Aeronautical Research Institute, University of Tokyo
Report No. 385, March 1964

On the Dynamic Stability of Rockets

By

Bunji TOMITA

Summary. The general equations of motion are derived for spinning elastic rocket with respect to axes attached to the rocket. The equations contain a force and moment system that includes, in addition to the usual quasisteady forces and moments, those due to Magnus effects. As a result, the stability criteria for spinning elastic rocket are approximately derived, including as special cases, those such as for the case of rigid body and for the case of nonspinning elastic rocket, which have been analyzed. Comparing these analytical results with the digital computational results, the applicability of these stability criteria is examined and from these results, empirical formulas for stability boundaries are also presented. Furthermore, clasifying the stability boundaries into seviral types according to the characteristics of motion of rocket at the critical speed, and the physical meaning of each stability boundary is also explained.

LIST OF SYMBOLS

$m(x)$; mass per unit length of rocket, kg·sec²/m^2

M; total mass of rocket, kg·sec²/m

$l=l_1+l_2$; total length of rocket, m

S; cross sectional area of rocket, m^2

l_1; $\overline{\text{C. G., Cp}_{\text{nose}}}$, m

l_2; $\overline{\text{C. G., Cp}_{\text{tail}}}$, m

$d_i l$; $\overline{\text{C. G., Cp}_{\text{total}}}$, m

$EI(x)$; flexural rigidity, kg·m^2

C_{Li}; lift coefficient,

C_{mi}; restoring moment coefficient,

u, v, w; linear velocity components along the x, y, and z axes, m/sec

a_x, a_y, a_z; linear acceleration components along the x, y and z axes, m/sec²

X, Y, Z; force components along the x, y, and z axes, kg

p, q, r; angular velocity components about the x, y, and z axes, rad/sec

L', M', N'; moments about the x, y and z axes, kg·m

I_x, $I_y=I_0$, $I_z=I_0$; moments of inertia about the x, y and z axes, kg·m·sec²

$\sigma(x, t)=\sigma_0(t)f_\sigma(x)$; displacement of bending vibration in $o-xy$ plane, m

$\tau(x, t)=\tau_0(t)f_\tau(x)$; displacement of bending vibration in $o-xz$ plane, m

$f_\sigma(x), f_\tau(x)$; normal functions,

t; time, sec

$f_i=\{f(x)\}_{x=xi}$

$f_i'=\{\partial f(x)/\partial x\}_{x=xi}$

ω_B; natural frequency of bending vibration, 1/sec
ρ; air density, $kg \cdot sec^2/m^4$
$\mu=\rho sl/2M$; density ratio,
.(dot); derivative with respect to t, 1/sec
,(prime); derivative with respect to x/l,
$j=\sqrt{-1}$,
$\xi=v+jw$; complex value of the linear velocity,
$\eta=q+jr$; complex value of the angular velocity,
$\zeta=\sigma+j\tau$; complex value of the displacement of bending vibration,
P; constant angular velocity about the x axis, rad/sec
U; constant linear velocity along the x axis, m/sec
$T=Ut/l$; nondimensional time,
$\bar{\omega}_B=\omega_B l/U$; reduced frequency,
$\bar{P}=Pl/U$; reduced frequency,
$Q=\frac{1}{2}\rho U^2 S$; kg
$R(x)$; radius of rocket body, m
$S_0=\int \pi R^2 dx/l$; mean cross sectional area, m^2
Y_M, Z_M; Magnus forces along the y and z axes, kg
M'_{MM}, N'_M; Magnus moments about the y and z axes, $kg \cdot m$
Y_P, Z_P; force components along the y and z axes due to spinning combined with yawing motion, kg
M'_P, N'_P; moment components about the y and z axes due to spinning combined with yawing motion, $kg \cdot m$
$d_p l=\overline{C.\ G.,\ C_{p_{Z_P}}}$,
$d_m l=\overline{C.\ G.,\ C_{p_{Y_M}}}$,
$\dot{\bar{v}}=v/l$; nondimensional linear velocity along the y axis,
$\dot{\bar{w}}=w/l$; nondimensional linear velocity along the z axis,
$\dot{\bar{q}}=q$; angular velocity about the y axis,
$\dot{\bar{r}}=r$; angular velocity about the z axis,
$\bar{\sigma}=\sigma/l$; nondimensional displacement,
$\bar{\tau}=\tau/l$; nondimensional displacement,
$\dot{\bar{\xi}}=\xi/l$; complex value of the nondimensional linear velocity,
$\dot{\bar{\eta}}=\eta$; complex value of the angular velocity,
$\bar{\zeta}=\zeta/l$; complex value of the nondimensional displacement,
U_{cr}; critical speed, m/sec
$k=\sqrt{\frac{I_0}{Ml^2}}$; nondimensional radius of gyration,
$C.\ G.$,; center of gravity,
C_p; center of pressure,
$i_0=I_x/I_0$

1. Introduction

In the field of dynamic stability of rocket, many analytical results under appropriate limitation of motion of rocket have been obtained such as for the case of nonspinning elastic rocket* and for the case of rigid body**. In the former case, the effects of flexural rigidity of rocket upon the dynamic stability are mainly explained, and in the latter case, the effects of spinning rate upon the dynamic stability are investigated. Since an actual rocket body is elastic and accompanied by a spinning motion, it is necessary to analyze the case of spinning elastic rocket.

In the case of elastic spinning rocket, the freedoms of motion are the translations along the three axes, the rotations about the three axes and the bending vibrations in two planes perpendicular to each other, so the analysis is very complicated.

To solve this complicated problem, simplification can be made without significant effect introducing appropriate assumptions as follows:

(a) Configuration of rocket is symmetrical.

(b) Gravitational effects are neglected.

(c) Forward velocity of rocket is constant. By the usual assumption, we have specified that the angle of yaw and the yawing velocities are small disturbance values, and we write $u=U\cos\theta$, where $\cos\theta\to 1$, and $\sin\theta\to\theta$; then $\dot{u}=\dot{U}-U\sin\theta\dot{\theta}$ or $\dot{u}\cong\dot{U}$ and $u\cong U$

(d) The spinning rate is constant. The angular velocity of spin motion p is actually the sum of the spinning rate P of the rocket about its own axis and the component of angular velocity p' about the x axis due to the precessional motion.

However, it is considered that p' is, in general, a small magnitude compared with that of P. Then, we neglect p' and replace p by P without significant effect on the stability results.

However, under these assumptions for simplification, the complete analytical expressions for stability boundaries of the spinning elastic rocket can be hardly expected, so the following procedures are adapted for the analysis of this problem: 1st procedure is the approximate analysis for some special cases under appropriate limitation for the magnitude of P and U, 2nd procedure is the digital computational research for many actual examples, and 3rd procedure is that the set up of empirical formulas of stability boundaries for the whole range of P and U by combining the results obtained by above mentioned procedures, and that the explanation of physical meaning of the various stability boundaries of spinning elastic rocket.

2. Formulation of the Equations of Motion

In this section, the general equations of motion of spinning rocket having the finite flexural rigidity are derived. The rocket under consideration is assumed to be slender and to possess geometric, elastic and inertial characteristics having nearly circular symmetricity with respect to all points on a longitudinal axis through the body of rocket. The right-handed orthogonal o–xyz system of coordinates is fixed in rocket body; it coincides with the principal axes, and the x-axis is positive in the forward direction.

The nomenclature and sign convention of the rigid body degrees of freedom and of certain kinematical and dynamic quantities relating to them are defined in Table 1. All velocities, forces and moments are positive in the direction of positive displacements.

TABLE 1.

Coordinate	Parallel to Coordnate Axes			About Coordnate Axes			
Axes	Velocity	Acceleration	Forces	Displacement	Velocity	Acceleration	Moment
x	$u=U$	a_x	X	φ	$p=P$	a_φ	L
y	v	a_y	Y	θ	q	a_θ	M
z	w	a_z	Z	ψ	r	a_ψ	N

In addition to the degrees of freedom defined in Table 1, the rocket body is considered to be elastic in the sense that it can execute beam bending vibrations. A vibration in the xy plane leads to a displacement $\sigma(x, t)$, that in the xz plane leads to a displacement $\tau(x, t)$. Both vibrations are restricted to occur in the fundamental mode only, and no other mode of elastic deformation is considered. $\sigma(x, t)$ and $\tau(x, t)$ are positive in the directions of positive y and z respectively.

(1) The Equations of Motion along the x, y and z Axes [1]

Derivation of linear accelerations presents no great difficulty. They are easily shown to be, respectively,

$$\left.\begin{aligned} a_x&=\dot{u}+\dot{q}(z+\tau)+2q(w+\dot{\tau})-\dot{r}(y+\sigma)-2r(v+\dot{\sigma})\\ &\quad+r(pz-xr+p\tau)+q(py+p\sigma-xq)\\ a_y&=\dot{v}-\dot{p}(z+\tau)-2p(w+\dot{\tau})+ur+x\dot{r}+\ddot{\sigma}\\ &\quad+r(qz+q\tau-yr-\sigma r)-p(yp-xq+\sigma p)\\ a_z&=\dot{w}-x\dot{q}-uq+\dot{p}(y+\sigma)+2p(v+\dot{\sigma})\\ &\quad+\ddot{\tau}-q(qz-yr+q\tau-\sigma r)-p(pz-xr+p\tau) \end{aligned}\right\} \tag{1}$$

Eqs. (1) are obtainable from purely kinematical considerations. Then, the equations of motion along the three axes are obtained as follows:

$$\left.\begin{aligned} \sum X&=\int_l m(x)a_x dx\\ \sum Y&=\int_l m(x)a_y dx\\ \sum Z&=\int_l m(x)a_z dx \end{aligned}\right\} \tag{2}$$

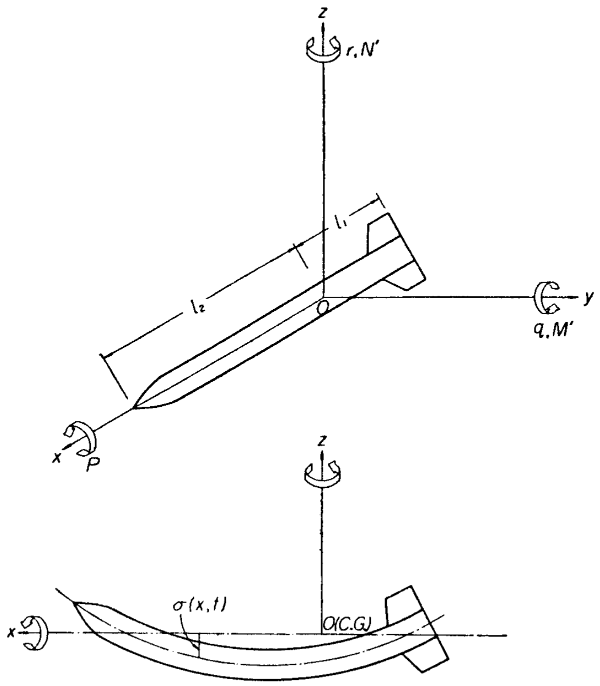


FIGURE 1.

(2) The Equations of Motion about the x, y and z Axes [2]

The angular accelerations must be derived from dynamical relations. They are obtained from Euler's equations of inertia moments of a body rotating about a fixed point coinciding with the center of gravity, and their simple form used here requires that the coordinate axes coincide with the principal axes. The angular momentums of a mass $m(x)$ about the x, y and z axes, respectively, are

$$\left.\begin{aligned}\lambda_x&=m(x)\{(y+\sigma)(w-xq+yp+\sigma p+\dot{\tau})-(z+\tau)(v-zp+xr-\tau p+\dot{\sigma})\}\\ \lambda_y&=m(x)\{(z+\tau)(u+zq-yr+\tau q-\sigma r)-x(w-xq+yp+\sigma p+\dot{\tau})\}\\ \lambda_z&=m(x)\{x(v-zp+xr-\tau p+\dot{\sigma})-(y+\sigma)(u+zq-yr+\tau q-\sigma r)\}\end{aligned}\right\} \quad (3)$$

Then, the inertia moments due to mass $m(x)$ about x, y and z axes, respectively, are

$$\left.\begin{aligned}\frac{dL''}{dt}&=\frac{d\lambda_x}{dt}-r\lambda_y+q\lambda_z\\ \frac{dM''}{dt}&=\frac{d\lambda_y}{dt}-p\lambda_z+\gamma\lambda_x\\ \frac{dN''}{dt}&=\frac{d\lambda_z}{dt}-q\lambda_x+p\lambda_y\end{aligned}\right\} \quad (4)$$

