \right]. \end{aligned} \quad (5.7)$$

Neglecting the higher order of infinitesimals after the estimation of the magnitudes of terms in the left hand side, Eq. (5.7) is reduced to

$$\begin{aligned} \frac{d^4 w}{dx^4} + 4\beta^4 R^2 C_1 \frac{d^2 w}{dx^2} + 4\beta^4 w \\ = -4\beta^4 R \left[\alpha \bar{T}_0 + \nu C_1 + \frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) \right] \\ - 4\beta^4 R \left\{ \alpha \sum_{i=2}^{\infty} \bar{T}_i \cos \frac{i\pi x}{2L} + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \left[1 - \frac{j^2}{4} \left(\frac{\pi R}{L} \right)^2 \right] \tilde{T}_j \cos \frac{j\pi x}{2L} \right\}, \end{aligned} \quad (5.8)$$

where

$$\beta^4 = \frac{Ed}{4DR^2} = \frac{3(1-\nu^2)}{d^2 R^2}. \quad (5.9)$$

The general solution of w obtained after integrating Eq. (5.8) is given by

$$\begin{aligned}
-\frac{w}{R} = & \left[\alpha \bar{T}_0 + \nu C_1 + \frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) \right] \\
& + 16\xi^4 \left\{ 4\alpha \sum_{i=2}^{\infty} \frac{\bar{T}_i}{(i^4\eta^4 - 16i^2C_1\xi^4\eta^2 + 64\xi^4)} \cos \frac{i\pi x}{2L} \right. \\
& \left. + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{(4-j^2\eta^2)\tilde{T}_j}{(j^4\eta^4 - 16j^2C_1\xi^4\eta^2 + 64\xi^4)} \cos \frac{j\pi x}{2L} \right\} + A(x), \quad (5.10)
\end{aligned}$$

where

$$\xi = R\beta, \quad \eta = \frac{\pi R}{L}, \quad (5.11)$$

and the complementary solution $A(x)$ can be given as

i) the case of $1 - \xi^2 C_1 > 0$,

$$\begin{aligned}
A(x) = & A_1 e^{-\sqrt{1-\xi^2 C_1} \beta x} \cos \sqrt{1+\xi^2 C_1} \beta x + A_2 e^{-\sqrt{1-\xi^2 C_1} \beta x} \sin \sqrt{1+\xi^2 C_1} \beta x \\
& + A_3 e^{\sqrt{1-\xi^2 C_1} \beta x} \cos \sqrt{1+\xi^2 C_1} \beta x + A_4 e^{\sqrt{1-\xi^2 C_1} \beta x} \sin \sqrt{1+\xi^2 C_1} \beta x \\
= & A_1 e^{-\sqrt{1-\xi^2 C_1} \beta x} \cos \sqrt{1+\xi^2 C_1} \beta x + A_2 e^{-\sqrt{1-\xi^2 C_1} \beta x} \sin \sqrt{1+\xi^2 C_1} \beta x \\
& + A_3 e^{\sqrt{1-\xi^2 C_1} \beta (x-2L)} \cos \sqrt{1+\xi^2 C_1} \beta (x-2L) \\
& + A_4 e^{\sqrt{1-\xi^2 C_1} \beta (x-2L)} \sin \sqrt{1+\xi^2 C_1} \beta (x-2L), \quad (5.12-1)
\end{aligned}$$

ii) the case of $1 - \xi^2 C_1 = 0$,

$$\begin{aligned}
A(x) = & (A_1 + A_3 x) \cos \sqrt{1+\xi^2 C_1} \beta x + (A_2 + A_4 x) \sin \sqrt{1+\xi^2 C_1} \beta x \\
= & (A_1 + A_3 x) \cos \sqrt{2} \beta x + (A_2 + A_4 x) \sin \sqrt{2} \beta x, \quad (5.12-2)
\end{aligned}$$

iii) the case of $1 - \xi^2 C_1 < 0$,

$$\begin{aligned}
A(x) = & A_1 \cos [(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1}) \beta x + A_3] \\
& + A_2 \cos [(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1}) \beta x + A_4]. \quad (5.12-3)
\end{aligned}$$

