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Boundary Layer Growth on a Flat Plate in the Presence of a Transverse Magnetic Field*

By

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Summary. Incompressible viscous flow solutions are found for an electrically conducting fluid flow over an infinite flat plate moving impulsively in the presence of a transverse magnetic field.

The equations are solved in the form of the Laplace transform under the boundary conditions taking into account the thickness and the conductivity of the plate and neglecting displacement currents.

It is found that the formulae for the skin friction of the plate include those calculated by Rossow as a special case of magnetic Prandtl number P_m zero, in the two limiting cases of relative conductivity of the fluid to the plate. If the magnetic field or the fluid conductivity is increased, the skin friction is increased in the case of insulating plate and is reduced in the case of conducting plate. The magnetic drag acting in the latter case is found to be finite and is shown to increase the total resistance. The velocity and magnetic fields are calculated for limiting values of mt, and are given in a closed form for the case of $P_m=1$, where m is the magnetic parameter and t is the time.

The stationary field formed after a long time is obtained for arbitrary conductivity of the plate.

1. INTRODUCTION

Much discussion has concerned the use of magnetic field to control the motion of an electrically conducting fluid. As far as the author is aware, there are few examples which satisfy the fundamental equation and the boundary conditions perfectly.

Recently, Rossow [1] has discussed the incompressible boundary layer solutions for the flow over a flat plate in the presence of a uniform magnetic field. He simplifies the fundamental equations by the assumption that " the induced magnetic field is negligible ", and avoids difficulties due to the uncertainty of boundary conditions for the electromagnetic field by physical considerations on two special cases: i) magnetic field fixed relative to the flow, and ii) magnetic field

* Principal part of this paper has been presented at the 8th Internal Congress for Applied Mechanics of Japan (Sept. 1958). Abridged review with additional results has appeared in Supplement of Progress of Theoretical Physics No. 24 (1962) 35. These additional results in an amplified form are given in §7. The skin friction and the magnetic resistance on a thick wall of finite conductivity are newly obtained in closed forms.

(1) V.J. Rossow: N.A.C.A. Tech. Rep. 1358 (1958), also N.A.C.A. TN 3971 (1957).

fixed relative to the plate. Starting from discussions for the impulsive motion of an infinite flat plate which serve to give suggestions for later study, he investigated precisely the steady flow over a semi-infinite flat plate, for small values of magnetic parameter. Except for the infinite values of magnetic drag in the second case, his results seem to be useful at least near the plate if we add appropriate interpretations, though there are some points to be clarified.

One purpose of this paper is to clarify these results, from another view point, relying on the full fundamental equations neglecting displacement currents which are given in §3 and §4. For this purpose, the case of impulsive motion is studied precisely.

According to our solutions, the events are as follows.

At time $t=0$ there are an infinite flat plate, and a transverse magnetic field. Let us move our plate parallel to its plane. If the plate is an insulator it does not disturb the magnetic field at first. But if it is a conductor it brings, so to speak, magnetic lines of force frozen in itself, corresponding to the induction current. This current forms a current sheet within the surface of the perfect conductor.

In the first stage of movement, there is formed a thin viscous boundary layer of the fluid on the surface of the plate. There then appears a component of magnetic field parallel to the plate, which is induced by this fluid motion and also partly by the current in the conducting plate. Then the fluid motion is also disturbed by the force acting on this current. These interactions are transferred to infinity as decaying hydromagnetic waves, which are damped by the complicated action of viscosity and resistance (magnetic viscosity) of the fluid. These waves introduce inflection points in the velocity and the magnetic field profiles. In the last stage, there is left a peculiar kind of quasi-stationary field near the plate which is generally non-uniform except for the case of perfectly conducting plate, §5.

These features are shown quite well in the case of the magnetic Prandtl number $P_m=1$, which allows closed solutions for the velocity and magnetic field, § 6.

If P_m is small or the magnetic viscosity is large corresponding to small conductivity, the induced current in the fluid is so small that our formulae for the skin friction are in accord with those of Rossow in the limit of vanishing P_m. For this small value of P_m, induced magnetic fields extend further away than the viscous boundary layer in which Rossow's velocity profile is shown to afford good approximation. If we calculate the magnetic resistance to the plate as a reaction to the plate by using extrapolated values of the induced magnetic field in this layer (which can be regarded as almost constant), it is obvious that we get infinite values. It may be regarded as a kind of Stokes paradox, similar to that for the flow of low Reynolds number past a two-dimensional body.