If the vibrational displacements σ and τ are sufficiently small, higher terms of the small quantities and products of inertia with regard to the vibrations can be neglected, and then, the equations of motion about the three axes are obtained as follows:

$$\left.\begin{aligned}\sum L'&=\int_l \frac{dL''}{dt}dx=I_x\dot{p}+(I_z-I_y)qr\\ \sum M'&=\int_l \frac{dM''}{dt}dx=I_y\dot{q}+(I_x-I_z)pr\\ \sum N'&=\int_l \frac{dN''}{dt}dx=I_z\dot{r}+(I_y-I_x)pq\end{aligned}\right\}\tag{5}$$

where, the following relations for the mode shapes of vibration are introduced:

$$\left.\begin{aligned}&\int_l m(x)\sigma(x,t)\,dx=\sigma_0(t)\int_l m(x)f_\sigma(x)dx=0\\ &\int_l m(x)\tau(x,t)\,dx=\tau_0(t)\int_l m(x)f_\tau(x)dx=0\\ &\int_l m(x)x\sigma(x,t)dx=\sigma_0(t)\int_l m(x)xf_\sigma(x)dx=0\\ &\int_l m(x)x\tau(x,t)dx=\tau_0(t)\int_l m(x)xf_\tau(x)dx=0\end{aligned}\right\}\tag{6}$$

(3) The Bending Vibrations

The derivation of the equation of bending vibration in the xy plane will be shown, and the equation in the xz plane will be derived by the same method. The intensity of inertia force due to the bending vibration distributed along the length of the rocket body is given by

$$\Delta T=-m(x)\{\ddot{\sigma}(x,t)-\sigma(x,t)P^2\}\tag{7}$$

Assuming a virtual displacement,

$$\delta D=f_\sigma(x)\delta\sigma_0(t)\tag{8}$$

it can be shown that the virtual work of inertia forces is given by

$$-\left\{\ddot{\sigma}\int_l m(x)\,f_\sigma^2(x)dx-\sigma_0 p^2\int_l m(x)f_\sigma^2(x)dx\right\}\delta\sigma_0(t)\tag{9}$$

The strain energy of bending of the rocket body at any instance is

$$V=\int_l \Delta V dx=\frac{1}{2}\int_l EI(x)\left\{\frac{\partial^2\sigma(x,t)}{\partial x^2}\right\}^2 dx\tag{10}$$

and for the virtual work of elasticity forces, we obtain as follows:

$$-\int_l \frac{\partial\Delta V}{\partial\sigma_0}\delta\sigma_0 dx=-\left\{\sigma_0\int_l EI(x)\left\{\frac{d^2f_\sigma(x)}{dx^2}\right\}^2 dx\right\}\delta\sigma_0(t)\tag{11}$$

The virtual work of the external forces applying to the position of $x=x_i$ is

$$\sum_i Y\{\delta D\}_{x=x_i}=\sum_i Yf_i(x)\delta\sigma_0(t) \tag{12}$$

Summing up the expressions (9), (11) and (12), we obtain the following equation:

$$\ddot{\sigma}_0(t)\int_l m(x)f_\sigma^2(x)dx-\sigma_0(t)p^2\int_l m(x)f_\sigma^2(x)dx+\sigma_0(t)\int_l EI(x)\left\{\frac{d^2f_\sigma(x)}{dx^2}\right\}^2dx$$
$$-\sum Y\{f_\sigma(x)\}_{x=x_i}=0 \tag{13}$$

Now, to determine the absolute values of $f_\sigma(x)$ and $f_\tau(x)$, the following normalizing conditions are used:

$$\int_l m(x)f_\sigma^2(x)dx=M,\ \int_l m(x)f_\sigma^2(x)dx=M \tag{14}$$

And, introducing the assumption of axial symmetricity of rocket structure, we obtain the following relations:

$$\left.\begin{aligned}&f_\sigma(x)=f_\tau(x)=f(x)\\&\int_l EI(x)\left\{\frac{d^2f_\sigma}{dx^2}\right\}^2\bigg/\int_l m(x)f_\sigma^2(x)dx=\int_l EI(x)\left\{\frac{d^2f_\tau}{dx^2}\right\}^2\bigg/\int_l m(x)f_\tau^2(x)dx=\omega_B^2\end{aligned}\right\} \tag{15}$$

Then, the equations of motiom of bending vibration in xy plane and xz plane are written as follows:

$$\left.\begin{aligned}&M\ddot{\sigma}_0-Mp^2\sigma_0+M\omega_B^2\sigma_0-\sum_i Yf_i=0\\&M\ddot{\tau}_0-Mp^2\tau_0+M\omega_B^2\tau_0-\sum_i Zf_i=0\end{aligned}\right\} \tag{16}$$

(4) The Aerodynamic Forces

Generally, the forward velocity of rocket u is much greater than v and w, and by the assumption for simplification we can write $u=U$. Then, the aerodynamic forces due to the change of angle of attack along the y and z axes are written as follows:

$$\left.\begin{aligned}&\sum Y_A=Q_0\sum_i C_{Li}\left\{-\frac{v}{U}+\frac{\sigma_0 f_i'}{l}-\frac{rd_il}{U}-\frac{\dot{\sigma}_0 f_i}{U}\right\}\\&\sum Z_A=Q_0\sum_i C_{Li}\left\{-\frac{w}{U}+\frac{\tau_0 f_i'}{l}+\frac{qd_il}{U}-\frac{\dot{\tau}_0 f_i}{U}\right\}\end{aligned}\right\} \tag{17}$$

And, the aerodynamic restoring moments about the y and z axes are written as follows:

$$\left.\begin{aligned}&\sum M'_A=Q_0l\sum C_{mi}\left\{\frac{w}{U}-\frac{\tau_0 f_i'}{l}-\frac{qd_il}{U}+\frac{\dot{\tau}_0 f_i}{U}\right\}\\&\sum N'_A=Q_0l\sum C_{mi}\left\{-\frac{v}{U}+\frac{\sigma_0 f_i'}{l}-\frac{rd_il}{U}-\frac{\dot{\sigma}_0 f_i}{U}\right\}\end{aligned}\right\} \tag{18}$$

In addition to the forces and moments represented by Eqs. (17) and (18) respectively, the aerodynamic forces and moments due to the Magnus effect [5] and due to spinning combined with pitching motion must be taken into account. The Magnus force is caused by the circulation of air around the rocket body due to the spin. If the air next to the rocket adhered tightly, the circulation velocity is equal to the surface velocity of rolling of rocket, but actually there is considerable slip which depends upon the inertia of the air, the roughness of the rocket surface, the magnitude of surface velocity, e.t.c. Then, introducing the coefficient ν, represent the magnitude of slip, the Magnus forces along the y and z axes are written by

$$\left.\begin{aligned} Y_M &= -2\nu\rho S_0 lwP \\ Z_M &= 2\nu\rho S_0 lvP \end{aligned}\right\} \tag{19}$$

where S_0 is the mean cross sectional area of rocket.

Then, the Magnus moments about y and z axes are written by

$$\left.\begin{aligned} M_M' &= 2\nu\rho S_0 l^2 d_M vP \\ N_M' &= 2\nu\rho S_0 l^2 d_M wP \end{aligned}\right\} \tag{20}$$

where $d_M l$ is the distance between the center of gravity and the center of pressure of the Magnus force.

The aerodynamic forces due to spinning combined with pitching motion [6] along the y and z axes are written by

$$\left.\begin{aligned} Y_P &= \varepsilon C_{Li}\frac{1}{2}\rho lwP \\ Z_P &= -\varepsilon C_{Li}\frac{1}{2}\rho SlvP \end{aligned}\right\} \tag{21}$$

where ε is the coefficient depends upon the shape of the fin, the magnitude of the spinning rate, e.t.c.

Then, the moments about y and z axes due to Z_p and Y_p are respectively

$$\left.\begin{aligned} M_P' &= \varepsilon C_{Li}\frac{1}{2}\rho Sl^2 d_P Pv \\ N_P' &= \varepsilon C_{Li}\frac{1}{2}\rho Sl^2 d_P Pw \end{aligned}\right\} \tag{22}$$

where $d_P l$ is the distance between the center of gravity and the center of pressure of Y_P (or Z_P).

Now, introducing the complex variables $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, and nondimensional time T, and using the Eqs. (2), (6), (16)~(22), the eqations of motion are obtained as follows:

$$\begin{aligned} &\frac{d^2\bar{\xi}}{dT^2}+\left\{\mu\sum C_{Li}+j\frac{Pl}{U}(1+\mu k_f)\right\}\frac{d\bar{\xi}}{dT}-j(1+\mu\sum C_{mi})\frac{d\bar{\eta}}{dT} \\ &+\mu\sum C_{Li}f_i\frac{d\bar{\zeta}}{dT}-\mu\sum C_{Li}f_i'\bar{\zeta}=0 \end{aligned}$$

$$\left.\begin{aligned}&\left(\mu\frac{k_m}{k^2}\cdot\frac{Pl}{U}+j\mu\frac{\sum C_{mi}}{k^2}\right)\frac{d\bar{\xi}}{dT}+\frac{d^2\bar{\eta}}{dT^2}+\left\{j(1-i_0)\frac{Pl}{U}+\mu\frac{\sum C_{mi}d_i}{k^2}\right\}\frac{d\bar{\eta}}{dT}\\&\quad+j\mu\frac{\sum C_{mi}f_i}{k^2}\cdot\frac{d\bar{\zeta}}{dT}-j\mu\frac{\sum C_{mi}f_i'}{k^2}\bar{\zeta}=0\\&\left(\mu\sum C_{Li}f_i+j\mu k_f f_i\frac{Pl}{U}\right)\frac{d\bar{\xi}}{dT}-j\mu\sum C_{mi}f_i\frac{d\bar{\eta}}{dT}+\frac{d^2\bar{\zeta}}{dT^2}\\&\quad+\left\{\mu\sum C_{Li}f_i^2+2j\frac{Pl}{U}\right\}\frac{d\bar{\zeta}}{dT}+\left\{\left(\frac{w_Bl}{U}\right)^2-\left(\frac{Pl}{U}\right)^2-\mu\sum C_{Li}f_if_i'\right\}\bar{\zeta}=0\end{aligned}\right\}\quad(23)$$

3. Stability Criteria of Nonspinning Rocket

In this section, the stability criteria of nonspinning rocket are analyzed. The equations of motion for nonspinning rocket are easily derived from Eqs. (23) putting $P=0$, $\bar{\xi}=\bar{v}$, $\bar{\eta}=\bar{\gamma}$ and $\bar{\zeta}=\bar{\sigma}$ as follows:

$$\left.\begin{aligned}&\frac{d^2\bar{v}}{dT^2}+\mu\sum C_{Li}\frac{d\bar{v}}{dT}+(1+\mu\sum C_{mi})\frac{d\bar{\gamma}}{dT}+\mu\sum C_{Li}f_i\frac{d\bar{\sigma}}{dT}-\mu\sum C_{Li}f_i'\bar{\sigma}=0\\&\mu\frac{\sum C_{mi}}{k^2}\cdot\frac{d\bar{v}}{dT}+\frac{d^2\bar{\gamma}}{dT^2}+\mu\frac{\sum C_{mi}d_i}{k^2}\cdot\frac{d\bar{\gamma}}{dT}+\mu\frac{\sum C_{mi}f_i}{k^2}\cdot\frac{d\bar{\sigma}}{dT}-\mu\frac{\sum C_{mi}f_i'}{k^2}\bar{\sigma}=0\\&\mu\sum C_{Li}f_i\frac{d\bar{v}}{dT}+\mu\sum C_{mi}f_i\frac{d\bar{\gamma}}{dT}+\frac{d^2\bar{\sigma}}{dT^2}+\mu\sum C_{Li}f_i^2\frac{d\bar{\sigma}}{dT}+(\bar{w}_B^2-\mu\sum C_{Li}f_if_i')\bar{\sigma}=0\end{aligned}\right\}$$

(24)

To obtain the stability criteria of rocket, whose motion is governed by Eqs. (24), as trial solutions we may choose

$$\bar{v}=\bar{v}_0e^{\lambda T},\qquad \bar{\gamma}=\bar{\gamma}_0e^{\lambda T},\qquad \bar{\sigma}=\bar{\sigma}_0e^{\lambda T}\tag{25}$$

which lead, by expanding the determinantal equation, to a quartic frequency equation with real coefficients.

$$\lambda^4+A_3\lambda^3+A_2\lambda^2+A_1\lambda+A_0=0\tag{26}$$

where

$$\left.\begin{aligned}&A_3=\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}+\sum C_{Li}f_i^2\right)\\&A_2=\bar{w}_B^2-\mu\left(\sum C_{Li}f_if_i'+\frac{\sum C_{mi}}{k^2}\right)\\&A_1=\mu\bar{w}_B^2\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)+\mu^2\left\{-\sum C_{Li}\sum C_{Li}f_if_i'-\frac{\sum C_{mi}d_i}{k^2}\sum C_{Li}f_if_i'\right.\\&\quad\left.+\sum C_{Li}f_i\frac{\sum C_{mi}f_i}{k^2}+\sum C_{Li}f_i\sum C_{Li}f_i'-\frac{\sum C_{mi}}{k^2}\sum C_{Li}f_i^2+\sum C_{mi}f_i\frac{\sum C_{mi}f_i'}{k^2}\right\}\end{aligned}\right\}$$

$$A_0=-\bar{\omega}_B^2\mu\frac{\sum C_{mi}}{k^2}+\mu^2\left\{\sum C_{Li}f_i\frac{\sum C_{mi}f_i'}{k^2}-\sum C_{Li}f_if_i'\frac{\sum C_{mi}}{k^2}\right\} \tag{27}$$

The conditions for stability are given by

$$\left.\begin{array}{l} A_0,\ A_1,\ A_2,\ A_3>0 \\ A_1A_2A_3-A_1{}^2-A_0A_3{}^2>0 \end{array}\right\} \tag{28}$$

(a) Inspection of the condition $A_0>0$;
From Eqs. (27), the critical reduced velocity is easily derived as follows:

$$\left\{\frac{U_{cr}}{\omega_B l}\right\}_{A_0,P_0}=\frac{1}{\sqrt{\mu}\left\{\sum C_{Li}f_if_i'-\frac{\sum C_{mi}f_i'}{\sum C_{mi}}\sum C_{Li}f_i\right\}^{1/2}} \tag{29}$$

It will be considered.that the critical speed $(U_{cr})_{A_0}$ coresponding to the stability condition $A_0>0$ always exist, and in the speed region smaller than the value of $(U_{cr})_{A_0}$ the rocket is stable, but in the speed region greater than $(U_{cr})_{A_0}$ the rocket is unstable. The frequency of motion at this speed is zero, then, this means that the critical speed $(U_{cr})_{A_0}$ physically can be defined as the divergence speed of rocket in a narrow sense.

