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On the Finite Deformation Mode in the Nonlinear Snap Buckling

(Snap Buckling of Circular Beams Having Small Curvature under External Pressure)

By

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Summary. The nonlinear snap buckling problem of curved beams having small curvature due to external pressure, in which it is possible to obtain the exact solution, was taken up as an example to discuss the questions (1) whether or not the approximate methods usually used in the analysis of nonlinear elastic problems are proper to define the buckling load, and, (2) whether or not an unsymmetrical deflection might give a lower buckling value, as pointed out previously.

The comparison of the exact solution with approximate solutions showed that the assumed deflection mode has a great influence on the buckling load. Further, the analytical result that the buckling loads obtained in the unsymmetrical mode are lower than those in symmetrical modes is found to be true only within the degree of precision of the approximate solutions themselves. The reason is that,the symmetrical deflection mode derived from the exact solution of the nonlinear equation gives the minimum buckling load.

These analytical results seem to suggest the need for scrupulous attention to the assumed deflection mode in the approximation method necessary for two-dimensional problems, such as problems of plates and shells.

1. INTRODUCTION

The critical buckling values of shells obtained on the basis of the linear small deflection theory are much higher than the experimental values. In view of these situations, many research works based on the finite deformation theory have been explored since Kärmän and Tsien [*1*, *2*] had published their papers more than two decades ago in a most spectaculor manner to explain this discrepancy between them. However, we have much mathematical difficulty in solving nonlinear partial simultanous differential equations in these problems and accordingly, some deflection mode which satisfies the boundary conditions are usually assumed in the approximate solution. Also in the previous papers by the present author and others on the snap buckling of spherical shell [*3*] and cylindrical shell [*4*] under external pressure, we limited ourselves to the simplest case under an assumption of symmetrical deflection mode. In the cases of structural members having large curvature, however, the inextensional deformation is expected to occur actually, presenting unsymmetrical deflection mode and giving the lower value of buckling load [*5*,*6*,*7*]. But these views were derived on the basis of the analyses where the deflection modes were assumed.

In view of the above-mentioned considerations, this paper is concerned with a shallow circular beam having small curvature so as to be able to obtain the exact solution, avoiding the mathematical difficulty encountered in the two-dimensional problems and to discuss the following questions; that is, taking up the snap buckling problem under external pressure, as an example, the following questions will be discussed comparing with the approximate solutions: (1) Whether or not the simple symmetrical deformation usually assumed is reasonable to define the upper buckling value in comparison with the unsymmetrical deflection mode. (2) In the case where we cannot help relying on the approximate method in two-dimensional plates or shells, how much will the buckling value be influenced with the assumption of deflection mode. (3) Whether the approximate solutions are proper or not compared with the exact solution. (4) Is it possible or not to exhibit an unsymmetrical deflection mode.

Through these discussions, the importance of the finite deflection mode in the solution of nonlinear large deformation problems and the matters that demand special attention in the analysis will be pointed out.

2. Equilibrium Equations of Circular Beams Having Small Curvature under Uniformly Distributed Pressure

For the simplicity's sake, a circular arch beam which has uniform cross-section (R; radius of curvature) is considered here as shown in Fig. 1. The uniformly distributed pressure per unit length q acts over the arch beam. The opening angle of arch beam θ is assumed to be not so large that it can be seen as a small quantity of the first order. By estimating the orders of magnitudes of the deformation components and their derivatives and by retaining only terms of the first order of magnitude, the extensional strain of the neutral axis $\bar{\varepsilon}$ and the change of curvature κ can be expressed by

$$\left.\begin{aligned} \bar{\varepsilon} &= \dot{u} + (w/R) + \dot{w}^2/2 \\ \kappa &= \ddot{w} \end{aligned}\right\} \qquad (1)$$

where "·" indicates the differentiation with respect to the coordinate s along the circular arch, and

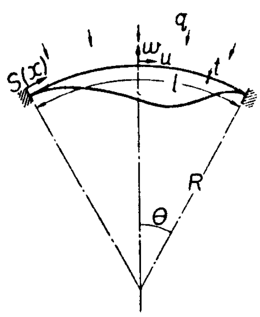

FIGURE 1. Circular beam having small curvature under external pressure and its notations.

u ; deformation component in the axial (s-) direction
w ; deformation component in the outward direction normal to the beam.