2. SYMBOLS.

(x, y, z)	rectangular coordinates
t	time
$\boldsymbol{V}=(u, v, 0)$	fluid velocity
$(U, 0, 0)$	velocity of flat plate
$\boldsymbol{H}=(h, h_y, 0)$	strength of magnetic field
$(0, H, 0)$	strength of magnetic field perpendicular to flat plate
$\boldsymbol{E}=(0, 0, E)$	strength of electric field
$\boldsymbol{J}=(0, 0, j)$	electric current density
ρ	fluid density
P	total pressure=fluid pressure$+\mu \boldsymbol{H}^2/2$
ν	kinematic viscosity
μ	magnetic permeability
σ	electric conductivity
$\kappa=1/(\mu\sigma)$	magnetic viscosity
$\alpha=(\mu/\rho)^{\frac{1}{2}}H$	velocity of Alfvén wave
$P_m=\nu/\kappa=\mu\nu\sigma$	magnetic Prandtl number
$m=\alpha^2/(\nu^{\frac{1}{2}}+\kappa^{\frac{1}{2}})^2$, $m_\kappa=\alpha^2/\kappa$	magnetic parameter
$M=(\sigma/\rho\nu)^{\frac{1}{2}}\mu HL=\alpha L/(\nu\kappa)^{\frac{1}{2}}$	Hartman number
$2L$	thickness of plate
Y or y (in § 7)	$y-L$
subscript 0	quantities for plate

3. FUNDAMENTAL EQUATIONS.

The magnetohydrodynamic equations for an electrically conducting viscous incompressible fluid are given by (in MKS units and conventional notations)

$$\frac{D\boldsymbol{V}}{Dt}=-\frac{1}{\rho}\nabla P+\nu\nabla^2\boldsymbol{V}+\frac{\mu}{\rho}(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}, \tag{3.1}$$

$$\nabla\cdot\boldsymbol{V}=0, \tag{3.2}$$

$$\frac{D\boldsymbol{H}}{Dt}=\kappa\nabla^2\boldsymbol{H}+(\boldsymbol{H}\cdot\nabla)\boldsymbol{V}, \tag{3.3}$$

$$\nabla\cdot\boldsymbol{H}=0, \tag{3.4}$$

$$\boldsymbol{E}=-\mu\boldsymbol{V}\times\boldsymbol{H}+\boldsymbol{J}/\sigma, \tag{3.5}$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+(\boldsymbol{V}\cdot\nabla), \tag{3.6}$$

and

$$\boldsymbol{J}=\nabla\times\boldsymbol{H}. \tag{3.7}$$

(3.3) is derived from (3.5) by taking its curl and making use of another equation for $\boldsymbol{E}$:

$$\nabla\times\boldsymbol{E}=-\mu\partial\boldsymbol{H}/\partial t. \tag{3.8}$$

The magnetic permeability and the electric conductivity are assumed to be constant and uniform.

Let us consider the unsteady symmetrical flow* due to the motion of an infinite flat plate of thickness $2L(|y|<L)$, starting at time $t=0$, with uniform velocity $U(t)$ parallel to the x-axis, in the presence of a uniform transverse magnetic field of strength H parallel to the y-axis.

In this case (3.1)-(3.6) are reduced to

$$v=0, \tag{3.9}$$
$$h_y=\text{const.}=H, \tag{3.10}$$
$$\partial u/\partial t=\nu\partial^2 u/\partial y^2+(\mu H/\rho)\partial h/\partial y, \tag{3.11}$$
$$P=\text{const.}, \tag{3.12}$$
$$\partial h/\partial t=\kappa\partial^2 h/\partial y^2+H\partial u/\partial y, \tag{3.13}$$
$$E_x=E_y=0,\ E_z=E=-\mu Hu+j/\sigma,\ j=-\partial h/\partial y, \tag{3.14}$$

if we take into account two-dimensionality and independence of the fields on x and z.