(b) Inspection of the conditions A_1, A_2 and $A_3>0$;
Generally, it can be considered that these stability conditions are always satisfied in the statically stable rocket at any speed region. This means that any criticl speed corresponding to these conditions does not exist.

(c) Inspection of the condition $A'=A_1A_2A_3-A_0A_3^2-A_1^2>0$;
This stability condition is reduced to the following inequality.

$$\left.\begin{array}{l} \bar{\omega}_B^4+\bar{\omega}_B^2\mu F_1+\mu^2F_2>0 \\ \text{where} \\ F_1=F_1(\mu,\ C_{Li},\ C_{mi},\ f_i,\ f_i',\ d_i,\ k) \\ F_2=F_2(\mu,\ C_{Li},\ C_{mi},\ f_i,\ f_i',\ d_i,\ k) \end{array}\right\} \tag{30}$$

Then, if the values of F_1 and F_2 satisfy the conditions of

$$F_1<0 \qquad F_1{}^2-4F_2>0, \tag{31}$$

the critical reduced velocities exist, and they are given by the following values:

$$\left.\begin{array}{l} \left\{\dfrac{U_{cr1}}{\omega_B l}\right\}_{A'}=\left[\dfrac{1}{\frac{1}{2}\mu\{-F_1+\sqrt{F_1{}^2-4F_2}\}}\right]^{1/2} \\ \left\{\dfrac{U_{cr2}}{\omega_B l}\right\}_{A'}=\left[\dfrac{1}{\frac{1}{2}\mu\{-F_1-\sqrt{F_1{}^2-4F_2}\}}\right]^{1/2} \end{array}\right\} \tag{32}$$

The stability regions are given by the following boundaries:

$$\left.\begin{array}{ll} U<\{U_{cr1}\}_{A'} & : \text{ stable region,} \\ \{U_{cr1}\}_{A'}\leq U\leq\{U_{cr2}\}_{A'} & : \text{ unstable region,} \\ \{U_{cr2}\}_{A'}<U & : \text{ stable region.} \end{array}\right\} \quad (33)$$

The condition of $A'>0$ is equivalent to the real parts of complex root of frequency equation (26) must be negative, this means that the motions of rocket at the critical speed given by Eqs. (32) are oscillatory. Hence, the critical speeds $\{U_{cr1,2}\}_{A'}$ can be defined physically as the flutter speeds of rocket in a narrow sense.

These critical speeds exist only in the case where the conditions (31) are satisfied, but these conditions are not always satisfied even in the case of statically stable rocket, so these critical speeds do not always exist. And according to the results of many examples, it can be considered that the values of flutter speeds much larger than the value of divergence speed given by Eq. (29), so we can consider the value of divergence speed as the critical speed of nonspinning rocket.

Discussion of Stability of Nonspinning Rocket

From the Eq. (29), the following results may be stated as applied to a nonspinning rocket.

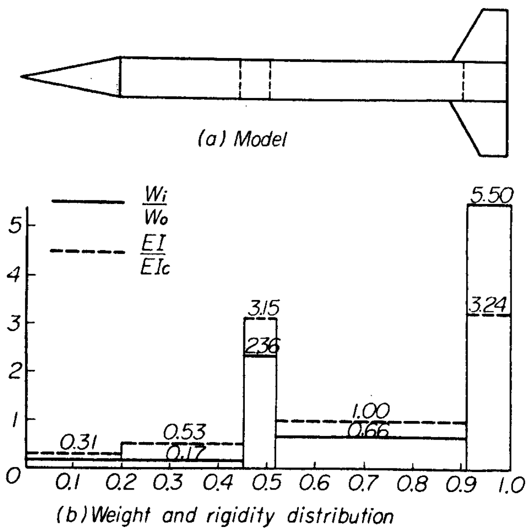
(a) The critical speed is proportional to $1/\sqrt{\rho}$, then the value of U_{cr} at high altitude larger than that of at low altitude.

(b) The natural frequency ω_B is used in Eq. (29) as the value representing the rigidity of rocket body, and the flexural rigidity EI is proportional to $\sqrt{\omega_B}$, hence it is considered that $U_{cr}\propto\sqrt{EI}$. It is noticed that the critical speed U_{cr} is affected by not only EI but also by vibration mode *i.e.* $f(x)$ and $f'(x)$ which are functions of structural characteristics, mass distribution and e.t.c. Then, for the calculation of U_{cr}, it is necessary to estimate both natural frequency and vibration mode. A convenient way to determine these values is the vibration test of rocket model, then the natural frequency ω_B is easily determined, and the vibration mode can be obtained by normalizing the amplitude along the rocket axis at resonant frequency.

(c) The critical speed is affected by the aerodynamic coefficients C_{Li} and C_{mi} by the power of inverse square root.

Example 1.

As an example, the critical speed of the actual rocket, shown in Figure 2, will be calculated using the results in this section. f_i and f_i' are obtained by normalizing the amplitude, which is measured by vibration test, and they are shown in Figure 3.



W_0: mean weight per unit length
EI_c: rigidity of chamber

FIGURE 2.

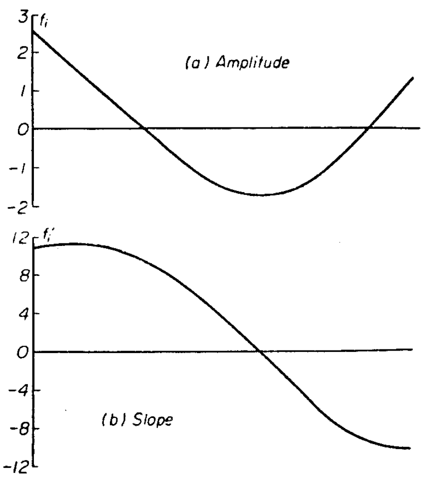


FIGURE 3. Vibration shape

$$\sum C_{Li} = C_{LN} + C_{LT} = 10\,.$$

$C_{LN} = 2.0$: lift coefficient of nose

$C_{LT} = 8.0$: lift coefficient of tail

$$\sum C_{mi} = C_{mN} + C_{mT} = -1.9\,,$$

$C_{mN} = 0.9$: moment coefficient of nose

$C_{mT} = -2.8$: moment coefficient of tail

Hence

$$\left.\begin{aligned} &\sum C_{mi} f_i = -0.6\,, & &\sum C_{Li} f_i f_i' = -19.4\,, \\ &\sum C_{mi} f_i' = 37.8\,, & &\sum C_{Li} f_i^2 = 5.76\,, \\ &\sum C_{Li} f_i = 7.2\,, & &\sum C_{mi} d_i = 1.39\,, \\ &\sum C_{Li} f_i' = -55.4\,, & &k^2 = \frac{1}{10}\,. \end{aligned}\right\}$$

Substituting these values into Eq. (29), divergence speed given by

$$\left\{\frac{U_{cr}}{\omega_B l}\right\}_{P_0} = 2.83$$

and

$$F_3 = -31 < 0\,, \qquad F_3 - 4F_4 = -46{,}590 < 0$$

Hence, in this case, the flutter phenomenon does not occur at any speed.

4. Stability Criteria of Spinning Rocket of Rigid Body

In this section, the stability criteria of spinning rocket of rigid body [7], [8], *, ** are analyzed as the special case where the flexural rigidty of rocket body is extremely large.

The equations of motion are easily derived from Eqs. (23) putting $\omega_B=\infty$, $f_i=f_i'=0$ as follows:

$$\left.\begin{aligned}&\frac{d^2\bar{\xi}}{dT^2}+\{\mu\sum C_{Li}+j\bar{P}(1+\mu k_f)\}\frac{d\bar{\xi}}{dT}-j(1+\mu\sum C_{mi})\frac{d\bar{\eta}}{dT}=0\\&\mu\left(\frac{k_m}{k^2}\bar{P}+j\frac{\sum C_{mi}}{k^2}\right)\frac{d\bar{\xi}}{dT}+\frac{d^2\bar{\eta}}{dT^2}+\left\{j(1-i_0)\bar{P}+\mu\frac{\sum C_{mi}d_i}{k^2}\right\}\frac{d\bar{\eta}}{dT}=0\end{aligned}\right\}\tag{34}$$

To obtain the stability criteria of rocket, whose motion is governed by Eqs. (34), as trial solutions we may choose

$$\bar{\xi}=\bar{\xi}_0e^{\lambda T},\qquad \bar{\eta}=\bar{\eta}_0e^{\lambda T}\tag{35}$$

which lead, by expanding the determinantal equation, to a frequency equation.

$$\left.\begin{aligned}&\lambda^2+(c_1+jd_1)\lambda+c_0+jd_0=0\\&\text{where}\\&c_1=\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)\\&d_1=(2-i_0)\bar{P}\\&c_0=\mu^2\sum C_{Li}\frac{\sum C_{mi}d_i}{k^2}-\mu^2\sum C_{mi}\frac{\sum C_{mi}}{k^2}-\mu\frac{\sum C_{mi}}{k^2}-(1-i_0)\bar{P}^2\\&d_0=\mu\bar{P}\left\{\frac{\sum C_{mi}d_i}{k^2}+(1-i_0)\sum C_{Li}+\frac{k_m}{k^2}\right\}\end{aligned}\right\}\tag{36}$$

The necessary and sufficient conditions [9] for stability are given by:

$$\left.\begin{aligned}&c_1>0\\&c_1^2c_0+c_1d_1d_0-d_0^2>0\end{aligned}\right\}\tag{37}$$

(a) Inspection of the condition $c_1>0$;
For the statically stable rocket, the values of $\sum C_{Li}$ and $\sum C_{mi}d_i$ are both positive, then c_1 is always positive. This means, any critical speed, at which the rocket becomes unstable divergently, does not exist.

(b) Inspection of the condition $c'=c_1^2c_0+c_1d_1d_0-d_0^2>0$
This stability condition easily can be reduced to the following inequality.

$$\begin{aligned}&\bar{P}^2\left\{\left(\frac{k_m}{k^2}\right)^2+i_0\frac{k_m}{k^2}\left(\frac{\sum C_{mi}d_i}{k^2}-\sum C_{Li}\right)-i_0^2\sum C_{Li}\frac{\sum C_{mi}d_i}{k^2}\right\}\\&\quad+\mu\frac{\sum C_{mi}}{k^2}\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)<0\end{aligned}\tag{38}$$

For the statically stable rocket $\sum C_{mi}/k^2<0$ and $(\sum C_{Li}+\sum C_{mi}d_i/k^2)>0$, then if condition

$$\left(\frac{k_m}{k^2}\right)^2+i_0\left(\frac{k_m}{k^2}\right)\left(\frac{\sum C_{mi}d_i}{k^2}-\sum C_{Li}\right)-i_0^2\sum C_{Li}\frac{\sum C_{mi}d_i}{k^2}>0 \qquad (39)$$

is satisfied, then the critical speed exists, whose reduced velocity is given by following formula:

$$\left\{\frac{U_{cr}}{Pl}\right\}_{c'}>\frac{\left\{\left(\frac{k_m}{k^2}\right)^2+i_0\left(\frac{k_m}{k^2}\right)\left(\frac{\sum C_{mi}d_i}{k^2}-\sum C_{Li}\right)-i_0^2\sum C_{Li}\frac{\sum C_{mi}d_i}{k^2}\right\}^{1/2}}{\left\{-\mu\frac{\sum C_{mi}}{k^2}\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)\right\}} \qquad (40)$$

At which speed, the rocket becomes unstable oscillatory, so this critical speed is considered physically to be the flutter speed of rocket of rigid body in a narrow sense.

From the inequality (38), under the condition (39) is satisfied, we can obtain another critical speed, but that is of a negative value and then it is meaningless physically.