Basing on the stationary theorem, the equilibrium equations can be derived by use of the minimum condition of the total potential energy Π. Π can be given by

$$\Pi=\frac{EA}{2}\int_0^l \bar{\varepsilon}^2 ds+\frac{EI}{2}\int_0^l \kappa^2\, ds+\int_0^l qwds$$

$$=\frac{EA}{2}\int_0^l\left(\dot{u}+\frac{w}{R}+\frac{\dot{w}^2}{2}\right)^2 ds+\frac{EI}{2}\int_0^l \ddot{w}^2 ds+\int_0^l qwds \quad . \qquad (2)$$

After the variation with u and w, the following two equilibrium equations can be obtained from the condition of $\delta\Pi=0$

$$s\text{-direction}\,;\quad \frac{d}{ds}\left(\dot{u}+\frac{w}{R}+\frac{\dot{w}^2}{2}\right)=0 \qquad (3.1)$$

$$n\text{-direction}\,;\quad EI\,\ddot{\ddot{w}}-EA\frac{d}{ds}\left\{\dot{w}\left(\dot{u}+\frac{w}{R}+\frac{\dot{w}^2}{2}\right)\right\}+\frac{EA}{R}\left(\dot{u}+\frac{w}{R}+\frac{\dot{w}^2}{2}\right)+q=0 \qquad (3.2)$$

and the so-called natural boundary conditions are given as the by-product of the variational procedure as follows.

$$\left.\begin{aligned} &s=0,\, l:\\ &\delta u=0 \quad \text{or} \quad \dot{u}+\frac{w}{R}+\frac{\dot{w}^2}{2}=0\\ &\delta w=0 \quad \text{or} \quad A\dot{w}\left(\dot{u}+\frac{w}{R}+\frac{\dot{w}^2}{2}\right)-I\dddot{w}=0\\ &\delta\dot{w}=0 \quad \text{or} \quad \ddot{w}=0 \end{aligned}\right\} \qquad (4)$$

The simple integration of Eq. (3.1) in the axial direction gives the following equation.

$$\dot{u}+\frac{w}{R}+\frac{\dot{w}^2}{2}=\text{const}=-e(e \text{ is positive when compressive}) \qquad (5)$$

This means that the axial strain on the neutral axis is constant along the beam in the deformation state. By use of Eq. (5), the nonlinear differential equation in the normal direction can be linearized as follows.

$$\ddot{\ddot{w}}+\frac{e}{i^2}\ddot{w}=\frac{1}{Ri^2}e-\frac{q}{EAi^2} \qquad (6)$$

where $i^2=I/A$, i ; radius of gyration
Thus the problem is attributed to the solution of Eq. (6) subject to the boundary conditions.

3. Approximate Solutions

The exact solution of Eq. (6) can be obtained as shown later in the following chapter. However, for the reason that we cannot help relying on the approximate solution in two-dimensional cases such as cases of plates or shells, some approximate methods of solution will be shown, first of all, in order to check the propriety of the assumed deflection modes and matters that demand special attention in it.

The following boundary conditions of the clamped edges at bothends are considered here.

$$x=s/l=0,\ 1 \quad ; \quad w=w_x=0 \tag{7}$$

The deflection mode w which satisfies the above conditions is now expressed by the following polynominal series.

$$w=\sum_{i=0}^{\infty} a_i x_1^i \tag{8}$$

Besides the boundary conditions, we put the deflection at the mid-point of the beam ($x=1/2$) as $-\delta$.
(i) The case of $i=4$.

In this case, the deflection w which satisfies Eq. (7) is expressed by

$$w_s=-16\delta x^2(1-x)^2 \quad . \tag{9}$$

w_s denotes the simplest deflection mode which is symmetrical about the mid-point.

Now, we make use of the Galerkin method which corresponds to the stationary theorem. That is,

$$\int_0^1\left(w_{xxxx}+\frac{el^2}{i^2}w_{xx}-\frac{l^4}{Ri^2}e+\frac{ql^4}{EAi^2}\right)wdx=0\,, \tag{10}$$

from which the following equation can be obtained

$$\frac{ql}{EA}=e\left(\frac{l}{R}-\frac{64}{7}\frac{\delta}{l}\right)+384\frac{\delta}{l}\frac{i^2}{l^2} \quad . \tag{11}$$

The unknown variable of e can be determined by integrating $\dot{u}$ in Eq. (5) and using the following boundary condition for u ; $x=0,\ 1$; $u=0$ that is,

$$u(1)/l=-e-\frac{1}{R}\int_0^1 wdx-\frac{1}{2l^2}\int_0^1 w_x^2dx=0 \quad . \tag{12}$$

e is determined as

$$e=\frac{16}{15}\theta\left(\frac{\delta}{l}\right)-\frac{256}{105}\left(\frac{\delta}{l}\right)^2 \quad , \tag{13}$$

where $\theta=l/2R$

Substitution of Eq. (13) into Eq. (11) leads to the equilibrium relation between the external pressure (q) and the maximum deflection (δ) as follows

$$\frac{ql}{EA}=\frac{32}{15}\theta^2\left(\frac{\delta}{l}\right)+384\frac{i^2}{l^2}\left(\frac{\delta}{l}\right)-\frac{512}{35}\theta\left(\frac{\delta}{l}\right)^2+\frac{16{,}384}{735}\left(\frac{\delta}{l}\right)^3 \ . \tag{14}$$

One group of the equilibrium curves (ql/EA versus δ/l) are shown in Fig. 2 with the parameter of θ in the case of $i^2/l^2=10^{-5}/4$ (which is the same in the following numerical examples). The maximum and the minimum values in a curve given by the algebraic equation of third order are given by the process of differentiation as follows.