These equations must be solved under appropriate boundary conditions, among which those associated with electromagnetic components are particularly important.

Kakutani [2] has discussed the periodic solution of (3.11) and (3.13) and has solved the boundary value problem assuming $h=0$ on the surface of the plate. According to the present author's interpretation, Rossow has eliminated the term $j=-\partial h/\partial y$ in (3.11) by taking E in (3.14) to be 0 or $-\mu HU$ according as "H is fixed to the fluid" or "fixed to the plate". He has confined his study to the approximate equation obtained in this way, disregarding the small magnetic field induced by the flow.

In order to apply (3.9)-(3.14) to our problem completely, however, it is essential to take into account the induced magnetic field in the plate. In this paper, we assume that this inner magnetic field is governed by the induction equation

$$\partial h_0/\partial t=\kappa_0\partial^2 h_0/\partial y^2 \tag{3.15}$$

and

$$E_0=-\mu HU-\sigma_0^{-1}\partial h_0/\partial y \qquad \text{at} \qquad y=\pm L \tag{3.16}$$

which we can derive from $\mu_0 H_0=\mu H$, Maxwell equations and Ohm's law, neglecting displacement current. This assumption is surely adequate except for a small initial time interval of the order not larger than L/c and κ^2/c, where c is the light velocity. (3.11)-(3.16) are subject to the usual boundary conditions expressing the continuity of fields:

$$u(\pm L,\ t)=U(t), \tag{3.17}$$
$$h(\pm L,\ t)=h_0(\pm L,\ t), \tag{3.18}$$
$$E(\pm L,\ t)=E_0(\pm L,\ t), \tag{3.19}$$

* A flow over a very thick rigid domain $\infty<y\leq 0$ is discussed in §7.

(2) T. Kakutani: J. Phys. Soc. JAPAN 13 (1958) 1504.

$$u(\pm\infty, t)=0, \tag{3.20}$$

$$h(\pm\infty, t)=0. \tag{3.21}$$

In these equations (3.19) can be replaced by

$$\sigma^{-1}\partial h/\partial y=\sigma_0^{-1}\partial h_0/\partial y, \tag{3.19'}$$

if we make use of (3.14), (3.16) and (3.17).

4. Laplace-Transform Solution of the Fundamental Equations and its Specialization.

Multiplying (3.11)-(3.19) by $\exp(-pt)$ and integrating with respect to t from 0 to infinity, we obtain

$$\left(\frac{d^2}{dy^2}-\frac{p}{\nu}\right)u^*=-\frac{\mu H}{\rho\nu}\frac{dh^*}{dy}, \tag{4.1}$$

$$\left(\frac{d^2}{dy^2}-\frac{p}{\kappa}\right)h^*=-\frac{H}{\kappa}\frac{du^*}{dy}, \tag{4.2}$$

$$\left(\frac{d^2}{dy^2}-\frac{p}{\kappa_0}\right)h_0^*=0, \tag{4.3}$$

$$u^*=U^* \quad \text{at} \quad y=\pm L, \tag{4.4}$$

$$h^*=h_0^*, \quad \sigma^{-1}dh^*/dy=\sigma_0^{-1}dh_0^*/dy, \quad \text{at} \quad y=\pm L, \tag{4.5}$$

$$u^*\to 0, \quad h^*\to 0 \quad \text{as} \quad y\to\pm\infty, \tag{4.6}$$

where f^* denotes the Laplace transform of f, *i.e.*

$$f^*=\int_0^\infty e^{-pt}f(t)dt, \tag{4.7}$$

and we have used initial conditions

$$u(y, 0)=0, \quad h(y, 0)=0. \tag{4.8}$$

Taking into account the symmetry of (4.1)-(4.6), we have only to consider the half space $y>0$. In this region, solutions of (4.1)-(4.3) satisfying (4.6) are found to be

$$u^*=U^*[A_1e^{-k_1Y}+A_2e^{-k_2Y}], \tag{4.9}$$

$$h^*=-HU^*\left[\frac{k_1A_1}{p-\kappa k_1^2}e^{-k_1Y}+\frac{k_2A_2}{p-\kappa k_2^2}e^{-k_2Y}\right], \quad (Y=y-L,\ R_ek_j>0), \tag{4.10}$$