Hence, in this case, there exist the only one critical speed, *i.e.* the flutter speed, and the the stable region is given as follows:

$$\left.\begin{array}{ll} 0\leq U\leq\{U_{cr}\}_{c'}; & \text{unstable} \\ \{U_{cr}\}_{c'}<U; & \text{stable} \end{array}\right\} \qquad (41)$$

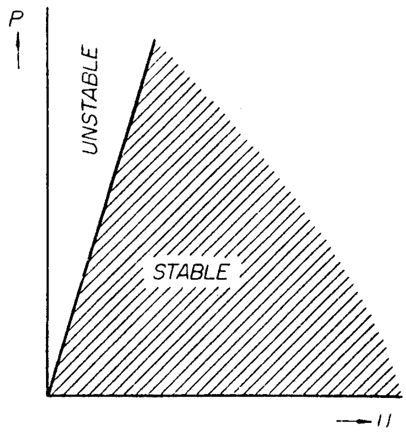


FIGURE 4. Stability Boundary of Rigid Body.

Disscusion of Stability of Spinning Rocket of Rigid Body.

From the above analysis, the following results may be stated as applied to a spinning rocket of rigid body.

(a) The critical speed is proportional to $1/\sqrt{\rho}$, then the value of U_{cr} at high altitude is larger than that at low altitude, *i.e.* the unstable rigion at high altitude is wider than that at low altitude.

(b) If the aerodynamic characteristics are constant at any speed of rocket, the critical speed is proportional to the spinning rate P.

(c) For the rocket of slender body, actually it may be considered that the value of i_0 is much smaller the unity, then the critical speed is aproximately given by

$$\left\{\frac{U_{cr}}{Pl}\right\}_{c'}=\frac{\dfrac{k_m}{k^2}}{\left\{-\mu\dfrac{\sum C_{mi}}{k^2}\left(\sum C_{Li}+\dfrac{\sum C_{mi}d_i}{k^2}\right)\right\}^{1/2}} \tag{42}$$

(d) The stability condition (39) depends upon not only the aerodynamic coefficients, structural characteristics but also upon the ratio of moment of inertia i_0. Eq. (39) can be transformed into the following formula.

$$\left(\frac{k_m}{k^2}-i_0\sum C_{Li}\right)\left(\frac{k_m}{k^2}+i_0\frac{\sum C_{mi}d_i}{k^2}\right)>0 \tag{43}$$

i_0, $\sum C_{Li}$, and $\sum C_{mi}d_i/k^2$ are positive values, hence the condition (39) may be rewritten as follows:

$$\frac{k_m}{k^2}>i_0\sum C_{Li} \qquad \text{or} \qquad \frac{k_m}{k^2}<-i_0\frac{\sum C_{mi}d_i}{k^2} \tag{44}$$

5. Stability Criteria of Spinning Rocket at Low Speed

In this section, the stability criteria of spinning rocket of elastic body at very low speed are analyzed, and in an extreme case where the speed of rocket is tending to zero this problem coincides with the whirling phenomenon of elastic shaft.

First, in the special case where the speed of rocket is zero, the equations of motion of rocket are easily obtained by putting $U=0$ in Eqs. (23) as follows:

$$\left.\begin{aligned}&\dot{\xi}+jP(1+\mu k_f)\xi=0 && \text{(a)}\\ &\dot{\eta}+j(1-i_0)P\eta+\mu\frac{k_m}{k^2l}P\xi=0 && \text{(b)}\\ &\ddot{\zeta}+(\omega_B^2-P^2)\zeta+j\mu k_f f_i P\xi=0 && \text{(c)}\end{aligned}\right\} \tag{44}$$

From the Eqs. (44)-(a) and (b), it is easily found that ξ and η do not diverge at any spinning rate, then only the Eq. (44)-(c) will be considered under this condition.

As trial solutions we may choose

$$\zeta=\zeta_0e^{\lambda t}, \qquad \xi=\eta=0, \tag{45}$$

which lead to a quadratic frequency equation.

$$\lambda^2+\omega_B^2-P^2=0 \tag{46}$$

Hence, the critical spinning rate which is the whirling speed, is given as follows:

$$P_{crU_0}=\omega_B \tag{47}$$

Next, for the special case where the speed of rocket is very low, the equations of

motion are shown by Eqs. (23):
As trial solutions we way choose

$$\bar{\xi}=\bar{\xi}_0 e^{\lambda T}, \quad \bar{\eta}=\bar{\eta}_0 e^{\lambda T}, \quad \bar{\zeta}=\bar{\zeta}_0 e^{\lambda T} \tag{48}$$

which lead, by expanding the resulting determinantal equation, to the quartic frequency equation with complex coefficients as follows:

$$\left.\begin{aligned}
&\lambda^4+A_3\lambda^3+A_2\lambda^2+A_1\lambda+A_0=0\\
&\text{where}\\
&A_3=\mu\left(\frac{\sum C_{mi}d_i}{k^2}+\sum C_{Li}+\sum C_{Li}f_i^2\right)+4j\bar{P}\\
&A_2=\bar{\omega}_B^2-6\bar{P}^2+\mu\left(\sum C_{Li}f_if_i'+\frac{\sum C_{mi}}{k^2}\right)\\
&\qquad+j\mu\bar{P}\left(\frac{k_m}{k^2}+2\sum C_{Li}f_i^2+3\sum C_{Li}+3\frac{\sum C_{mi}d_i}{k^2}\right)\\
&A_1=(\bar{\omega}_B^2-3\bar{P}^2)\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)+\bar{P}^2\mu\left(2\frac{k_m}{k^2}-\sum C_{Li}f_i^2\right)\\
&\qquad+2j\bar{P}\left\{\bar{\omega}_B^2-2\bar{P}^2-\mu\left(\sum C_{Li}f_if_i'+\frac{\sum C_{mi}}{k^2}\right)\right\}\\
&A_0=-(\bar{\omega}_B^2-\bar{P}^2-\mu\sum C_{Li}f_if_i')\left(\mu\frac{\sum C_{mi}}{k^2}+\bar{P}^2\right)\\
&\qquad+j\bar{P}\mu(\bar{\omega}_B^2-\bar{P}-\mu\sum C_{Li}f_if_i')\left(\sum C_{Li}+\frac{k_m}{k^2}+\frac{\sum C_{mi}d_i}{k^2}\right)
\end{aligned}\right\} \tag{49}$$

Eq. (49) can be rewritten approximately as follows:

$$\begin{aligned}
&\left[\lambda^2+\left\{\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)+2j\bar{P}\right\}\lambda-\left(\bar{P}^2+\mu\frac{\sum C_{mi}}{k^2}\right)\right.\\
&\left.\quad+j\bar{P}\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}+\frac{k_m}{k^2}\right)\right][\lambda^2+(\mu\sum C_{Li}f_i^2+2j\bar{P})\lambda+\bar{\omega}_B^2-\bar{P}^2-\mu\sum C_{Li}f_if_i']\\
&=F_1(\lambda)\cdot F_2(\lambda)=0\,.
\end{aligned} \tag{50}$$

Hence, instead of the inspection of Eq. (49), which determine the critical speed of rocket, we can use the equations $F_1(\lambda)=0$ and $F_2(\lambda)=0$.

(a) Stability criteria derived from $F_1(\lambda)=0$.

$$\begin{aligned}
F_1(\lambda)=\lambda^2&+\left\{\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)+2j\bar{P}\right\}\lambda\\
&-\left(\bar{P}^2+\mu\frac{\sum C_{mi}}{k^2}\right)+j\bar{P}\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}+\frac{k_m}{k^2}\right)=0
\end{aligned} \tag{51}$$

Comparing Eq. (51) with Eq. (36), which is the frequency equation of rigid body, we can easily obtain the critical reduced velocity as follows:

$$\left\{\frac{U_{cr}}{Pl}\right\}^2=\frac{\left(\frac{k_m}{k^2}\right)^2}{-\mu\frac{\sum C_{mi}}{k^2}\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)} \tag{52}$$

(b) Stability criteria derived from $F(\lambda_2)=0$

$$F_2(\lambda)=\lambda^2+(\mu\sum C_{Li}f_i^2+2j\bar{P})\lambda+\bar{\omega}_B^2-\bar{P}^2-\mu\sum C_{Li}f_if_i'=0 \tag{53}$$

In this equation, $\sum C_{Li}f_i^2$ is always positive, then the critical spinning rate is easily given by

$$P_{cr}^2=\omega_B^2-\mu\sum C_{Li}f_if_i'\left(\frac{U}{l}\right)^2=P_{cr_{U_0}}^2-\mu\sum C_{Li}f_if_i'\left(\frac{U}{l}\right)^2 \tag{54}$$

Discussion of Stability of Spinning Rocket at Low Speed

From the above analysis, the following results may be stated as applied to a spinning rocket at low speed.

In this case, there are two critical values. One of them, which is represented by Eq. (52), coincides with the critical speed of rigid body, which is shown in previous section, and the other, represented by Eq. (54), shows the critical spinning rate at low speed.

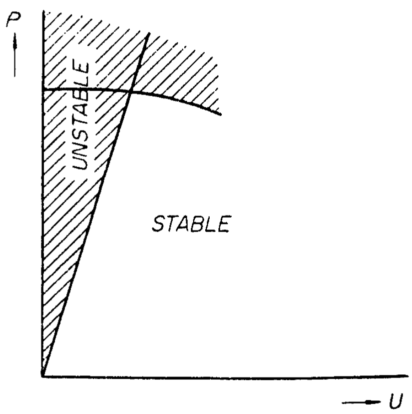


FIGURE 5. Stabilty Boundary of Elastic Rocket at Low Speed.

6. STABILITY CRITERIA OF SPINNING ROCKET OF ELASTIC BODY

In the previous sections, the dynamic stability of rocket for some special cases under appropriate assumptions for motion of rocket are investigated, but in this section, the stability criteria for spinning rocket of elastic body are analyzed. Hence, in this case the rocket has 6 degrees of freedom of motion, *i.e.* heaving or bouncing motions along the y and z axes, pitching or yawing motions about y and z axes and bending vibrations in xz and xz planes. The velocity along the x axis U and spinning rate about the x axis P are also both constant in this case. For this case, the equations of motion are given by Eq. (23) and hence, the frequency equation, from which the critical speeds are determined, is given by a quartic equation with complex coefficients as follows:

$$f(\lambda)=\lambda^4+A_1^{(0)}\lambda^3+A_2^{(0)}\lambda^2+A_3^{(0)}\lambda+A_4^{(0)}=0 \tag{55}$$

The stability criteria are obtained by the following procedure. That is, an iterative process [10] on the complex coefficients $A_K = A_{KR} + jA_{KI}$ is given by the following definitions:

$$\left.\begin{array}{lll}
A_1^{(0)} = A_{1R}^{(0)} + jA_{1I}^{(0)} & A_2^{(2)} = q_2^{(2)}/q_0^{(2)} & A_4^{(0)} = A_{4R}^{(0)} + jA_{4I}^{(0)} \\
A_1^{(1)} = q_1^{(1)}/q_0^{(1)} & A_3^{(3)} = q_2^{(3)}/q_0^{(3)} & A_4^{(1)} = A_4^{(2)} = A_4^{(3)} = 0 \\
A_1^{(2)} = q_1^{(2)}/b_0^{(1)} & A_3^{(0)} = A_{3R}^{(0)} + jA_{3I}^{(0)} & \\
A_1^{(3)} = q_1^{(3)}/q_0^{(3)} & A_3^{(1)} = q_3^{(1)}/q_0^{(1)} & \\
A_2^{(0)} = A_{2R}^{(0)} + jA_{2I}^{(0)} & A_3^{(2)} = q_3^{(2)}/q_0^{(2)} & \\
A_2^{(1)} = q_2^{(1)}/q_0^{(1)} & A_3^{(3)} = q_3^{(3)}/q_0^{(3)} &
\end{array}\right\} \quad (56)$$

where

$$\left.\begin{array}{ll}
q_0^{(1)} = A_{1R}^{(0)} A_1^{(0)} - jA_{2I}^{(0)} & q_2^{(1)} = A_{1R}^{(0)} A_3^{(0)} - jA_{4I}^{(0)} \\
q_0^{(2)} = A_{1R}^{(1)} A_1^{(1)} - jA_{2I}^{(1)} & q_2^{(2)} = A_{1R}^{(1)} A_3^{(1)} \\
q_0^{(3)} = A_{1R}^{(2)} A_1^{(2)} - jA_{2I}^{(2)} & q_2^{(3)} = A_{1R}^{(2)} A_3^{(2)} \\
q_1^{(1)} = A_{1R}^{(0)} A_2^{(0)} - A_{3R}^{(0)} & q_3^{(1)} = A_{1R}^{(0)} A_4^{(0)} \\
q_1^{(2)} = A_{1R}^{(1)} A_2^{(1)} - A_3^{(1)} & \\
q_1^{(3)} = A_{1R}^{(2)} A_2^{(2)} - A_{3R}^{(2)} & q_3^{(2)} = q_3^{(3)} = 0
\end{array}\right\} \quad (57)$$

Then, the necessary and sufficient stability conditions are shown by

$$A_{1R}^{(0)} > 0, \quad A_{1R}^{(1)} > 0, \quad A_{1R}^{(2)} > 0, \quad A_{1R}^{(3)} > 0 \qquad (58)$$

Hence, if the actual rocket, *i.e.* the aerodynamic coefficients, structural characteristics, flight conditions *e.t.c.* are given, we will be able to calculate the critical speed of rocket at the expence of much time and effort. But it is not expected to get the analytical representation for the critical speeds by the iteration method mentioned above. Therefore an approximate analysis for the critical speeds will be presented as follows:

The approximate analytical method consists of three procedures: 1st procedure is approximate analysis of the critical speed under the assuption of spinning rate P is much smaller than bending circular frequency ω_B, 2nd procedure is the calculation of critical speed by digital computer for many examples and comparing these results with the analytical results obtained by 1st procedure, the applicable limits of the analytical results for the value of P/ω_B can be estimated. The last procedure is the set up of empirical formulas for the critical speed in the whole $P \sim U$ plane comparing the analytical results of this section with the results by digital computer and the results of many special cases treated in previous sections.