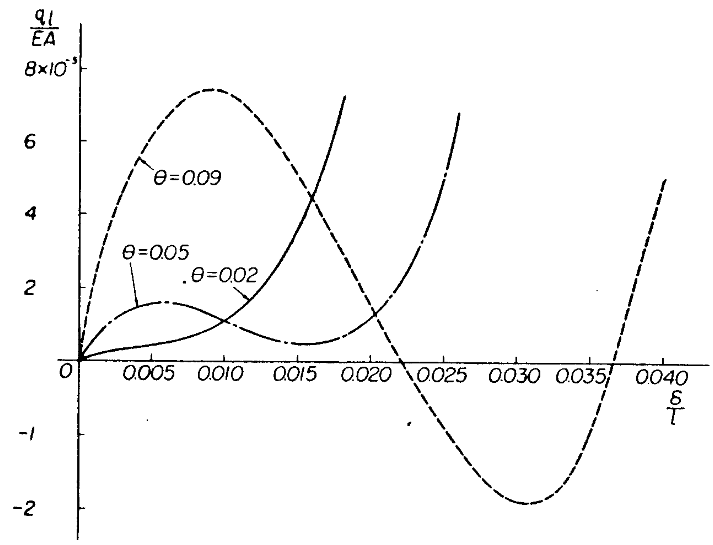

FIGURE 2. Relationship between the pressure and the central deflection with the parameter of θ.

$$\left(\frac{\delta}{l}\right)_{\substack{\min\\\max}}=\frac{7}{32}\theta\mp\frac{7}{32\sqrt{3}}\sqrt{\theta^2-360(i^2/l^2)} \tag{15}$$

If $\theta^2<360i^2/l^2$ ($\theta<0.03$ in the case of $i^2/l^2=10^{-5}/4$), (16)

there are no maximum and minimum extremes in the curve. This means the nonexistence of the snap-through buckling phenomenon. On the contrary, when θ^2 is greater than 360 (i^2/l^2), the beam will snap through at the maximum extreme in the unloading process. The maximum extreme value is here called as " upper buckling value " which is given by

$$\left(\frac{ql}{EA}\right)_u=840\theta\frac{i^2}{l^2}+\frac{7}{45\sqrt{3}}\left(\theta^2-360\frac{i^2}{l^2}\right)^{\frac{3}{2}} \tag{17}$$

Besides such an upper buckling value, we can define the "lower buckling value" where the values of Π before and after buckling are equal, as presented by Tsien [2], bearing in mind the inevitable external disturbances and the imperfections in the experiments. This value is given by

$$\left(\frac{ql}{EA}\right)_e = 840\theta\frac{i^2}{l^2} \tag{18}$$

The variations of the upper and the lower buckling values against θ are shown in Fig. 3. It should be noted that these values increase with an increase of θ without presenting a minimum even though there are minimum extremes in the

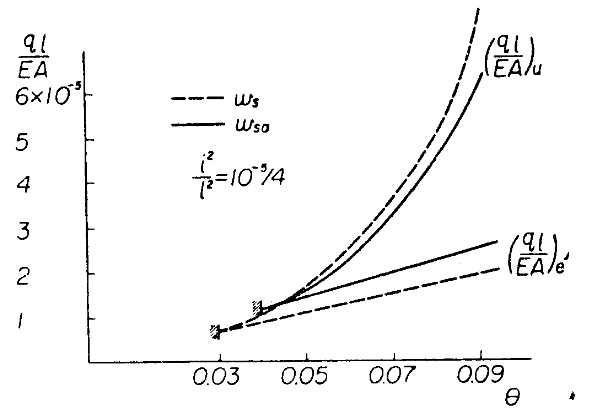

FIGURE 3. Variations of the upper and lower buckling pressures against θ.

curves of buckling values against θ in the case of spherical shells.

In this paper, only the upper buckling value will be discussed hereafter.

(ii) The case of $i=5$.

In this case, one constant of a_i remains unknown and the deflection mode can be devided in the two terms as follows.

$$\left.\begin{aligned} & w=w_1+w_2 \\ \text{where}\quad & w_1=w_s \\ & w_2=aw_s(1-2x)=a'(x^2-4x^3+5x^4-2x^5) \end{aligned}\right\} \tag{19}$$

w_1 denotes the same symmetrical deflection mode as Eq. (9), while w_2 denotes an anti-symmetrical deflection mode about the mid-point and so w means an unsymmetrical deflection mode. The general procedure of the Galerkin method is given by the following equation when the function is expressed by the sum of some terms containing arbitrary constants.

$$\int_0^1\left(w_{xxxx}+\frac{el^2}{i^2}w_{xx}-\frac{l^4}{Ri^2}e+\frac{ql^4}{EAi^2}\right)w_j dx=0 \quad (j=1,\ 2) \tag{20}$$

The final equilibrium relation can be obtained by eliminating the arbitrary constants from these equations of Eqs. (20). In this case of $i=5$, however, unfortunately we can not determine an arbitrary constant because of the orthogonality condition of both functions of w_1 and w_2.