The integral constants A_i 's in the complementary solution can be determined so as to satisfy the boundary conditions on w , that is,

$$(w)_{x=0, 2L} = 0, \quad (w_x)_{x=0, 2L} = 0, \quad (5.13)$$

as

i) the case of $1 - \xi^2 C_1 > 0$,

$$\begin{aligned}
A_1 = A_3 = & - \frac{1 - e^{-2\sqrt{1-\xi^2 C_1} \beta L} [\cos(2\sqrt{1+\xi^2 C_1} \beta L) - \bar{k} \sin(2\sqrt{1+\xi^2 C_1} \beta L)]}{1 - e^{-4\sqrt{1-\xi^2 C_1} \beta L} + 2\bar{k} e^{-2\sqrt{1-\xi^2 C_1} \beta L} \sin(2\sqrt{1+\xi^2 C_1} \beta L)} A_0, \\
A_2 = -A_4 = & - \frac{\bar{k} - e^{-2\sqrt{1-\xi^2 C_1} \beta L} [\bar{k} \cos(2\sqrt{1+\xi^2 C_1} \beta L) + \sin(2\sqrt{1+\xi^2 C_1} \beta L)]}{1 - e^{-4\sqrt{1-\xi^2 C_1} \beta L} + 2\bar{k} e^{-2\sqrt{1-\xi^2 C_1} \beta L} \sin(2\sqrt{1+\xi^2 C_1} \beta L)} A_0, \quad (5.14-1)
\end{aligned}$$

ii) the case of $1 - \xi^2 C_1 = 0$,

$$\left. \begin{aligned} A_1 &= -A_0, & A_2 &= -\frac{1 - \cos 2\sqrt{2}\beta L}{2\sqrt{2}\beta L + \sin 2\sqrt{2}\beta L} A_0, \\ A_3 &= \frac{\sqrt{2}\beta(1 - \cos 2\sqrt{2}\beta L)}{2\sqrt{2}\beta L + \sin 2\sqrt{2}\beta L} A_0, & A_4 &= -\frac{\sqrt{2}\beta \sin 2\sqrt{2}\beta L}{2\sqrt{2}\beta L + \sin 2\sqrt{2}\beta L} A_0, \end{aligned} \right\} \quad (5.14-2)$$

iii) the case of $1 - \xi^2 C_1 < 0$,

$$\left. \begin{aligned} A_1 &= \frac{(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1}) \sin [(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1})\beta L]}{\sqrt{\xi^2 C_1 - 1} \sin (2\sqrt{\xi^2 C_1 + 1}\beta L) + \sqrt{\xi^2 C_1 + 1} \sin (2\sqrt{\xi^2 C_1 - 1}\beta L)} A_0, \\ A_2 &= -\frac{(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1}) \sin [(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1})\beta L]}{\sqrt{\xi^2 C_1 - 1} \sin (2\sqrt{\xi^2 C_1 + 1}\beta L) + \sqrt{\xi^2 C_1 + 1} \sin (2\sqrt{\xi^2 C_1 - 1}\beta L)} A_0, \\ A_3 &= -(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1})\beta L, \\ A_4 &= -(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1})\beta L, \end{aligned} \right\} \quad (5.14-3)$$

where

$$\begin{aligned} A_0 &= \left[\alpha \bar{T}_0 + \nu C_1 + \frac{\alpha \bar{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) \right] + 16\xi^4 \left\{ 4\alpha \sum_{i=2}^{\infty} \frac{\bar{T}_i}{(i^4 \eta^4 - 16i^2 C_1 \xi^4 \eta^2 + 64\xi^4)} \right. \\ &\quad \left. + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{(4-j^2 \eta^2) \bar{T}_j}{(j^4 \eta^4 - 16j^2 C_1 \xi^4 \eta^2 + 64\xi^4)} \right\}, \end{aligned} \quad (5.15)$$

$$\bar{k} = \left(\frac{1 - \xi^2 C_1}{1 + \xi^2 C_1} \right)^{\frac{1}{2}}. \quad (5.16)$$

It can be seen in the above analysis that the deformation mode is expressed through the unknown Eigen value, C_1 . The problem will then be how to determine C_1 in Eq. (5.10) by the following process.