(1) Approximate Analysis of Critical Speed under the Assumption $P/\omega_B \ll 1$.

Under the assumption that the spinning rate P is much smaller than bending frequency ω_B, the critical speeds of rocket can be represented in an analytical formula by simplifying the frequency equation appropriately. The applicable limits of such the critical speed are not determined generally, but these will be estimated later by the comparison with the results of digital computer.

In previous section, for nonspinning rocket of elastic body, it was found that two critical speeds exist; one is the divergence speed and the other is the flutter speed. From the standpoint of the value of frequency of motion at each critical speed, the former is zero and the latter is the order of bending frequency. And also, from the standpoint of the value of the critical speed, in generally, the value of latter, which does not always exist, is much greater than the value of the former. Hence, at first, we will consider the divergence speed as the critical speed of dynamic stability of rocket. As far as considering the divergence phenomenon, even if the rocket is spinning, the frequency of motion at divergence speed may be considered as the order of the spinning rate. In other words, the imaginary part of λ, complex root of frequency equation, is the order of P and much smaller than the value of bending frequency. In accordance with this assumption, the neglections of higher order of λ, comparing with the lower order of λ, may be allowed as follows:

$$A_2\lambda^2 \gg A_3\lambda^3, \qquad A_2\lambda^2 \gg \lambda^4 \tag{59}$$

Hence the frequency equation is simplified as following quadratic equation with complex coefficients.

$$(c_2+jd_2)\lambda^2+(c_1+jd_1)\lambda+c_0+jd_0=0 \tag{60}$$

where

$$\left.\begin{aligned}
c_2&=\bar{\omega}_B^2-6\bar{P}^2-\mu\left(\sum C_{Li}f_if_i'+\frac{\sum C_{mi}}{k^2}\right)\\
d_2&=\mu\bar{P}\left(2\sum C_{Li}f_i^2+\frac{k_m}{k^2}+3\sum C_{Li}+3\frac{\sum C_{mi}d_i}{k^2}\right)\\
c_1&=\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)(\bar{\omega}_B^2-3\bar{P}^2)-\mu\bar{P}^2\left(2\frac{k_m}{k^2}+\sum C_{Li}f_i^2\right)\\
&\quad-\mu^2\left\{\sum C_{Li}f_if_i'\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)-\sum C_{Li}f_if_i'\left(\frac{\sum C_{mi}f_i'}{k^2}+\sum C_{Li}f_i'\right)\right.\\
&\quad\left.+\sum C_mf_i\frac{\sum C_{mi}f_i'}{k^2}-\frac{\sum C_{mi}}{k^2}\sum C_{Li}f_i^2\right\}\\
d_1&=2\bar{P}\left\{\bar{\omega}_B^2-\mu\left(\sum C_{Li}f_if_i'+\frac{\sum C_{mi}}{k^2}\right)-2\bar{P}^2\right\}\\
c_0&=\left(\mu\frac{\sum C_{mi}}{k^2}+\bar{P}^2\right)(\bar{\omega}_B^2-\bar{P}^2-\mu\sum C_{Li}f_if_i')-\mu^2\frac{\sum C_{mi}f_i'}{k^2}\sum C_{Li}f_i\\
d_0&=\mu\bar{P}\left\{(\bar{\omega}_B^2-\bar{P}^2)\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}+\frac{k_m}{k^2}\right)\right.\\
&\quad-\mu\sum C_{Li}f_if_i'\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}+\frac{k_m}{k^2}\right)+\mu\frac{\sum C_{mi}f_i}{k^2}\sum C_{mi}f_i'\\
&\quad\left.+\mu\sum C_{Li}f_i\sum C_{Li}f_i'\right\}
\end{aligned}\right\} \tag{61}$$

Then, the necessary and sufficient stability conditions are given by

$$\left.\begin{aligned}
&(\mathrm{a})\quad c'=\frac{c_1c_2+d_1d_2}{c_2^2+d_2^2}>0\\
&(\mathrm{b})\quad c''=\left\{\frac{c_1c_2+d_1d_2}{c_2^2+d_2^2}\right\}^2\left\{\frac{c_0c_2+d_0d_2}{c_2^2+d_2^2}\right\}\\
&\qquad+\left\{\frac{c_1c_2+d_1d_2}{c_2^2+d_2^2}\right\}\left\{\frac{d_1c_2-c_1d_2}{c_2^2+d_2^2}\right\}\left\{\frac{d_0c_2-c_0d_2}{c_2^2+d_2^2}\right\}-\left\{\frac{d_0c_2-c_0d_2}{c_2^2+d_2^2}\right\}^2>0
\end{aligned}\right\}\quad(62)$$

(a) Inspection of condition $c'>0$;

The dominator of c' may be considered as to be positive, unless the spinning rate becomes too large, such as the order of bending frequency, hence in the limitation of $P/\omega_B \ll 1$, the dominator of c' is positive. Then the stability criteria are determined from the condition $c_1c_2+d_1d_2>0$. Neglecting the terms of higher order of μ and P/ω_B, the critical speed is obtained approximately as follows:

$$\left\{\frac{U_{cr}}{\omega_B l}\right\}^2=\frac{k_1}{k_2}\left\{1+\left(\frac{P}{\omega_B}\right)^2(2K_2-3k_1-N)/k_1\right\}$$

where

$$\left.\begin{aligned}
&k_1=\mu\left\{\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right\}\\
&k_2=\mu^2\left\{\sum C_{Li}f_if_i'\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)-\sum C_{Li}f_i\left(\frac{\sum C_{mi}d_i}{k^2}+\sum C_{Li}f_i'\right)\right.\\
&\qquad\left.+\sum C_{mi}f_i\frac{\sum C_{mi}f_i'}{k_2}-\frac{\sum C_{mi}}{k^2}\sum C_{Li}f_i^2\right\}\\
&K_2=\mu\left\{2\sum C_{Li}f_i^2+\frac{k_m}{k^2}+3\sum C_{Li}+3\frac{\sum C_{mi}d_i}{k^2}\right\}\\
&N=\mu\left\{\sum C_{Li}f_i^2+2\frac{k_m}{k^2}\right\}
\end{aligned}\right\}\quad(63)$$

In the statically stable rocket k_1 is always positive, and $\{1+(P/\omega_B)^2(2K_2-3k_1-N)/k_i\}$ is positive, unless the spinning rate becomes extremely large. Then, if k_2 is positive, the critical speed exists, and if it is negative, the critical speed does not exist. According to the numerical calculation about many examples of actual rocket, k_2 may be considered to be negative, hence no critical speed determined from this condition exists practically.

(b) Inspection of condition $c''>0$;

In order to simplify the expression for calculation of c'', the following symboles are preliminarily introduced.

$$\left.\begin{aligned}
&c_2=\bar{\omega}_B^2-6\bar{P}^2-Q-E\\
&d_2=\bar{P}K_2\\
&c_1=k_1(\bar{\omega}_B^2-3\bar{P}^2)-N\bar{P}^2-k_2\\
&d_1=2\bar{P}(\bar{\omega}_B^2-2\bar{P}^2-Q-E)\\
&c_0=-(E+\bar{P}^2)(\bar{\omega}_B^2-\bar{P}^2-Q)-k_0\\
&d_0=\bar{P}k_1(1+\alpha)(\bar{\omega}_B^2-\bar{P}^2-Q)+\bar{P}K_0
\end{aligned}\right|$$

where

$$\left.\begin{aligned}
&Q=\mu\sum C_{Li}f_i f_i', \qquad E=\mu\frac{\sum C_{mi}}{k^2}\\
&k_0=\mu^2\frac{\sum C_{mi}f_i'}{k^2}\sum C_{Li}f_i\\
&\alpha=\frac{\dfrac{k_m}{k^2}}{\sum C_{Li}+\dfrac{\sum C_{mi}d_i}{k^2}}\\
&K_0=\mu^2\left\{\frac{\sum C_{mi}f_i'}{k^2}\sum C_{mi}f_i+\sum C_{Li}\sum C_{Li}f_i'\right\}
\end{aligned}\right\} \tag{64}$$

In the statically stable rocket, c_2 is always considered to be positive, so the condition $c''>0$, can be rewritten as follows:

$$\left.\begin{aligned}
&\frac{c_0}{c_2}+\frac{\dfrac{d_0}{c_2}-\dfrac{c_0}{c_2}\cdot\dfrac{d_2}{c_2}}{\dfrac{c_1}{c_2}+\dfrac{d_1}{c_2}\cdot\dfrac{d_1}{c_2}}\cdot\frac{d_1}{c_2}-\left\{\frac{\dfrac{d_0}{c_2}-\dfrac{c_0}{c_2}\cdot\dfrac{d_2}{c_2}}{\dfrac{c_1}{c_2}+\dfrac{d_1}{c_2}\cdot\dfrac{d_2}{c_2}}\right\}>0\\
&\text{where}\\
&\frac{c_0}{c_2}=-E\left\{1+\frac{4\bar{P}^2}{c_2}+\frac{E+\dfrac{k_0}{E}}{c_2}\right\}-\bar{P}^2\\
&\frac{d_0}{c_2}=k_1(1+\alpha)\bar{P}\left\{1+\frac{5\bar{P}^2}{c_2}+\frac{E+\dfrac{K_0}{k_1(1+\alpha)}}{c_2}\right\}\\
&\frac{c_1}{c_2}=k_1\left\{1+\frac{3\dfrac{N}{k_1}}{c_2}\bar{P}^2+\frac{Q+E-\dfrac{k_2}{k_1}}{c_2}\right\}\\
&\frac{d_1}{c_2}=2\bar{P}\left\{1+\frac{4\bar{P}^2}{c_2}\right\}\\
&\frac{d_2}{c_2}=\bar{P}\frac{K_2}{c_2}
\end{aligned}\right\} \tag{65}$$

Neglecting the terms of higher order of (P/ω_B), Eq. (65) is written as follows:

$$-E\left\{1+\frac{E+\dfrac{k_0}{E}}{c_2}\right\}-\bar{P}^2\alpha^2+\frac{2\bar{P}^2}{c_2}\left\{E+\alpha\left(-\frac{k_2}{k_1}+Q-\frac{K_0}{k_1}-\frac{EK_2}{k_1}\right)\right\}>0 \tag{66}$$

From Eq. (66), the following inequality, 4th power of $\bar{\omega}_B$, is derived:

$$\left.\begin{aligned}
&\bar{\omega}_B^4(1-6\beta)\beta\alpha^2+\bar{\omega}_B^2\{E-\beta(6E-2K)\}-EQ+k_0<0\\
&\text{where}\\
&\left(\frac{P}{\omega_B}\right)^2=\beta
\end{aligned}\right\} \tag{67}$$

$$K=E+\alpha\left(-\frac{k_2}{k_1}+Q-\frac{K_0}{k_1}-\frac{EK_2}{k_1}\right)$$

Then, the stability criteria are given by

$$\left.\begin{array}{ll}\bar{\omega}^2_{B(-)}<\dfrac{-\{E-\beta(6E-2K)\}}{\beta\alpha^2(1-6\beta)}-\dfrac{k_0-EQ}{E-\beta(6E-2K)} & \text{(a)}\\ \bar{\omega}^2_{B(+)}>\dfrac{k_0-EQ}{E-\beta(6E-2K)} & \text{(b)}\end{array}\right\} \tag{68}$$

where $\bar{\omega}^2_{B(-)}$ is the smaller one of the two roots of Eq. (67)=0, and $\bar{\omega}^2_{B(+)}$ is the larger one.

Next, the detailed discussion about the stability criteria, shown by Eq. (68), will be made.