We can anticipate the possibility that the arch beam takes an unsymmetrical deflection mode in the initial stage of deformation under the inextensional state, as has in fact been shown in the case for larger values of θ [*6*, *7*]. In order to check this expectation, we now specity a' as $a'=16\delta'$ and consider the following unsymmetrical deflection mode w_{sa}.

$$w_{sa}=-16\ \delta[(x^2-2x^3+x^4)-(x^2-4x^3+5x^4-2x^5)]$$
$$=-32\ \delta(x^3-2x^4+x^5) \tag{21}$$

Through the same calculation process as in (i), the relation between q and δ, and the axial strain e can be obtained as

$$\frac{ql}{EA}=\left(\frac{32}{15}\theta^2+\frac{288\times16}{7}\frac{i^2}{l^2}\right)\left(\frac{\delta}{l}\right)-\frac{256\times8}{105}\theta\left(\frac{\delta}{l}\right)^2+\frac{64\times256\times16}{21\times315}\left(\frac{\delta}{l}\right)^3 \tag{22}$$

$$e=\frac{16}{15}\theta\left(\frac{\delta}{l}\right)-\frac{1,024}{315}\left(\frac{\delta}{l}\right)^2\ , \tag{23}$$

respectively.

The upper and lower buckling values are given as follows.

$$\left(\frac{ql}{EA}\right)_u=108\theta\left(\frac{i^2}{l^2}\right)+\frac{7}{60\sqrt{3}}\left(\theta^2-\frac{4,320}{7}\frac{i^2}{l^2}\right)^{\frac{3}{2}} \tag{24}$$

$$\left(\frac{ql}{EA}\right)_e=108\theta\left(\frac{i^2}{l^2}\right) \tag{25}$$

The relation between (ql/EA) and (δ/l) in this case is shown by a solid line in Fig. 4 where $i^2/l^2=10^{-5}/4$ and $\theta=0.05$, and is compared with that in the case (i)

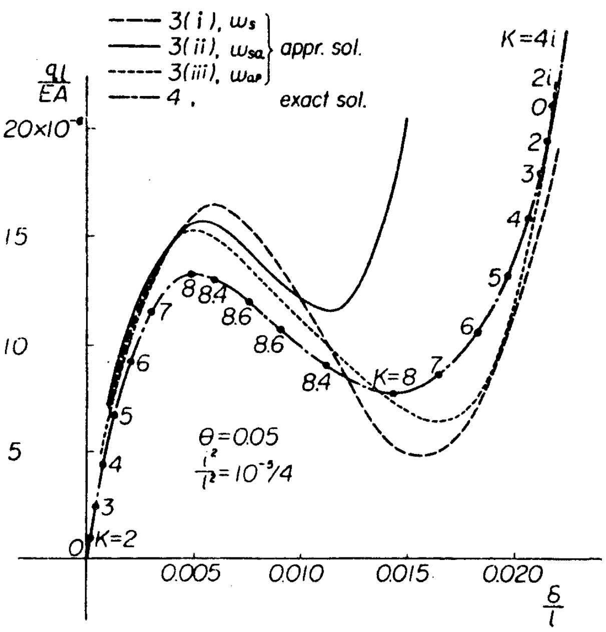

FIGURE 4. Comparison of the equilibrium curves obtained by various kinds of solutions.

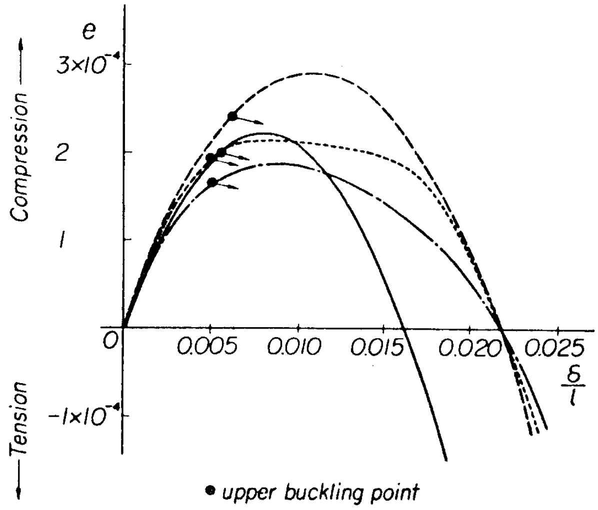

FIGURE 5. Comparison of the relationships between the axial strain and the central deflection.

shown by a dashed line. And the upper and lower buckling values in both cases are shown in Fig. 3. The upper buckling value with an unsymmetrical deflection mode is lower than that with a symmetrical deflection mode.

The variation of the axial strain (e) with an increase of deflection (δ/l) is shown in Fig. 5 from which we can find that the axial strain e is much lower than in the case (i). The black round dots marked on the curve indicate the onset of the upper buckling. But the external pressure in the postbuckling state is much higher, and this shows that it is impractical to keep such an unsymmetrical deflection mode after buckling. As far as the analytical results on the upper buckling values by the approximate methods are concerned, the assumption of unsymmetrical deflection seems to be more reasonable than the simple symmetrical mode. In the case (ii), the anti-symmetrical deflection component was specified, but it should be reasonably determined so as to minimize the total potential energy as lower as possible as will be shown later in the following case (iii).