Now, C_1 can be determined from the condition of compatibility and from the boundary conditions on the displacement in the shell plane as follows: By integrating the compatibility equation, Eq. (5.2) with Eqs. (5.3) and (5.10), the stress function can be expressed as Eq. (5.17).

$$\begin{aligned} \frac{\chi}{E} &= -\frac{C_1}{2} y^2 - \frac{C_2}{2} x^2 + 16\xi^4 \int \int \left\{ 4\alpha \sum_{i=2}^{\infty} \frac{\bar{T}_i}{(i^4 \eta^4 - 16i^2 C_1 \xi^4 \eta^2 + 64\xi^4)} \cos \frac{i\pi x}{2L} \right. \\ &\quad \left. + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{(4-j^2 \eta^2) \bar{T}_j}{(j^4 \eta^4 - 16j^2 C_1 \xi^4 \eta^2 + 64\xi^4)} \cos \frac{j\pi x}{2L} \right\} dx dx \\ &\quad + \int \int A(x) dx dx - \alpha \int \int \sum_{i=2}^{\infty} \bar{T}_i \cos \frac{i\pi x}{2L} dx dx. \end{aligned} \quad (5.17)$$

Then, $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$ and $\bar{\sigma}_{12}$ can be given as

$$\left. \begin{aligned} \frac{\bar{\sigma}_{11}}{E} &= \frac{\chi_{yy}}{E} = -C_1, \\ \frac{\bar{\sigma}_{22}}{E} &= \frac{\chi_{xx}}{E} = -C_2 + 16\xi^4 \left\{ 4\alpha \sum_{i=2}^{\infty} \frac{\bar{T}_i}{(i^4 \eta^4 - 16i^2 C_1 \xi^4 \eta^2 + 64\xi^4)} \cos \frac{i\pi x}{2L} \right. \\ &\quad \left. + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{(4-j^2 \eta^2) \bar{T}_j}{(j^4 \eta^4 - 16j^2 C_1 \xi^4 \eta^2 + 64\xi^4)} \cos \frac{j\pi x}{2L} \right\} \end{aligned} \right\} \quad (5.18)$$

$$\begin{aligned}
& + A(x) - \alpha \sum_{i=2}^{\infty} \bar{T}_i \cos \frac{i\pi x}{2L} \\
& = -\frac{w}{R} - \alpha \bar{T} - \left[\nu C_1 + C_2 + \frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) \right], \\
\frac{\bar{\sigma}_{12}}{E} & = -\frac{\chi_{xy}}{E} = 0.
\end{aligned}$$

The integral constants C_1 , which means the axial stress in the x -direction, and C_2 can be determined so as to satisfy the boundary conditions on the displacement in the shell plane, that is,

$$\int_0^{2L} u_x dx = 0, \quad v_y = 0, \quad (5.19)$$

as

$$-C_1 = -\alpha \bar{T}_0 + \frac{\nu \alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right) + \frac{\nu}{2L} \int_0^{2L} A(x) dx + \frac{1}{4L} \int_0^{2L} w_x^2 dx, \quad (5.20)$$

$$-C_2 = \frac{\alpha \tilde{T}_0}{(1-\nu)} \left(\frac{d}{R} \right), \quad (5.21)$$

where, the third and the fourth term in the right hand side in Eq. (5.20) are given as follows corresponding to the sign of $(1 - \xi^2 C_1)$:

$$\frac{1}{2L} \int_0^{2L} A(x) dx = F_1(A):$$

i) the case of $1 - \xi^2 C_1 > 0$,

$$\begin{aligned}
F_1(A) &= \frac{\sqrt{1 + \xi^2 C_1}}{2\beta L} \left[A_1 \{ \bar{k} + e^{-2\sqrt{1 - \xi^2 C_1} \beta L} [-\bar{k} \cos(2\sqrt{1 + \xi^2 C_1} \beta L) + \sin(2\sqrt{1 + \xi^2 C_1} \beta L)] \} \right. \\
&\quad \left. + A_2 \{ 1 - e^{-2\sqrt{1 - \xi^2 C_1} \beta L} [\cos(2\sqrt{1 + \xi^2 C_1} \beta L) + \bar{k} \sin(2\sqrt{1 + \xi^2 C_1} \beta L)] \} \right], \quad (5.22-1)
\end{aligned}$$

ii) the case of $1 - \xi^2 C_1 = 0$,

$$F_1(A) = -\frac{2\sqrt{2}(1 - \cos 2\sqrt{2}\beta L)}{2\beta L(2\sqrt{2}\beta L + \sin 2\sqrt{2}\beta L)} A_0, \quad (5.22-2)$$

iii) the case of $1 - \xi^2 C_1 < 0$,

$$F_1(A) = \frac{1}{\beta L} \left\{ \frac{\sin [(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1})\beta L]}{\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1}} A_1 + \frac{\sin [(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1})\beta L]}{\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1}} A_2 \right\}, \quad (5.22-3)$$