(i) Stability criteria $\bar{\omega}^2<\bar{\omega}^2_{B(+)}$:

Introducing the critical reduced frequency in the case of rigid body $(U_{cr}/Pl)_{\omega_B=\infty}$, Eq. (68)-(b) is written as follows:

$$\left\{\frac{U}{Pl}\right\}^2>\left\{\frac{U_{cr}}{Pl}\right\}^2_{\omega_B=\infty}\left\{1+2\left(\frac{P}{\omega_B}\right)^2\frac{K}{E}\right\}$$

or

$$\left\{\frac{U_{cr}}{Pl}\right\}=\left\{\frac{U_{cr}}{Pl}\right\}_{\omega_B=\infty}\left\{1+\left(\frac{P}{\omega_B}\right)^2(1+\alpha G)\right\}$$

where

$$G=\frac{\left[\begin{array}{l}2\sum C_{Li}f_if_i'\left(\sum C_{Li}+\dfrac{\sum C_{mi}d_i}{k_2}\right)-\left(\dfrac{\sum C_{mi}f_i}{k_2}\sum C_{Li}f_i'\right.\\ \left.+2\sum C_{Li}f_i\sum C_{Li}f_i'\right)-\dfrac{\sum C_{mi}}{k^2}\left(3\sum C_{Li}f_i^2+\dfrac{k_m}{k^2}\right)\end{array}\right]}{\dfrac{\sum C_{mi}}{k^2}\left(\sum C_{Li}+\dfrac{\sum C_{mi}d_i}{k^2}\right)} \tag{69}$$

This means physically that (U_{cr}/Pl) is one of the critical reduced velocities, at which the motion of rocket, which is principally rigid body motion, becomes unstable, and that how the flexural rigitity affect the reduced velocity of rigid body.

(ii) Stability criteria $\bar{\omega}^2_B>\bar{\omega}^2_{B(-)}$:

Introducing the critical reduced frequency in the case of nonspinning elastic rocket $\{U_{cr}/\omega_Bl\}_{P=0}$, Eq. (68)-(a) is written as follows:

$$\left\{\frac{U}{\omega_Bl}\right\}^2<\left\{\frac{U_{cr}}{\omega_Bl}\right\}^2_{P_0}\cdot\left\{1-2\left(\frac{P}{\omega_B}\right)^2\left(3-\frac{K}{E}\right)\right\}$$

or

$$\left\{\frac{U}{\omega_Bl}\right\}<\left\{\frac{U_{cr}}{\omega_Bl}\right\}_{P_0}\left\{1-\left(\frac{P}{\omega_B}\right)^2(2+\alpha G)\right\} \tag{70}$$

This means physically that U/ω_Bl is one of the critical reduced velocities, at which the motion of rocket, which is principally the bending vibration, becomes unstable, and that how the spinning rate affect the nonspinning reduced frequency.

Next, the frequency at the critical speed will be discussed. In the analysis of stability criteria of spinning rocket of elastic body mentioned above, for the simplification of analysis, an assumption for frequency at critical speed is introduced, *i.e.*, it will be an order of spinning rate. Then, now the propriety of this assumption will be discussed. Two roots of quadratic equation with complex coefficients are given by

$$\left.\begin{aligned}&\lambda^2+(c_1'+jd_1')\lambda+(c_0'+jd_0')=0\\&2R\{\lambda\}=-c_1'\pm\frac{1}{\sqrt{2}}\{\sqrt{(c_1'^2-d_1'^2-4c_0')^2+4(c_1'd_1'-2d_0')^2}+(c_1'^2-d_1'^2-4c_0')\}^{1/2}\\&2I\{\lambda\}=-d_1'\pm\frac{1}{\sqrt{2}}\{\sqrt{(c_1'^2-d_1'^2-4c_0')^2+4(c_1'd_1'-2d_0')^2}-(c_1'^2-d_1'^2-4c_0')\}^{1/2}\end{aligned}\right\}\quad(71)$$

At the critical speed, $R(\lambda)=0$, *i.e.*, $c_1'^2c_0'+c_1'd_1'd_0'-d_0'^2=0$, hence under this condition Eq. (71) is rewritten as follows:

$$2I\{\lambda\}=-d_1'+\sqrt{d_1'^2+4c_0'}\qquad(72)$$

Only the positive sign is used corresponding to the condition $R(\lambda)=0$. Comparing Eq. (71) with Eq. (60),

$$\left.\begin{aligned}d_1'&=\frac{d_1c_2-c_1d_2}{c_2^2+d_2^2}\doteqdot\frac{d_1}{c_2}\\c_0'&=\frac{c_0c_2+d_0d_2}{c_2^2+d_2^2}\doteqdot\frac{c_0}{c_2}\end{aligned}\right\}\qquad(73)$$

Hence

$$I\{\lambda\}=-\frac{d_1}{2c_2}+\sqrt{\left(\frac{d_1}{2c_2}\right)^2+\frac{c_0}{c_2}}\qquad(74)$$

Substituting Eq. (64) into Eq. (74), and introducing the condition at critical speed, it is easily found that

$$I\{\lambda\}\doteqdot-\bar{P}\qquad(75)$$

This means that the frequency at critical speed is the order of spinning rate. Substituting this result into the quartic frequency equation, and the propriety of assumption, Eq. (59), will be discussed. At critical speed,

$$R\{\lambda\}=0,\qquad I\{\lambda\}=\pm\bar{P}$$

Hence

$$\begin{aligned}\frac{\lambda^4}{A_2\lambda^2}&=\frac{-\bar{P}^4}{\bar{P}^2\bar{\omega}_B^2+j\bar{P}^3f_2}=\frac{-\bar{P}^2}{\bar{\omega}_B^2+j\bar{P}f_2}\\&=-\frac{\bar{\omega}_B^2\bar{P}^2}{\bar{\omega}_B^4+\bar{P}^2f_2^2}+j\frac{f_2\bar{P}^3}{\bar{\omega}_B^4+\bar{P}^2f_2^2}\\\frac{A_3\lambda^3}{A_2\lambda^2}&=\frac{4\bar{P}^4+jf_1\bar{P}^3}{\bar{P}^2\bar{\omega}_B^2+j\bar{P}^2f_2}=\frac{4\bar{P}^2+jf_1\bar{P}}{\bar{\omega}_B^2+jf_2\bar{P}}\end{aligned}$$

$$\left.\begin{aligned}&=\frac{4\bar{\omega}_B^2\bar{P}^2+f_1f_2\bar{P}^2}{\bar{\omega}_B^4+f_2{}^2\bar{P}^2}+j\frac{f_1\bar{\omega}_B^2\bar{P}-4f_2\bar{P}^3}{\bar{\omega}_B^4+f_2{}^2\bar{P}^2}\\&\text{where}\\&f_1=\mu\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}+\sum C_{Li}f_i^2\right)\\&f_2=\mu\left\{2\sum C_{Li}f_i^2+\frac{k_m}{k_2}3\sum C_{Li}+3\frac{\sum C_{mi}d_i}{k^2}\right\}\end{aligned}\right\} \quad (76)$$

As $\bar{\omega}_B \gg f_1$, f_2 and $\bar{P}$ then both the real and imajinary parts of Eq. (76) are much smaller than unity. Hence the neglection of terms of higer order of λ *i.e.*, λ^4 and $A_3\lambda^3$, will be allowed in the approximate analysis as mentioned above.

The appilacable limit of the results by this approximate analysis will be estimated later comparing with the numerical results by digital computer.

Discussion of Stability of Spinning Rocket of Elastic Body

From the above analysis, the following results may be stated as applied to a spinning rocket at low speed:

1) In this case, there are two critical reduced velocities, *i.e.*, $U_{cr}/\omega_B l$ and U_{cr}/Pl, and the former shows how the spinning rate affects the critical reduced velocity of nonspinning rocket and the latter shows how the flexural rigidity affects the critical reduced velocity of spinning rocket of rigid body. Physically, this means, that the main motion of rocket at critical speed is bending vibration in the former and is rigid body motion in the latter.

2) Various characteristics affect $\{U_{cr}/\omega_B l\}_{P=0}$ and $\{U_{cr}/Pl\}_{\omega_B=\infty}$ are discussed in the previous sections, hence the correction terms of critical speeds are discussed. $\{1-(P/\omega_B)^2(2+\alpha G)\}$ is the correction term for $\{U_{cr}/\omega_B l\}_{P=0}$ by the existence of spinning motion, and can be separated into two part, *i.e.* the one is $\{1-2(P/\omega_B)^2\}$, representing the effect of centrifugal force due to spinning and the other is $\{1-(P/\omega_B)^2\alpha G\}$, representing the effect of Magnus effect combined with the bending motin of rocket body. The ratio of these effects is given by $\alpha G/2$, whose value will be considered to be the order of $0\sim0.1$ from the results of numerical calculation about many actual rocket. Hence it can be said that the main part of correction term is one due to the centrifugal force.

3) $\{1+(P/\omega_B)^2(1+\alpha G)\}$ is the correction term for $\{U_{cr}/Pl\}_{\omega_B=\infty}$, and if the Magnus effect is neglected, this value is unity. (Neglecting the Magnus effect, $\{U_{cr}/Pl\}_{\omega_B=\infty}$ does not exist)

The ratio of the above two correcting values is given by αG and can be considered to be the order of less than 0.1, Hence, U_{cr}/Pl is roughly approximated by $\{U_{cr}/Pl\}_{\omega_B=\infty}$ in the limitation $P\ll\omega_B$, and $\{U_{cr}/\omega_B l\}$ is roughly approximated by $\{U_{cr}/\omega_B l\}_{P=0}\{1-2(P/\omega_B)^2\}$ introducing only the effect of centrifugal force in the limitation $P\ll\omega_B$.

Next, the effect of spinning rate on the flutter speed will be discussed. As shown in the previous section, the frequency of motion at flutter speed is the

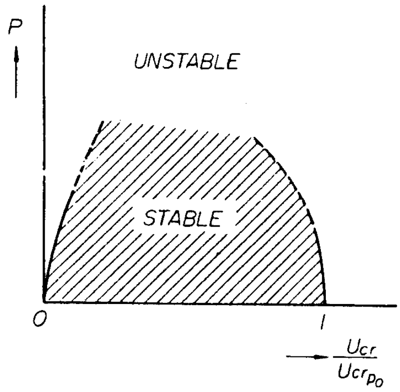


FIGURE 6. Stability Boundary of Elastic Rocket at Low Spinning Rate.

order of bending frequency and it is much greater than that of motion at divergence speed. Hence, as far as the high frequency phonomena are taken into consideration, we can rewrite the quartic frequency equation as follows:

$$\left.\begin{aligned}
&\lambda^4+A_3\lambda^3+A_2\lambda^2+A_1\lambda+A_0=(\lambda^2+B_1\lambda+B_0)(\lambda^2+D_1\lambda+D_0)+E_1\lambda+E_0\\
&\quad=F(\lambda)\cdot D(\lambda)+E_1\lambda+E_0=0\\
&\text{where}\\
&B_1=\left(A_3-\frac{A_1}{A_2}-\frac{A_0A_3^2}{A_1A_2}\right)\\
&B_2=A_2-\frac{A_0}{A_2}-\left(A_3-\frac{A_1}{A_2}-\frac{A_0A_3}{A_1A_2}\right)\left(\frac{A_1}{A_2}+\frac{A_0A_3^2}{A_1A_2}\right)\\
&D_1=\frac{A_1}{A_2}+\frac{A_0A_3^2}{A_1A_2}\\
&D_0=\frac{A_0}{A_2}\\
&E_1=-\frac{A_0}{A_2}A_3+2\frac{A_0}{A_2}\left(\frac{A_1}{A_2}+\frac{A_0A_3^2}{A_1A_2}\right)-\frac{A_0A_3^2}{A_1}+\left(A_3-\frac{A_1}{A_2}-\frac{A_0A_3^2}{A_1A_2}\right)\left(\frac{A_1}{A_2}+\frac{A_0A_3^2}{A_1A^2}\right)^2\\
&E_0=\frac{A_0^2}{A_2}+\frac{A_0}{A_2}\left(A_3-\frac{A_1}{A_2}-\frac{A_0A_3^2}{A_1A_2}\right)\left(\frac{A_1}{A_2}+\frac{A_0A_3^2}{A_0A_2}\right)
\end{aligned}\right\}\tag{77}$$

$F(\lambda)$ is the frequency equation for the motion of high frequency (equivalent to the short period motion of airplane), and $D(\lambda)$ is the frequency equation for the motion of low frequency (equivalent to the phugoid motion of airplane), and so the stability criteria for flutter phenomena will be derived from $F(\lambda)=0$.