(iii) The case of $i=6$.

In the case of $i=6$, the deflection mode w includes two arbitrary constants in addition to δ. And so, by applying the Galerkin method such as shown in Eq. (20), three algebraic equations of third order are obtained after eliminating e, which make it much difficult to solve them. Accordingly, in order to make the analysis easier, we express w by deleting the fifth term ($i=5$) as follows.

$$w=\sum_{i=1}^{4}a_i x^i + a_6 x^6 \tag{26}$$

By determining the four constants of them so as to satisfy the boundary conditions of Eq. (7), leaving one constant indifinite, w can be given by

$$\left.\begin{aligned} w_{ap}&=w_1+w_2\\ w_1&=-\frac{64}{5}\delta x^3(1-x)^2(2+x)\\ w_2&=\frac{a_2}{5}x^2(1-x)^2(5-8x-4x^2)\end{aligned}\right\} \tag{27}$$

Both w_1 and w_2 denote the unsymmetrical deflection modes, but $w_1=-\delta$ and $w_2=0$ at the midpoint $x=1/2$, and the former one is near to a symmetical deflection mode, while the latter is near to an unsymmetrical mode. Both w_1 and w_2 can be decomposed into the two components, one is symmetrical and the other is antisymmetrical, but w_1 and w_2 do not satisfy the orthogonality condition. Hence, the application of the Galerkin method by use of w_1 and w_2 leads to the following equations.

$$\left.\begin{aligned} \frac{ql}{EA}&=2e\theta+\frac{24}{5}\left(\frac{i^2}{l^2}\right)\left(3{,}008\frac{\delta}{l}+183\frac{a_2}{l}\right)-\frac{532}{55}\,e\left(16\frac{\delta}{l}+\frac{a_2}{l}\right) \\ \frac{ql}{EA}&=2e\theta+\frac{8}{15}\left(\frac{i^2}{l^2}\right)\left(1{,}472\frac{\delta}{l}+47\frac{a_2}{l}\right)-\frac{1}{495}\,e\left(6{,}528\frac{\delta}{e}+133\frac{a_2}{l}\right) \end{aligned}\right\} \tag{28}$$

e can be obtained from the condition of Eq. (12) as follows.

$$e=\frac{\theta}{525}\left(576\frac{\delta}{l}+\frac{a_2}{l}\right)-\frac{1}{1{,}375}\left[\frac{34{,}816}{7}\left(\frac{\delta}{l}\right)^2+\frac{19}{3}\left(\frac{a_2}{l}\right)^2+\frac{608}{3}\left(\frac{\delta}{l}\right)\left(\frac{a_2}{l}\right)\right] \tag{29}$$

Substitution of Eq. (29) into both of Eqs. (28) and elimination of (a_2/l) from these two equations lead to the relation between the external pressure (ql/EA) and the deflection (δ/l). The numerical results obtained by use of the same specified values of δ and i^2/l^2 are shown in Figs. 4 and 5 by the dotted lines.

In this case, the deflection mode was chosen so as to satisfy the minimum con-

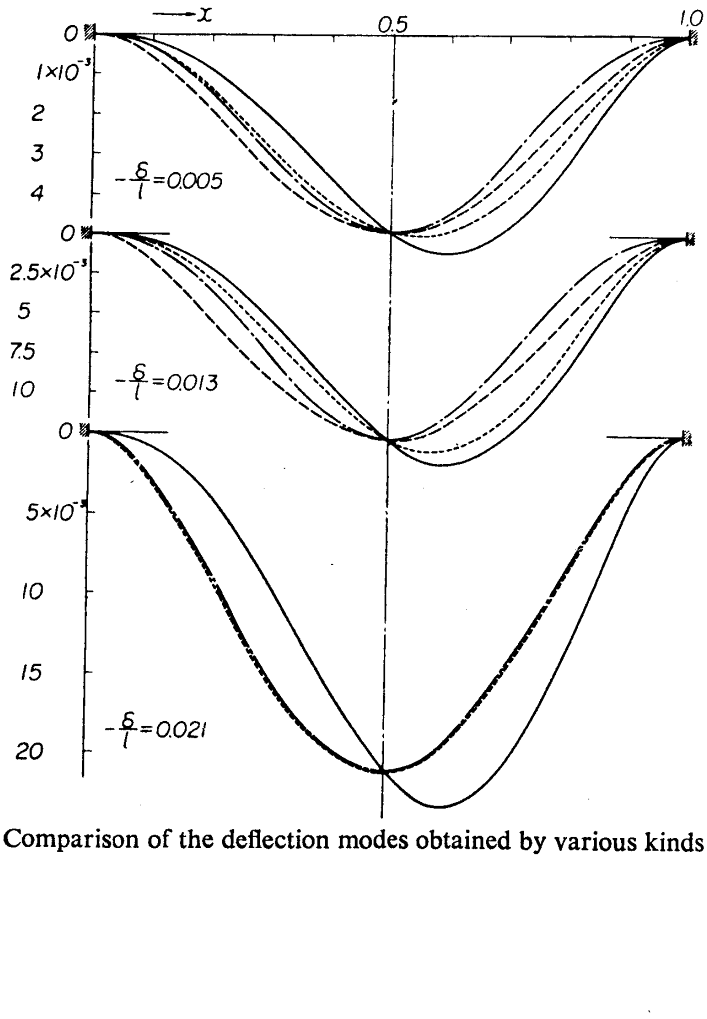

FIGURE 6. Comparison of the deflection modes obtained by various kinds of solutions.