$$\frac{1}{4L} \int_0^{2L} w_x^2 dx = F_2\left(\frac{1}{2} w_x^2\right):$$

i) the case of $1 - \xi^2 C_1 > 0$,

$$\begin{aligned}
F_2\left(\frac{1}{2}w_x^2\right) = & \frac{1}{2}(1+\xi^2 C_1)\xi^2 \left\{ (1-e^{-4\sqrt{1-\xi^2 C_1}\beta L}) \frac{(B_1^2+B_2^2)}{4\sqrt{1-\xi^2 C_1}\beta L} \right. \\
& + (1-e^{-4\sqrt{1-\xi^2 C_1}\beta L} \cos [4\sqrt{1+\xi^2 C_1}\beta L]) \\
& \quad \times \frac{2\sqrt{1+\xi^2 C_1}\beta L}{16(\beta L)^2} [\bar{k}(B_1^2-B_2^2)+2B_1B_2] \\
& + e^{-4\sqrt{1-\xi^2 C_1}\beta L} \sin [4\sqrt{1+\xi^2 C_1}\beta L] \\
& \quad \times \left[\frac{2\sqrt{1+\xi^2 C_1}\beta L}{16(\beta L)^2} [(B_1^2-B_2^2)-2\bar{k}B_1B_2] \right] \\
& - e^{-2\sqrt{1-\xi^2 C_1}\beta L} \left[(B_1^2-B_2^2) \cos (2\sqrt{1+\xi^2 C_1}\beta L) \right. \\
& \quad \left. + \left(\frac{B_1^2+B_2^2}{2\sqrt{1+\xi^2 C_1}\beta L} + 2B_1B_2 \right) \sin (2\sqrt{1+\xi^2 C_1}\beta L) \right] \Big\} \\
& + 8\sqrt{1+\xi^2 C_1}\xi^2 \eta \left[4\alpha \sum_{i=2}^{\infty} \frac{i\bar{T}_i}{(i^4\eta^4-16i^2C_1\xi^4\eta^2+64\xi^4)} f_1(i) \right. \\
& \left. + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{j(4-j^2\eta^2)\bar{T}_j}{(j^4\eta^4-16j^2C_1\xi^4\eta^2+64\xi^4)} f_1(j) \right] + g(i, j), \quad (5.23-1)
\end{aligned}$$

ii) the case of $1-\xi^2 C_1=0$,

$$\begin{aligned}
F_2\left(\frac{1}{2}w_x^2\right) = & \frac{R^2}{4} \left[[(A_3^2+A_4^2)+(A_2A_3-A_1A_4)2\sqrt{2}\beta+2(A_1^2+A_2^2)\beta^2] \right. \\
& - [A_3(A_4-A_1\sqrt{2}\beta)-A_4(A_3+A_2\sqrt{2}\beta)]2\sqrt{2}\beta L + \frac{8}{3}(A_3^2+A_4^2)(\beta L)^2 \\
& + \frac{1}{4\sqrt{2}\beta L} \left\{ [A_3A_4-(A_1A_3-A_2A_4)\sqrt{2}\beta-4A_1A_2\beta^2] \right. \\
& + [(A_1A_3-A_2A_4)\sqrt{2}\beta+[(A_3^2-A_4^2)+[A_2A_3+A_1A_4]2\sqrt{2}\beta)2\sqrt{2}\beta L \\
& + 4A_1A_2\beta^2+A_3A_4(16[\beta L]^2-1)] \cos 4\sqrt{2}\beta L \\
& + [(A_2A_3+A_1A_4)\sqrt{2}\beta+(2A_3A_4-[A_1A_3-A_2A_4]2\sqrt{2}\beta)2\sqrt{2}\beta L \\
& + 2(A_2^2-A_1^2)\beta^2-\frac{1}{2}(A_3^2-A_4^2)(16[\beta L]^2-1)] \sin 4\sqrt{2}\beta L \Big\} \Big] \\
& + 8\xi^4\eta R \left[4\alpha \sum_{i=2}^{\infty} \frac{\pi i^2\bar{T}_i}{(i^4\eta^4-16i^2C_1\xi^4\eta^2+64\xi^4)[(\pi i)^2-8(\beta L)^2]} f_2(i) \right. \\
& + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{\pi j^2(4-j^2\eta^2)\bar{T}_j}{(j^4\eta^4-16j^2C_1\xi^4\eta^2+64\xi^4)[(\pi j)^2-8(\beta L)^2]} f_2(j) \Big] \\
& + g(i, j), \quad (5.23-2)
\end{aligned}$$