$$h_0^*=-HU^*\ a\sinh k_0y \quad (k_0=(p/\kappa_0)^{\frac{1}{2}},\ R_ek_0>0), \tag{4.11}$$

where k_1 and k_2 are defined by

$$k_{\substack{1\\2}}=\frac{1}{2}[(\nu^{-\frac{1}{2}}+\kappa^{-\frac{1}{2}})p_+^{\frac{1}{2}}\pm(\nu^{-\frac{1}{2}}-\kappa^{-\frac{1}{2}})p_-^{\frac{1}{2}}] \tag{4.12}$$

with

$$p_\pm=p+m_\pm, \qquad m_\pm=\frac{\alpha^2}{(\nu^{\frac{1}{2}}\pm\kappa^{\frac{1}{2}})^2} \tag{4.13}$$

and

$$\alpha=(\mu/\rho)^{\frac{1}{2}}H \tag{4.14}$$

which is the speed of the Alfvén wave propagating along the initial magnetic lines of force.

(4.9) and (4.10) are essentially the same formulae as that obtained by Kakutani (2), except (4.11) and compact expressions for k_1 and k_2, which are found from the characteristic equation for (4.1) and (4.2):

$$(k^2)^2-\left[\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)p+\frac{\alpha^2}{\nu\kappa}\right]k^2+\frac{p^2}{\nu\kappa}=0$$

yielding

$$k_1^2+k_2^2=\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)p+\frac{\alpha^2}{\nu\kappa}, \quad k_1k_2=\frac{p}{(\nu\kappa)^{\frac{1}{2}}}. \tag{4.15}$$

The functional forms of u^* and h^* for $y<0$ are determined by

$$u^*(-y)=u^*(y) \quad \text{and} \quad h^*(-y)=-h(y). \tag{4.16}$$

Introducing (4.9)-(4.11) into (4.4) and (4.5) we obtain

$$A_1+A_2=1, \tag{4.17}$$

$$\frac{k_1A_1}{p-\kappa k_1^2}+\frac{k_2A_2}{p-\kappa k_2^2}=a\sinh k_0L, \tag{4.18}$$

$$\frac{k_1^2A_1}{p-\kappa k_1^2}+\frac{k_2^2A_2}{p-\kappa k_2^2}=-\frac{\sigma}{\sigma_0}k_0a\cosh k_0L, \tag{4.19}$$

which yield A_1, A_2 and a as follows

$$A_1=-\frac{k_2(1+\beta k_2)(p-\kappa k_1^2)}{(k_1-k_2)[p\{1+\beta(k_1+k_2)\}+\kappa k_1k_2]} \tag{4.20}$$

$$A_2=\frac{k_1(1+\beta k_1)(p-\kappa k_2^2)}{(k_1-k_2)[p\{1+\beta(k_1+k_2)\}+\kappa k_1k_2]} \tag{4.21}$$

$$a\sinh k_0L=\beta/(\nu^{\frac{1}{2}}+\kappa^{\frac{1}{2}})(\beta p_+^{\frac{1}{2}}+\kappa^{\frac{1}{2}}) \tag{4.22}$$

where

$$\beta=(\sigma_0/\sigma)k_0^{-1}\tanh k_0L. \tag{4.23}$$

The skin friction drag D_f per unit area of the plate is given by

$$D_f^*=-\rho\nu\left(\frac{\partial u^*}{\partial y}\right)_{y=L}=\rho\nu U^*(A_1k_1+A_2k_2)=\rho\nu U^*\frac{\beta p+(\kappa p_+)^{\frac{1}{2}}}{\nu^{\frac{1}{2}}(\beta p_+^{\frac{1}{2}}+\kappa^{\frac{1}{2}})} \tag{4.24}$$

deduced from (4.20), (4.21), (4.12), (4.24) and (4.15).