dition of Π as possible by use of the Galerkin method. Accordingly, the upper buckling value of (ql/EA) and the axial strain at upper buckling shown by a black dot are much decreased compared with the previous two cases. These three kinds of deflection modes obtained by the approximate solutions of (i), (ii) and (iii) are all shown with the three kinds of values of (δ/l) in Fig. 6. The deflection mode obtained in the case (iii) is intermediate between the two specified symmetrical and unsymmetrical modes corresponding to the cases (i) and (ii) respectively, and varies with an increase of the deflection. That is, even though it presents an unsymmetrical mode near by the upper buckling value, while it presents an almost symmetrical mode after buckling which is consistent with our intuition.

According to the above approximate solution, the upper buckling value obtained in this case (iii) is lower than those obtained with the specified deflection mode including the constraint effect and it seemed to be reasonable to include some unsymmetrical deflection component near by the upper buckling point. Now, whether such conclusions are correct or not will be checked by the exact solution in the following chapter.

4. Exact Solution

The nonlinear equilibrium equation of Eq. (3.2) in the normal direction could be linearized by an introduction of e in it and so it will be able to solve it exactly as an Eigenvalue problem.

Taking account of the fact that the compressive axial strain occurs in the initial stage of deformation under external pressure, Eq. (6) is now rewritten as

$$w_{xxxx}+K^2 w_{xx}=\frac{K^2 l^2}{R}-\frac{ql^4}{EAi^2} \tag{30}$$

where $K^2=e(i^2/l^2)$

and the general solution of w can be given by

$$w=a\sin Kx+b\cos Kx+d_1 x+d_2+\frac{l^2}{2}\left(\frac{1}{R}-\frac{ql^2}{EAK^2 i^2}\right)x^2 \tag{31}$$

The determination of the integral constants so as to satisfy the boundary conditions Eq. (7) of the clamped edges leads to the following equation, leaving K unknown.

$$\frac{w}{l}=\frac{\theta\left(1-\dfrac{X}{K^2}\right)}{K(2-2\cos K-K\sin K)}\times\begin{bmatrix}(2-2\cos K-K\sin K)(\sin Kx-Kx+Kx^2)\\+(2\sin K-K-K\cos K)(\cos Kx-1)\end{bmatrix} \tag{32}$$

where $X=\dfrac{ql}{2EA}\left(\dfrac{l^2}{i^2}\right)\dfrac{1}{\theta}$

It can be easily found from the examination of this expression that w is symmetrical about the mid-point $x=1/2$ independent of the value of K. The value of K can be determined from the boundary condition on u of Eq. (12) and is given by the following equation :

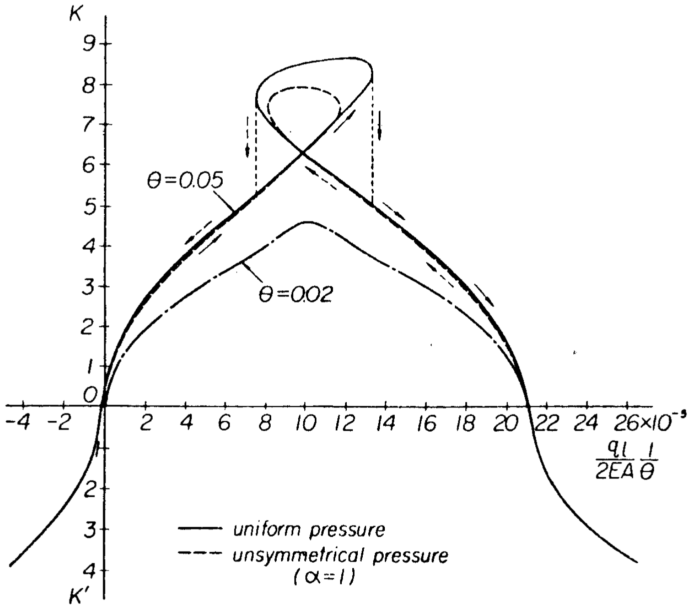

FIGURE 7. Relationship between K and the pressure in the case of larger value of $(i^2/l^2)/\theta^2$.

$$ⓐX^2-2ⓑX+ⓒ=0 \tag{33}$$

$$ⓐ=24(1-\cos K)-9K\cdot\sin K-(4-\cos K)K^2$$

$$ⓑ=3K^2\{4(1-\cos K)-K\sin K-K^2\}$$

$$ⓒ=K^4\{3K\sin K-(2+\cos K)K^2\}-6\frac{K}{\theta^2}\left(\frac{i^2}{l^2}\right)(1-\cos K)$$

The relation between K and $(ql/EA\cdot 1/\theta)$ is shown in Fig. 7 with the same value of $\theta=0.05$, $i^2/l^2=10^{-5}/4$ as previously. The loop as seen in the curve means the existence of snap-through buckling phenomenon, and the deformation proceed as indicated by the arrow-sign along the curve. In the case of small θ, for example, $\theta=0.02$, there exists no unstable equilibrium state.