iii) the case of $1-\xi^2 C_1 < 0$,

$$\begin{aligned}
F_2\left(\frac{1}{2}w_x^2\right) = & \frac{1}{2}\xi^2 \left[[(\xi^2 C_1+\sqrt{\xi^4 C_1^2-1})A_1^2+(\xi^2 C_1-\sqrt{\xi^4 C_1^2-1})A_2^2] \right. \\
& \left. - \frac{A_1A_2}{\beta L} \left[\frac{1}{\sqrt{\xi^2 C_1+1}} \sin (2\sqrt{\xi^2 C_1+1}\beta L) - \frac{1}{\sqrt{\xi^2 C_1-1}} \sin (2\sqrt{\xi^2 C_1-1}\beta L) \right] \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2\beta L} \left[\frac{(\xi^2 C_1 + \sqrt{\xi^4 C_1^2 - 1}) A_1^2}{(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1})} \sin [2(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1})\beta L] \right. \\
& \left. + \frac{(\xi^2 C_1 - \sqrt{\xi^4 C_1^2 - 1}) A_2^2}{(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1})} \sin [2(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1})\beta L] \right] \\
& - 8\xi^5 \eta \left[4\alpha \sum_{i=2}^{\infty} \frac{2\pi i^2 \bar{T}_i}{(i^4 \eta^4 - 16i^2 C_1 \xi^4 \eta^2 + 64\xi^4)} f_3(i) \right. \\
& \left. + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{2\pi j^2 (4-j^2 \eta^2) \tilde{T}_j}{(j^4 \eta^4 - 16j^2 C_1 \xi^4 \eta^2 + 64\xi^4)} f_3(j) \right] + g(i, j), \quad (5.23-3)
\end{aligned}$$

where

$$\left. \begin{aligned} B_1 &= \bar{k} A_1 - A_2, \\ B_2 &= A_1 + \bar{k} A_2, \end{aligned} \right\} \quad (5.24)$$

$$\begin{aligned}
f_1(k) &= (1 - e^{-2\sqrt{1-\xi^2 C_1} \beta L} \cos [2\sqrt{1+\xi^2 C_1} \beta L]) \left[\frac{(\pi k + 2\sqrt{1+\xi^2 C_1} \beta L) B_1 - 2\sqrt{1-\xi^2 C_1} \beta L B_2}{(\pi k)^2 + 8(\beta L)^2 + 4\pi k \sqrt{1+\xi^2 C_1} \beta L} \right. \\
& \quad \left. + \frac{(\pi k - 2\sqrt{1+\xi^2 C_1} \beta L) B_1 + 2\sqrt{1-\xi^2 C_1} \beta L B_2}{(\pi k)^2 + 8(\beta L)^2 - 4\pi k \sqrt{1+\xi^2 C_1} \beta L} \right] \\
& - e^{-2\sqrt{1-\xi^2 C_1} \beta L} \sin [2\sqrt{1+\xi^2 C_1} \beta L] \left[\frac{2\sqrt{1-\xi^2 C_1} \beta L B_1 + (\pi k + 2\sqrt{1+\xi^2 C_1} \beta L) B_2}{(\pi k)^2 + 8(\beta L)^2 + 4\pi k \sqrt{1+\xi^2 C_1} \beta L} \right. \\
& \quad \left. + \frac{-2\sqrt{1-\xi^2 C_1} \beta L B_1 + (\pi k - 2\sqrt{1+\xi^2 C_1} \beta L) B_2}{(\pi k)^2 + 8(\beta L)^2 - 4\pi k \sqrt{1+\xi^2 C_1} \beta L} \right], \quad (5.25-1)
\end{aligned}$$

$$\begin{aligned}
f_2(k) &= - \left[A_2 \sqrt{2} \beta + A_3 \frac{(\pi k)^2 - 24(\beta L)^2}{(\pi k)^2 - 8(\beta L)^2} \right] \\
& + \left[(A_2 + 2A_4 L) \sqrt{2} \beta + A_3 \frac{(\pi k)^2 - 24(\beta L)^2}{(\pi k)^2 - 8(\beta L)^2} \right] \cos 2\sqrt{2} \beta L \\
& + \left[(-A_1 + 2A_3 L) \sqrt{2} \beta + A_4 \frac{(\pi k)^2 + 8(\beta L)^2}{(\pi k)^2 - 8(\beta L)^2} \right] \sin 2\sqrt{2} \beta L, \quad (5.25-2)
\end{aligned}$$