We can also deduce the magnetic resistance D_m per unit area of the plate from the Maxwell stress on the surface or by integrating the magnetic force acting on the electric current in the plate*, as follows

* We note that the total electric currents in the fluid (J_f) and the plate (J_p) are of equal magnitude and of opposite sign :

$$J_f=-2\int_L^\infty(\partial h/\partial y)\,dy=2h\,(y=L)=-J_p=2\int_0^L(\partial h/\partial y)dy$$

which assures the balance of electric currents and shows that j must have at least one inversion point for $y>0$.

$$D_m^* = -\frac{1}{2}[\mu(Hh)_{y=L} - \mu(Hh)_{y=-L}] = -\frac{1}{2}\int_{-L}^{L}\mu H\frac{\partial h}{\partial y}dy = \mu H^2 U^* a \sinh k_0 L = \frac{1}{2}\mu H J_p \tag{4.25}$$

making use of (4.23).

Since general treatments of these solutions are laborious, it is convenient to confine our study to the following two limiting cases.

i) $\beta=0$ (4.26)

This will be attained by a thin insulator or a ferromagnetic plate* ($(\sigma_0/\sigma)L\to 0$ or $\mu_0 L^2=\infty$). The initial short stage and high frequency movements represented by $p \gg c/L$, c^2/κ_0 will be excluded.

ii) $\beta=\infty$ (4.26′)

This will be attained by the perfectly conductive plate of finite thickness $\sigma_0/\sigma \to \infty$). In this case the electric current in the plate is concentrated on the surface current sheets (skin effect).

Comparing (4.18) and (4.19) with (3.18) and (3.19), we find that i) and ii) are nothing but the assumptions about the boundary conditions for the magnetic field of the fluid given respectively by

i) $h=0$, (4.27)

ii) $\partial h/\partial y=0$ (4.27′)

on the surface of the plate.

Introducing (4.26) and (4.26′) into (4.9) (4.10), (4.20)-(4.25) and (3.14) we obtain for these two cases, respectively:

i)
$$\frac{u^*}{U^*} = \frac{(\nu\kappa)^{\frac{1}{2}}}{(\kappa-\nu)p_-^{\frac{1}{2}}}[(\kappa^{\frac{1}{2}}k_1 - \nu^{\frac{1}{2}}k_2)e^{-k_1 Y} + (\nu^{\frac{1}{2}}k_1 - \kappa^{\frac{1}{2}}k_2)e^{-k_2 Y}], \tag{4.28}$$

$$\frac{h^*}{HU^*} = \frac{\nu^{\frac{1}{2}}}{(\kappa-\nu)p_-^{\frac{1}{2}}}[e^{-k_1 Y} - e^{-k_2 Y}], \tag{4.29}$$

$$D_f^* = \rho\nu U^*(p_+/\nu)^{\frac{1}{2}}, \tag{4.30}$$

$$D_m^* = 0 \tag{4.31}$$

$$E_{(Y=0)} = -\frac{\nu^{\frac{1}{2}}}{\nu^{\frac{1}{2}} + \kappa^{\frac{1}{2}}}\mu H U(t) \tag{4.32}$$

ii)
$$\frac{u^*}{U^*} = \frac{\kappa}{(\kappa-\nu)(p_+ p_-)^{\frac{1}{2}}}[(p-\nu k_2^2)e^{-k_1 Y} - (p-\nu k_1^2)e^{-k_2 Y}], \tag{4.28′}$$

$$\frac{h^*}{HU^*} = \frac{(\nu\kappa)^{\frac{1}{2}}}{(\kappa-\nu)(p_+ p_-)^{\frac{1}{2}}}[k_2 e^{-k_1 Y} - k_1 e^{-k_2 Y}], \tag{4.29′}$$

$$D_f^* = \rho\nu U^* p/(\nu p_+)^{\frac{1}{2}} \tag{4.30′}$$

$$D_m^* = \rho a U^*(m_+/p_+)^{\frac{1}{2}} \tag{4.31′}$$

$$E_{(Y=0)} = -\mu H U(t) \tag{4.32′}$$

The magnetic drag is zero in the first case, because there is no current in the

* The description in 4) should be corrected.

plate, and the electric field on the surface is proportional to $-\mu HU(t)$ for the both cases.