The deflection amplitude at the mid-point in this symmetrical mode is given by

$$-\left(\frac{\delta}{l}\right)=\frac{\theta\left(1-\frac{X}{K^2}\right)}{K(2-2\cos K-K\sin K)}\left[\begin{array}{l}(2-2\cos K-K\sin K)\left(\sin\frac{K}{2}-\frac{K}{4}\right)\\+(2\sin K-K-K\cos K)\left(\cos\frac{K}{2}-1\right)\end{array}\right]. \tag{34}$$

Introduction of the relation between K and $X(q)$ of Eq. (33) into Eq. (34) leads to the relation between (ql/EA) and (δ/l) which is shown in Fig. 4 by the chained line and the corresponding values of K are referred along this curve. The relation between the axial strain $e=K^2i^2/l^2$ and the deflection (δ/l) is shown also in Fig. 5 by the chained line, Before the snap bucklidg happens, the axial compoessive strain (K) and the deflection increase with an increase of external pressure, but after the snap buckling at the upper buckling value, the axial strain decreases suddenly with the increase of deflection. Even though the deflection mode obtained by this exact solution is symmetrical, the upper buckling value and the axial complressive value at that point (shown by the black dot in Fig. 5) are lower than those obtained by all the approximate solutions. It should be noted that this is beyond our expectation described in the §3 (iii).

The deflection increases with an increase of the external pressure after buckling, but the value of K decreases, finally approaching the imaginary value which corresponds to the axial tensile strain. Beyond the point $K=0$, that is, for $e<0$, the analytical formulae can be rewritten by replacing K with iK'. The lower part of the ordinate in Fig. 7 denotes the value of K'.

The symmetrical deflection mode obtained by this exact solution is shown in Fig. 6 by the chained line. It can be seen from this figure that the exact deflection mode is much different from those obtained by approximate solutions in the range of small deflection around the upper buckling value, but is much similar with each other after buckling.

The exact solution just above mentioned satisfies the condition of minimum potential energy, giving the lowest upper buckling value and the lowest axial compressive strain at that point with the symmetrical deflection mode. Even though we usually try to make the value of Π as low as possible in the approximate solution, the solutions thus obtained give the larger buckling loads and the deflction modes that are quite different from the exact solution. In addition, if we assume a mode carelessly, even if it is symmetrical, the solution will give the large value of upper buckling. Therefore, we must be very careful in assuming the mode in the two-dimensional members such as plates or shells, where we can not help relying on the approximate solutions.

In the case of $\theta=0.05$ as shown in Fig. 7, there is no equilibrium state in the range $K>9$. It comes from that the discriminant of the algebraic equation (33) becomes negative under larger value of θ. However, if $\left(\frac{i^2}{l^2}\right)/\theta^2$ is sufficiently small, we can obtain another kind of curve in the range of larger K. As an example,

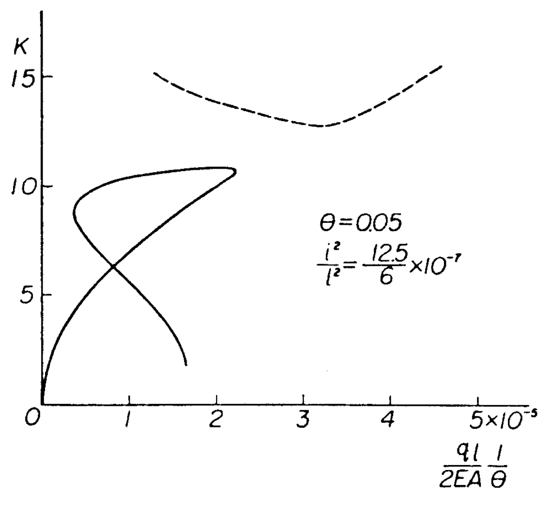

FIGURE 8. Relationship between K and the pressure in the case of smaller value of $(i^2/l^2)/\theta^2$.