$$\begin{aligned}
f_3(k) &= \frac{(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1}) A_1}{(\pi k)^2 - 8(\xi^2 C_1 + \sqrt{\xi^4 C_1^2 - 1})(\beta L)^2} \sin [(\sqrt{\xi^2 C_1 + 1} + \sqrt{\xi^2 C_1 - 1})\beta L] \\
& + \frac{(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1}) A_2}{(\pi k)^2 - 8(\xi^2 C_1 - \sqrt{\xi^4 C_1^2 - 1})(\beta L)^2} \sin [(\sqrt{\xi^2 C_1 + 1} - \sqrt{\xi^2 C_1 - 1})\beta L], \quad (5.25-3)
\end{aligned}$$

$$\begin{aligned}
g(i, j) &= 16\xi^5 \eta^2 \left[4\alpha \sum_{i=2}^{\infty} \frac{i \bar{T}_i}{(i^4 \eta^4 - 16i^2 C_1 \xi^4 \eta^2 + 64\xi^4)} \right. \\
& \left. + \frac{\alpha}{(1-\nu)} \left(\frac{d}{R} \right) \sum_{j=2}^{\infty} \frac{j(4-j^2 \eta^2) \tilde{T}_j}{(j^4 \eta^4 - 16j^2 C_1 \xi^4 \eta^2 + 64\xi^4)} \right]^2. \quad (5.26)
\end{aligned}$$

Finally, the deformation aspect can be made clear with Eqs. (5.10), (5.12), (5.14), (5.18), and from (5.20) to (5.26) inclusive under the given arbitrary symmetrical temperature rise. From the above equations it can be seen that C_1 is related to \bar{T} and \tilde{T} through the transcendental equations, and the relation cannot be expressed explicitly, so it must be solved numerically in every individual case.

5.2. Numerical example.

The numerical example will be given in order to show concretely the analytical result regarding the deformation aspect of the clamped cylindrical shell subjected to heating in the preceding section.

The temperature rise is assumed to be uniform over the shell neglecting the effect of temperature gradient through the thickness, that is,

$$\bar{T} \equiv \bar{T}_0, \quad \tilde{T} = 0. \quad (5.27)$$

As an example, the heated cylindrical shell, where $R/L=0.3254$ and $d/R=2 \times 10^{-2}$, will be considered. The relations between the temperature rise and the axial stress can be obtained by the process mentioned above and are shown in Fig. 7. That is, the solutions of transcendental equations exist only on the curve

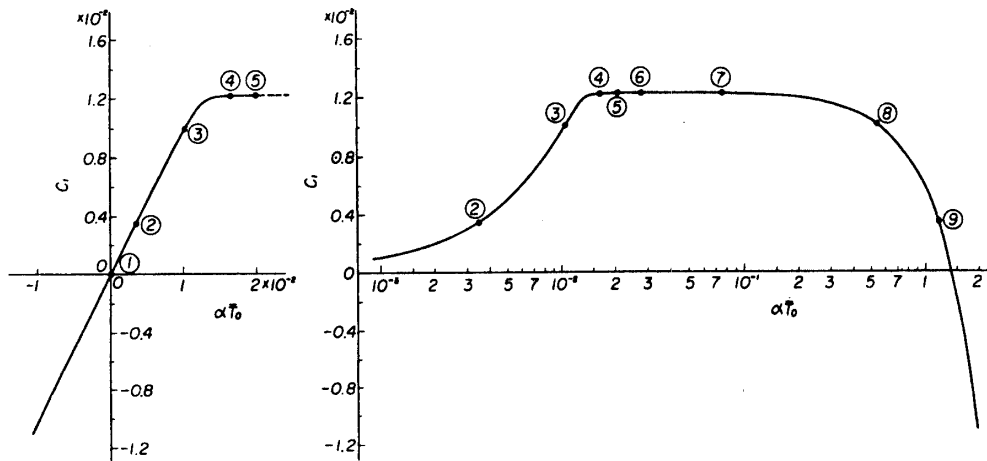


FIGURE 7. Relation between the temperature rise and the axial stress of the cylindrical shell, clamped edges, $R/L=0.3254$, $d/R=2 \times 10^{-2}$.

in this figure. In this numerical example, the higher order of solutions cannot be seen, and the reason for this is considered to be due to the elastic supporting effect by the circumferential stress, which exists in the two-dimensional cylindrical shell and the higher value of d/R assumed here; however, in the cases of the extremely thin shell or the like, the existence of the higher order of solutions, which was shown in the one-dimensional flat strip [3], [10], may be presumed depending on the values of R/L and d/R .