The stability condition for $F(\lambda)=0$ is given by

$$\left.\begin{aligned}
&B_{1R}^2B_{0R}+B_{1R}B_{1I}B_{0I}-B_{0I}^2>0\\
&B_{1R}>0\\
&\text{where}\\
&B_{1R}=R\{B_1\}=R\left\{A_3-\frac{A_1}{A_2}-\frac{A_0A_3^2}{A_1A_2}\right\}
\end{aligned}\right\}\tag{78}$$

$$\left.\begin{aligned}B_{1I}&=I\{B_1\}=I\left\{A_3-\frac{A_1}{A_2}-\frac{A_0A_3^2}{A_1A_2}\right\}\\B_{0R}&=R\{B_0\}\doteqdot R\{A_2\}\\B_{0I}&=I\{B_0\}\doteqdot I\{A_2\}\end{aligned}\right)$$

Eq. (78)-(a) is equivalent to the condition $A_2>0$ in the case of nonspinning elastic rocket {Eq. (27)}, then this condition will be satisfied in the range of small spinning rate. Hence the stability criteria for flutter phonomena will be derived from the condition $B_{1R}>0$.

$$\left.\begin{aligned}&\frac{B_{1R}}{\mu^2\sum C_{Li}f_i^2\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)}=\bar{\omega}_B^4+\mu\boldsymbol{F}_3\bar{\omega}_B^2+\mu^2\boldsymbol{F}_4\\&\qquad+4\bar{P}^2\left\{\bar{\omega}_B^2-\mu\left(\sum C_{Li}f_if_i'+\frac{\sum C_{mi}}{k^2}\right)\right\}^2\Big/\mu^2\sum C_{Li}f_i^2\left(\sum C_{Li}+\frac{\sum C_{mi}d_i}{k^2}\right)\\&\quad=J(\bar{\omega}_B^2)+\bar{P}^2H(\bar{\omega}_B^2)=0\end{aligned}\right\}\tag{79}$$

where $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$ are given by Eq. (30).

By the same method in the case of nonspinning elastic rocket, flutter speed can be calculated from Eq. (79).

Example 2.

As an example, the flutter speed of the actual rocket, shown in Fig. 2, will be calculated using the results obtained in this section.

$C_{LN}=2$, $\quad C_{LT}=8$, $\quad \mu=1\times10^{-3}$

$1/k^2=10$,

f_i, f_i'; shown in Fig. 3,

C.G.; midpoint of rocket,

$d_N=0.4$; $\overline{\text{C. G., Cp}_{\text{nose}}}$,

$d_T=-0.45\sim-0.20$; $\overline{\text{C. G., Cp}_{\text{tail}}}$,

Calculation results are shown in Figs. 7 and 8.

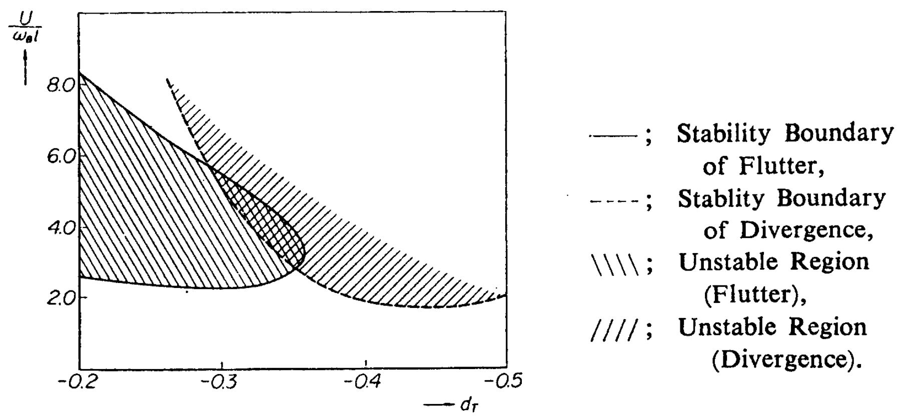
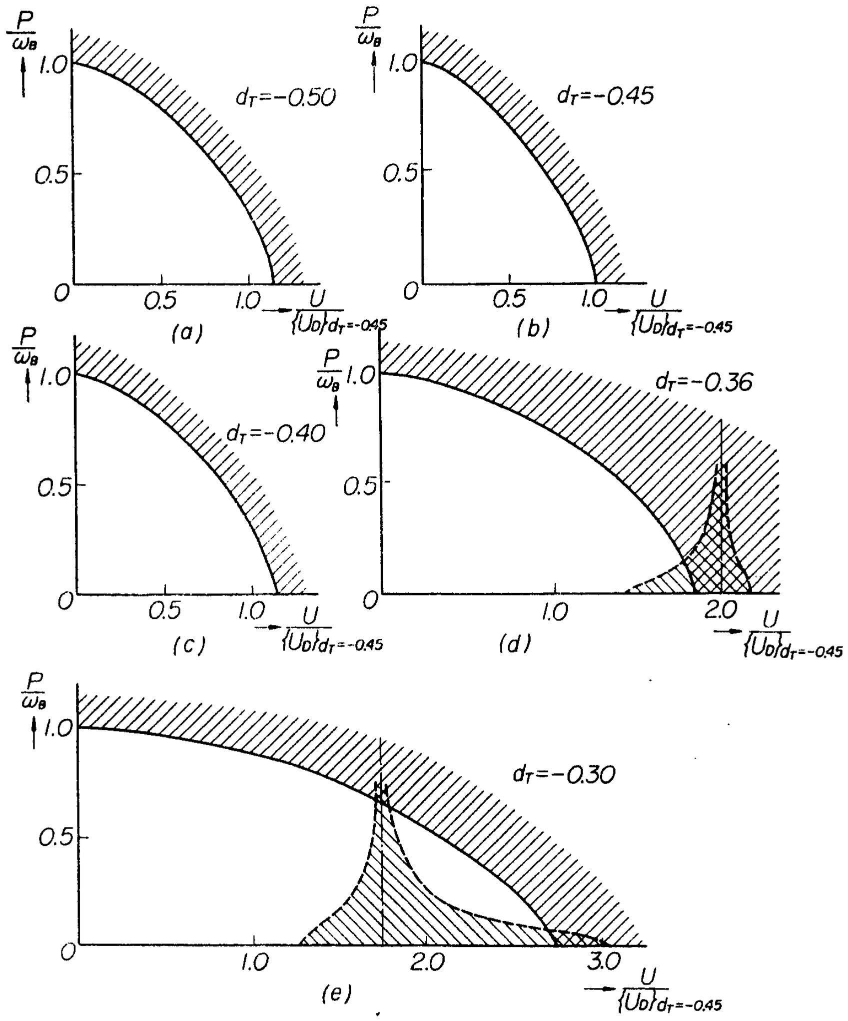


FIGURE 7. Stability Boundary of Nonspinning Elastic Rocket.



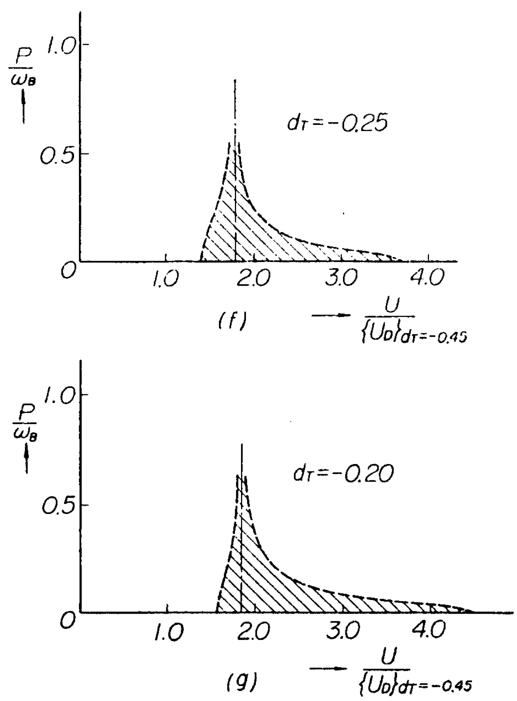


FIGURE 8. Stability Boundary of Spinning Rocket.

- - - - ; Stability Boundary of Flutter,
——; Stability Boundary of Divergence,
\\\\\\\\; Unstable Region (Flutter),
////; Unstable Region (Divergence).

7. Dynamic Stability of Rocket

(1) Stability Analysis by Digital Computation.

The dynamic stability of nonspinning rocket of elastic body is analyzed exactly, but taking into account the spinning motion, only the approximate analysis can be carried out under the limitation of $P/\omega_B \ll 1$. The exact analysis of dynamic stability of this case is carried out only by digital computational method.

The frequency equation, corresponding to the equation of motion, is given by the following polynomial with real coefficients.

$$\lambda^8+A_7\lambda^7+A_6\lambda^6+A_5\lambda^5+A_4\lambda^4+A_3\lambda^3+A_2\lambda^2+A_1\lambda+A_0=0 \tag{80}$$

The coefficients $A_0\sim A_7$ are functions of structural characteristics, mass distribution, aerodynamic coefficients, and flight condition *e.t.c.*

$$A_0\sim A_7=g(\mu, k, \sum C_{Li}, \sum C_{mi}, f_i, f_i' \cdots, \bar{\omega}_B, \bar{P}) \tag{81}$$

If the actual rocket is given, then μ, k, $\sum C_{Li}$, $\sum C_{mi}$, f_i, f_i', ω_B are determined calculatively or experimentally. Therefore, $A_0\sim A_7$ are functions of these known factors and another unknown factor U. Giving the spinning rate P as parameter, and searching for a minimum value of λ at which the real part of root of Eq. (80) becomes to zero, then the stability boundaries, *i.e.*, P_{cr}/P_{crU_0} *v.s.* U_{cr}/U_{crP_0} curves are obtained.

As an example, the computational results about a rocket shown in Fig. 2, whose characteristics are given as follows, are shown in Fig. 9.

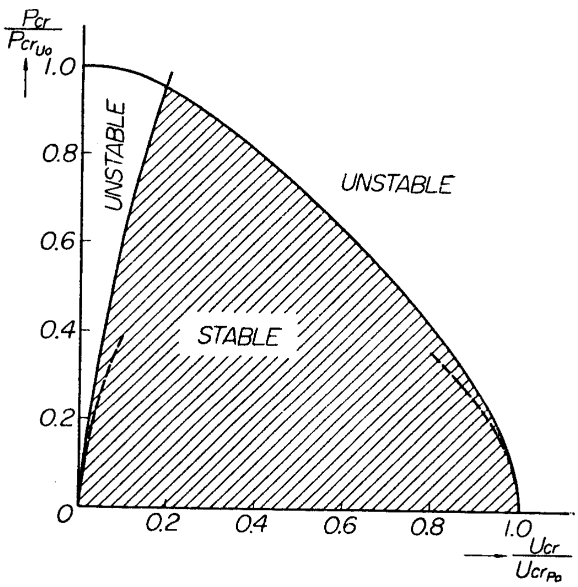


FIGURE 9. Stability Boundary of Spinning Elastic Rocket.

——; Digital Computer,

----; Approximate Analysis.

$$\left.\begin{array}{ll} \mu=1\times10^{-3} & \sum C_{Li}f_i^2=5.76 \\ \sum C_{Li}=10.0 & \sum C_{Li}f_i'=-55.4 \\ \sum C_{mi}=-1.90 & \sum C_{mi}f_i=-0.60 \\ \sum C_{mi}d_i=1.39 & \sum C_{mi}f_i'=37.8 \\ \sum C_{Li}f_i=7.20 & \sum C_{Li}f_if_i'=-19.4 \\ k^2=0.10 & \end{array}\right\} \tag{82}$$

(2) Empirical Formula for the Critical Speed of Spinning Rocket

As the special cases of dynamic stability of rocket, the critical values of non-spinning rocket $\{U_{cr}\}_{P=0}$, rigid body $\{U_{cr}\}_{\omega_B=\infty}$ and spinning rocket at low speed $\{P_{cr}\}_{U_{low}}$ and $\{U_{cr}\}_{U_{low}}$ were obatined in tne previous sections. In this section, the critical value of spinning rocket under the limitation of $P/\omega_B \ll 1$ is approximately calculated and for some examples of rocket digital computational results are obtained under no limitation. Combining with these results, the empirical formulas representing the critical values of rocket in the form of U_{cr} *v.s.* P_{cr}, can be obtained. Two stability boundries exist, as shown above, *i.e.* at the critical speed, one is that the main motion of rocket is bending vibration, which is designated as the stability boundry of A type, and the other is that the main motion of rocket is rigid body motion, which is designated as the stability boundary of B type.

(a) Emprical formula for the stability boundary of A type

In this type, the main motion of rocket at the critical speed is bending vibration. In the extreme case where P tends to zero, the critical value is shown by Eq. (29), and where U tends to zero, the critical value is shown by Eq. (47) and in the case of $U \neq 0$, $P \neq 0$ the critical values are given by Eqs. (69) and (70). The existence of spinning rate P mainly affectes the critical value by the form of centrifugal forces, and the other effect, for example the Magnus effect, is much smaller than that of centrifugal forces. Hence, in the empirical formula for the stability boundary roughly approximated, the Magnus effect can be neglected. And now, as trial formula we may choose

$$\left\{\frac{P_{cr}}{P_{crU_0}}\right\}^n+\left\{\frac{U_{cr}}{U_{crP_0}}\right\}^n=1 \tag{83}$$

From the analytical and digital computation results, fairly good approximation can be obtained by putting $n=3/2$ as shown in Fig. 10, then using the equation,

$$\left\{\frac{P_{cr}}{P_{crU_0}}\right\}^{3/2}+\left\{\frac{U_{cr}}{U_{crP_0}}\right\}^{3/2}=1 \tag{84}$$

we can easily calculate the critical value of A type approximately.