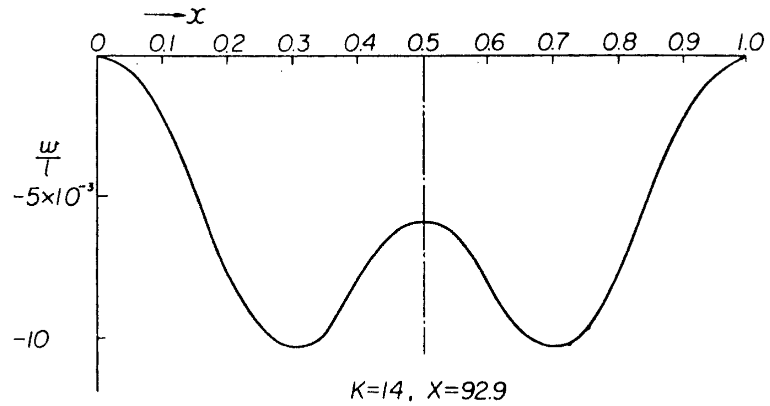

FIGURE 9. Deflection mode in the case of smaller value of $(i^2/l^2)/\theta^2$.

the relationship curves in the case of $i^2/l^2=(12.5\times10^{-7})/6$ with the same value of $\theta(\theta=0.05)$ is shown in Fig. 8. The upper curve given by dashed line under the larger value of K denotes the higher order of symmetrical deflection mode and the deflection mode of $K=14$ is shown in Fig. 9 as an example. It will be necessary to examine further whether such a higher order of deflection mode is actually realized or not.

5. BUCKLING UNDER UNSYMMETRICAL EXTERNAL PRESSURE

It is concluded from the analytical exact solution that the symmetrical deflection mode gives the lowest buckling value. The unsymmetrical deflection mode obtained in the actual experiments giving a lower buckling load seems to be due to the imperfection in the loading system. It is naturally expected that the unsymmetrical mode will occur under unsymmetrical external pressure. In order to examine how much the non-uniformity of external pressure will contribute to the lowering of the buckling value, an example for the case under linearly distributed external pressure will be analysed here, taking into account that the deformation mode has much effect on the buckling value.

The external pressure is assumed to be expressed by

$$q=q_0+q_1(2x-1) \tag{35}$$

Substituting Eq. (35) into Eq. (30), w can be obtained by use of the clamped boundary condition as

$$\frac{w}{l}=\frac{1}{K(2-2\cos K-K\sin K)}\left[\left\{\theta-\frac{q_0 l(1-\alpha)}{2EAK^2}\frac{l^2}{i^2}\right\}\left\{\begin{matrix}(2-2\cos K-K\sin K)(\sin Kx-Kx+Kx^2)\\+(2\sin K-K-K\cos K)(\cos Kx-1)\end{matrix}\right\}\right.$$
$$\left.+\frac{q_0\alpha l^3}{3\,EAK^2 i^2}\left\{\begin{matrix}(K\sin K+3\cos K-3)(\sin Kx-Kx)\\+(K\cos K+2K-3\sin K)(\cos Kx-1)-K(2-2\cos K-K\sin K)x^3\end{matrix}\right\}\right] \tag{36}$$

where $\alpha = q_1/q_0$.
Using Eq. (36), the value of K can be determined from the condition of Eq. (12) and given by the following equation.

$$\left\{ⓐ+\frac{\alpha^2}{15}ⓐ'\right\}X^2-2ⓑX+ⓒ=0 \tag{37}$$

were

$$ⓐ'=(1-\cos K)\left\{\frac{480(1-\cos K)-480K\sin K+48K^2(2+3\cos K)+19K^3\sin K-K^4(6+\cos K)}{4(1-\cos K)-4K\sin K+(1+\cos K)K^2}\right\}$$

In comparison with Eq. (33), the only difference is the addition of the term containing ⓐ′ as a correction term. The relationship between K and $ql/2EA\cdot 1/\theta$ is shown in Fig. 7 by the dashed line for the case of $\alpha=1$. It is natural that the buckling value decreases compared with under uniform external pressure.

6. Conclusions

In this paper, the buckling problem for the circular arch beam having small curvature was analysed as an example in order to examine the effect of finite deflection modes on the nonlinear snap-through buckling. The main results can be summarized as follows.

(1) The nonlinear equilibrium equations for the circular beam having small curvature under external pressure, were derived, taking account of finite deformations and their exact solutions were obtained.

(2) With a view towards the two-dimensional cases of plate and shells, where we cannot obtain the exact solutions analytically, the present problem was also treated approximately by assuming some deflection modes. According to the approximate solutions obtained, the unsymmetrical deflection mode gives the lower value of the upper buckling load and hence this mode appears to be the actual one.

(3) The exact solution, however, predicts the lowest values of the upper buckling pressure and of the axial compressive strain at upper buckling notwithstanding the symmetry of the deflection mode. Consequently, one must be particularly careful when assuming the deflection mode in the approximate solution of two-dimensional problems.

(4) It is questionable from the analytical standpoint to assume an unsymmetrical mode and inextensional deformation under uniform pressure. Unsymmetrical deflection modes in buckling experiments should be attributed to imperfections in the specimen and to the prevailing actual loading conditions. For this reason, an analysis for the case of non-uniform pressure was added to check how much the upper buckling value would be decreased.

(5) This paper is limited to the circular arch beams having small curvature. In the case of large curvature (θ is large), we have to take account of the additional terms of higher order deformation components. The above conclusions cannot

be applied to such a case which will need more detailed research.
(6) Lastly, when $(i^2/l^2)/\theta^2$ is small, that is, when the beam is very thin or θ is large, we encountered the higher order of deflection mode analytically. We will have to examine further whether such a deformation mode is actually realized or not.

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