Next, the relations between the temperature rise and the deflection, w can be determined with the axial stress, C_1 thus obtained and are shown in Fig. 8. That is, the cylindrical shell starts to deflect from the beginning of heating which is the initial stress- or strain-free state marked ①, and the deflection under a comparatively small temperature rise is nearly uniform except in the vicinity of the edge. And this deflection wave is extended to the centre as ② → ③ → ④ → ⑤ according as the temperature rises. The deflection further increases with the increase of the temperature, which results in the decrease of the axial stress beyond the point of $\alpha \bar{T}_0 \doteq 2.9 \times 10^{-2}$ for this case as seen in Fig. 8. After that, the

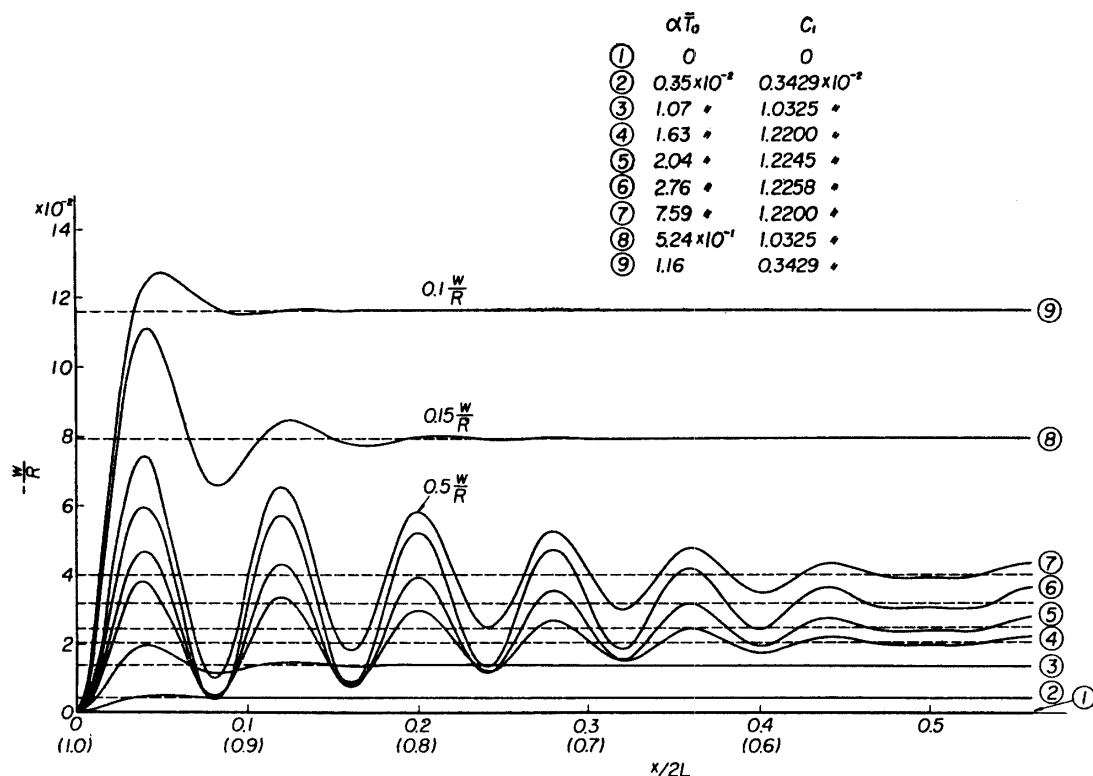


FIGURE 8. Variation of the thermal deformation of the cylindrical shell with the temperature rise, clamped edges, $R/L=0.3254$, $d/R=2 \times 10^{-2}$.

deflection wave decreases with the increase of the temperature approaching the initial state ① as ⑦ → ⑧ → ⑨.

In this analysis, the three types of complementary solution are given as Eqs. (5.12) corresponding to the sign of $(1-\xi^2 C_1)$, but they are smoothly connected through the point of $1-\xi^2 C_1=0$.

The numerical example has been given for the special values of R/L and d/R , but the same process will be used for the other cases.