(b) Empirical formula for the stability boundary of B type

In this type, the motion of rocket at the critical speed is a rigid body motion. In the extreme case where $\omega_B=\infty$, the critical value is shown by Eq. (40). For

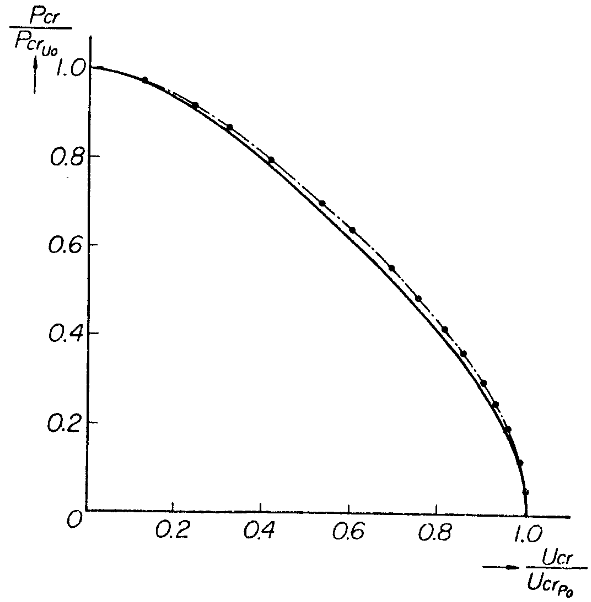


FIGURE 10.

–·–·–; Digital Computer (Not Incude Magnus Effect),
——; Digital Computer (Include Magnus Effects),
····; Empiricrl Formula.

this case, if we choose the following formula

$$\frac{U_{cr}}{\{U_{cr}\}_{\omega_B=\infty}}=1+K\left(\frac{P}{\omega_B}\right)^2 \tag{85}$$

and the estimation of K is order of 0.1~0.2, fairly good approximation can be obtained.

Further Discussion of the Stability of Spinning Rocket of Elastic Body

From the approximate analytical results and the digital computational results, it has been found that two sorts of critical speed will exist in spinning rocket of elastic body. However, the frequency equation, which is led from the general equations of motion (23), is a quartic polynamial with complex coefficients, then four pairs of complex roots will generally exist, and this means that corresponding to these values four sorts of motion of rocket will exist. It is necessary to investigate two other types of motion, but it is hardly expected to solve the quartic equation with complex coefficients. Hence, in the following discussion, the chracteristis of four motions of rocket will be qualitatively explained.

Generally, four stability boundaries corresponding to four pairs of roots can exist, and these will be classified into two sorts; *i.e.*

Stability boundaries (S.B.) of 1st kind; at these boundaries, the main motion of rocket is rigid body motion.

Stability boundaries (S.B.) of 2nd kind; at these boundaries, the main motion of rocket is bending vibration.

Furthermore, these will be devided into four classes as follows,

$$\left.\begin{array}{ll} \text{Stability boundaries of 1st kind} & \left\{\begin{array}{l}\text{Stability boundary of A type}\\ \text{Stability boundary of C type}\end{array}\right. \\ \text{Stability boundaries of 2nd kind} & \left\{\begin{array}{l}\text{Stability boundary of B type}\\ \text{Stability boundary of D type}\end{array}\right. \end{array}\right\} \tag{86}$$

S.B. of 1st kind are S.B. as the rigid body motion affected by the elastic characteristics of rocket. In the extreme case where the flexural rigidity tends to the infinite, S.B. coincide with the results shown by Eq. (40). Generally, there are two S.B. in the case of rigid body, one is oscillatory motion the other is unoscillatory motion, and as far as considering the statically stable rocket, S.B. corresponding to unoscillatory motion cannot exist.

Then, in the case of finite flexural rigidity of rocket, it cannot be considered that the S.B. of C type, which is unoscillatory motion at the critical value, exists. Hence, in the S.B. of 1st kind, only the S.B. of A type, which is oscillatory motion at the critical value, shown by Eq. (69), exist.

Next, we will consider the physical meaning of S.B. of 2nd kind. In the extreme case where the spinning rate tends to zero, the S.B. of B type and D type coincide with the S.B. of divergence and flutter in a narrow sense shown by Eqs. (29) and (32). However, as far as taking into account the spinning rate, unoscillatory motion at critical value, *i.e.*, divergence, cannot exist, then the designations of divergence and flutter are not adequate in this case. But, for the clear distinction of physical meaning of B type and D type, these designations will be often used in the following discussion corresponding to B type and D type respectively.

For the S.B. of B type, as an example shown in Fig. 9, $P_{cr} \sim U_{cr}$ curve are obtained in the whole range of the spinning rate, but the S.B. of D type are not cleared analytically except the case of $P=0$, whose value is given by $\{U_F\}_{P=0}$.

In the case of very low speed, the frequency equation given by Eq. (53) is the quadratic equation with real coefficients, and this means that there is no distinction between flutter and divergence.

From the above considerations, for the 4 types of S.B., physical explanation can be briefly made as follows:

S.B. of 1st kind:

- A type: The main motion of rocket at critical value, which is secondary modified by elastic property, is rigid body motion and the motion is oscillatory.
- C type: In the statically stable rocket, no S.B. of this type exist.

- B type: The main motion of rocket at critical value is elastic bending vibration.
  In the extreme cases of P and U tend to zero, the critical values are $U_{cr}=\{U_D\}_{P=0}$, divergence speed, and $P_{cr}=\omega_B$, whirling speed, respectively.

S.B. of this type always exist.

D type: The main motion of rocket at critical value is elastic bending vibration.
In the extreme case of P tend to zero, the critical values are $U_{cr}=\{U_F\}_{P=0}$, flutter speed.
S.B. of this type do not always exist, and even if it exists, it will be considered that the critical value of this type is much large than that of B type.

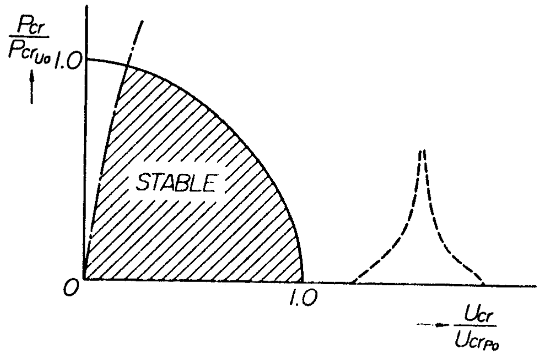


FIGURE 11.
-·-·-; S.B. of A Type,
——; S.B. of B Type,
- - - -; S.B. of D Type.

These relations are schematically shown in Fig. 11. In this figure, the actual stable region is that enclosed by the abscissa and the boundaries of A and B types. If the S.B. of A type and D type do not exist owing to the characteristics of rocket, the value of the stable region is that enclosed by the abscissa, the ordinate and the boundary of B type. If the flutter speed is smaller than that of divergence speed in nonspinning rocket, *i.e.* $\{U_F\}_{P=0}<\{U_D\}_{P=0}$, though such case hardly exists actually, the stable region is that enclosed by the obscissa and the boundaries of A and D types as shown in Figs. 12 and 13.

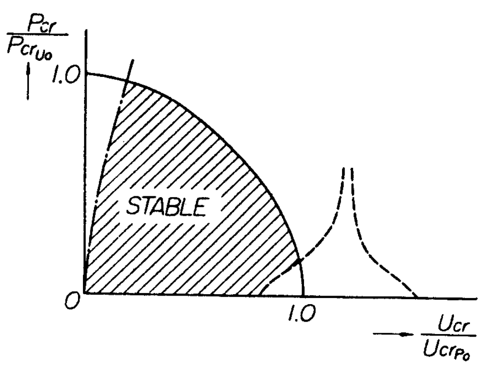


FIGURE 12.
-·-·-; S.B. of A Type,
——; S.B. of B Type,
- - - -; S.B. of D Type.

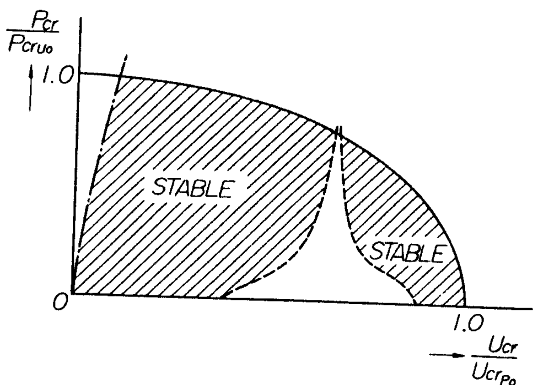


FIGURE 13.
-·-·-; S.B. of A Type,
——; S.B. of B Type,
- - - -; S.B. of D Type.

In the above discussion, the aerodynamic coefficients and the coefficients of Magnus effect are considered to be constant, but actually these coefficients are functions of the velocity and spinning rate. Then, in such case, if the structural characteristics and flight conditions are given and as far as considering the quasi-steady air forces, the stability bonndary P'_{cr} *v.s.* U'_{cr} curve will be obtained under the conditions that the above coefficients are functions of velocity and spinning rate. The primes of P'_{cr} and U'_{cr} mean P_{cr} and U_{cr} in the case of the coefficients, $\sum C_{Li}$, $\sum C_{mi}$, k_m *e.t.c.*, are functions of U and P. Comparing $P'_{cr} \sim U'_{cr}$ curve with $P_{cr}/P_{cr U_0} \sim U_{cr}/U_{cr U_0}$ curve, $P'_{cr}/P'_{cr U_0} \sim U'_{cr}/U'_{cr P_0}$ curve will be easily obtained. In comparison $P'_{cr}/P'_{cr U_0} \sim U'_{cr}/U'_{cr P_0}$ curve with $P_{cr}/P_{cr U_0} \sim U_{cr}/U_{cr P_0}$ curve, the following two cases are considered. First, on the S.B. of B type, the magnitude of modification by $\sum C_{Li}$, $\sum C_{mi}$, k_{mi}, *e.t.c.* is to be considerably small, then the basic form of S.B. of B type is hardly affected, but $U_{cr P_0}$ is much affected by these values. Second, on the S.B. of A type, the modification by these values is considerably small, but $\{U'_{cr}\}_{\omega_E=\infty}$ is much affected. Then, in this case, it will be considered that the form of S.B. is considerably affected. Futhermore, the predominant term deciding the critical value of A type is k_m, wich is small value comparing with the other aerodynamic coefficients, and this means that if more strict evaluation of k_m is made, then the posibility of considerable change of S.B. of A type is remained.

Such analysis about the stability of rigid body is made by D.R. Davis, but about the stability of elastic body, it is expected in future.

CONCLUSION

The general equations of motion for the spinning rocket of elastic body using body axes were presented. The stability criteria under some limitation of spinning rate were approximotely analyzed including the case of rigid body and also of nonspinning elastic rocket, as special cases. Comparing these analytical approximate results with the digital computational results, the applicability of these results was examined as a results, empirical formulas for stability boundaries were also presented and furthermore the physical meanings of stability boundaries of spinning elastic rocket were given.

ACKNOWLEAGEMENT

The author wishes to express his sincere thanks to professors K. Washizu, K. Ikeda, T. Hayashi, M. Yamana, M. Sanuki and M. Hosaka for their kind advice and encouragement throughout this work.

Department of Structures,
Aeronautical Research Institute,
University of Tokyo, Tokyo.
February 25, 1964.

REFERENCES

* K. Ikeda: Jour. Japan Soc. Aero. Engg., Vol. 7, No. 68, 69, 70.
T. Ichikawa: Jour. Japan Soc. Aero. Engg., Vol. 8, No. 72.
B. Tomita: Aero. Res. Inst., Univ. of Tokyo, Rocket Note Vol. 1, No. 19.
** R.E. Bolz: Dynamic Stability of a Missile in Rolling Flight, JAS, Vol. 19, No. 6.
R.A. Davis: The Response of a Bisymmetric Aircraft a Small Combined Ditch, Yaw and Roll Control Actions, J.A.S. Vol. 24, No. 12.
R.A. Dosenberg: On the Flight Dynamics of Slender Special Purpose Aircraft, JAS, Jan., 1952.
[1, 2] C.D. Perkins: Airplane Performance, Stability and Control, John Wiley & Sons, 1958.
W.F. Durand: Aerodynamic Theory, Vol. 5, 1943.
[3, 4] S. Timoshenko: Vibration Problems in Engneering, D. Van Nostrand, 1955.
[5] L. Davis, Jr: Exterior Ballistics of Rocket, pp. 236~239, D. Van Nostrand, 1958.
[6] S.M. Harmon: Stability Derivatives of Thin Rectangular Wing at Supersonic Speeds, NACA TN, No. 1706.
[7] C.H. Murphy Jr: The Prediction of Nonlinear Pitching and Yawing Motion of Symmetric Missiles, JAS, Jul., 1957.
[8] C.H. Murphy Jr: Criterion for the Generalized Dynamic Stability of a Rolling Symmetric Missiles, JAS, Oct., 1957.
[9, 10] R.A. Scanlan: Introduction to the Study of Aircraft Vibration and Flutter, pp. 116~122, Macmillan, 1951.