6. CONCLUSIONS

In this paper, the fundamental equations for the thermoelastic problems of a cylindrical shell were derived taking the finite deformation into account. And, from the discussions on the fundamental equations, it was shown that the cylindrical shell started to deflect from the beginning of heating and that there existed a possibility of thermal buckling due to the effect of temperature gradient through the thickness, which acted equivalent to the external pressure for the case of the internal heating, even if the thermal stress caused by the external constraint was absent. And, the critical buckling value for this case was obtained and shown graphically. It was also shown that, for the case where the external constraint upon the thermal expansion existed, the effect of induced thermal stress was greatly remarkable and the effect of temperature gradient through the thickness could be neglected.

Then, two examples of the analyses for the cases where the thermal expansion was restrained were shown. The one was the case where the cylindrical shell was longitudinally unrestrained. After showing that the deformation behaviour for this case was characterized by the local buckling, the critical temperature was determined, and it was concluded that this phenomenon was of great importance only for the very thin cylindrical shell.

Next, the thermal deformation of the cylindrical shell clamped at the edges was analyzed and a numerical example was given. For this case, the relation between the temperature rise and the axial stress was given through the transcendental equations and so the relation had to be solved numerically in every individual case. Then, the relations between the temperature rise and the deflection were calculated for the special combination of values of R/L and d/R , and shown graphically. The relations between the temperature rise and the axial stress were given by a set of curves, each of which corresponded respectively to different deformation mode for the one-dimensional case as a flat strip, while it was shown, in the case of two-dimensional cylindrical shell calculated in this paper where the values of R/L and d/R were specified, that the axial stress corresponded to the temperature rise one to one and consequently the deflection mode could be determined immediately for the temperature rise when the geometrical dimensions of the shell were given. Furthermore, the possibilities of the buckling with the Diamond-type waves and of the Euler type buckling resembling to the case of a long column may be reserved depending on the geometrical dimensions of the shell.

Moreover, it was assumed in the above analyses that the cylindrical shells were clamped at the edges; however, for the case of simple support, the same processes can be used except that the boundary condition, $D[w_{xx} + \nu(w_{yy} + w/R^2)] + Ed^2\alpha\tilde{T}/(1-\nu) = 0$ must be replaced by $w_x = 0$ in Eqs. (3.15), (4.6) and (5.13).

The author would like to express his sincere thanks to Professors K. Ikeda and M. Uemura of the Aeronautical Research Institute, University of Tokyo, for their instructive suggestions and discussions on the present study.

*Department of Structures,
Aeronautical Research Institute,
University of Tokyo, Tokyo.
July 4, 1962.*

REFERENCES

- [1] Zuk, W.: Thermal Buckling of Clamped Cylindrical Shells, *Jour. Aero. Sci.*, **24** (1957), 389.
- [2] Hoff, N.J.: Buckling of Thin Cylindrical Shell Under Hoop Stresses Varying in Axial Direction, *Jour. Appl. Mech.*, **24** (1957), 405.
- [3] Uemura, M.: Deformation and Thermal Stress of Rectangular Beam or Flat Strips Heated

- at one Surface, Aero. Res. Inst., Univ. of Tokyo, Report **26** (1960), 33.
- [4] Sunakawa, M. and Uemura, M.: Deformation and Thermal Stress in a Rectangular Plate Subjected to Aerodynamic Heating (For the Case of Simply Supported Edges), Aero. Res. Inst., Univ. of Tokyo, Report **26** (1960), 195.
 - [5] Sunakawa, M.: Thermal Deformation of a Clamped Rectangular Plate Subjected to Kinetic Heating, Jour. Japan Soc. Aero. Space Sci., **9** (1961), 37.
 - [6] Sunakawa, M.: Thermal Deformation of a Circular Plate, Aero. Res. Inst., Univ. of Tokyo, Bulletin **2**, 5 (1961/3), 269.
 - [7] Sunakawa, M.: Large-Deflection Analysis for a Circular Flat Plate Subjected to Heating and Normal Pressure, Aero. Res. Inst., Univ. of Tokyo, Bulletin **2**, 7 (1961/9), 415.
 - [8] For example,
Timoshenko, S.: "Theory of Elastic Stability", McGraw-Hill, 1936.
Bijlaard, P.P.: Buckling Stress of Thin Cylindrical Clamped Shells Subjected to Hydrostatic Pressure, Jour. Aero. Sci., **21** (1954), 853.
 - [9] For example,
Timoshenko, S.: "Theory of Plates and Shells", McGraw-Hill, 1940.
 - [10] Williams, M.L.: Further Large-Deflection Analysis for a Plate Strip Subjected to Normal Pressure and Heating, Jour. Appl. Mech., **25** (1958), 251.