5. THE CASE OF RAYLEIGH'S PROBLEM.

Let us restrict ourselves to the special case corresponding to Rayleigh's problem, *i.e.* $U=$constant or $U^*=U/p$. For the general functional form of $U(t)$ we have only to use the convolution theorem

$$F(y, t)=\int_0^t U'(t-\tau)F_R(y, \tau)d\tau \tag{5.1}$$

in order to obtain the values of u, h, D_f and D_m represented by F from the corresponding values of F_R obtained in our case (const.$=1$).

The quantities easily obtained are the skin frictional drag D_f and the magnetic drag D_m on the surface of the plate. Transforming back (4.30)-(4.31′) by use of the inversion theorem for the Laplace transform, we obtain

i) $$D_f=\frac{\rho\nu U}{(\pi\nu t)^{\frac{1}{2}}}[e^{-mt}+(\pi mt)^{\frac{1}{2}}\mathrm{erf}(mt)^{\frac{1}{2}}], \tag{5.2}$$

$$D_m=0 \tag{5.3}$$

ii) $$D_f=\frac{\rho\nu U}{(\pi\nu t)^{\frac{1}{2}}}e^{-mt}, \tag{5.2'}$$

$$D_m=\frac{\rho\nu U}{(\pi\nu t)^{\frac{1}{2}}}\left[\left(1+\left(\frac{\kappa}{\nu}\right)^{\frac{1}{2}}\right)(\pi mt)^{\frac{1}{2}}\,\mathrm{erf}\,(mt)^{\frac{1}{2}}\right]=\rho\alpha U\,\mathrm{erf}\,(mt)^{\frac{1}{2}}, \tag{5.3'}$$

where

$$\mathrm{erf}\,\xi=\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\xi} e^{-\xi^2}d\xi \tag{5.4}$$

and

$$m=m_+=\frac{\alpha^2}{(\nu^{\frac{1}{2}}+\kappa^{\frac{1}{2}})^2}. \tag{5.5}$$

Fig. 1 shows the values of D_f, D_m and D_f+D_m divided by $\rho U(\nu m)^{\frac{1}{2}}$. One of the important non-dimensional parameters in these formulae is the magnetic Prandtl number

$$P_m=\nu/\kappa=R_m/R \tag{5.6}$$

giving the ratio of magnetic and viscous Reynolds numbers R_m and R. Allowing P_m (or R_m for fixed R) to tend to zero we get

$$m\to\alpha^2/\kappa=m_\kappa \tag{5.7}$$

and (5.2) and (5.2′) are reduced respectively to the values obtained by Rossow for the two cases i) "H being fixed to the fluid" and ii) "H being fixed to the plate." In this respect Rossow's approximation is valid in the limit of small P_m or R_m, which is very common in the usual aerodynamic problem owing to the small conductivity of the air.

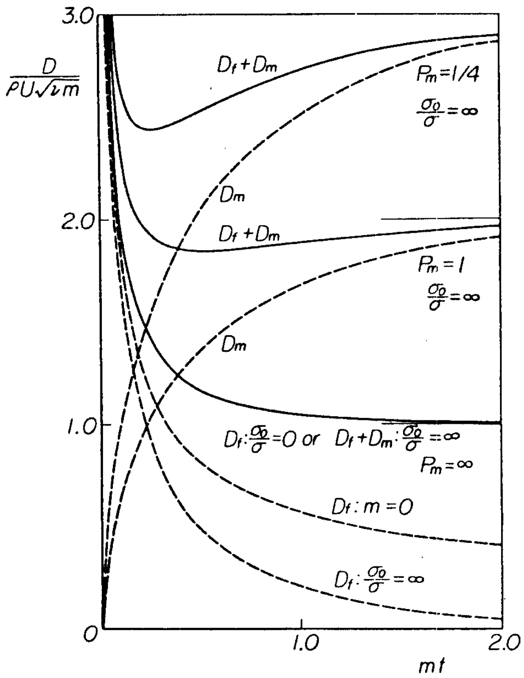

FIGURE 1. Frictional and magnetic drag against mt.

We also note that Rossow's second case is realized quite naturally by our conducting plate, in which the magnetic lines are, so to speak, frozen. The curves for $D_f/(\rho U(\nu m)^{\frac{1}{2}})$ coincide with those of Rossow, if we replace his $m_1=m_c$ by our m. This replacement is considered to be very useful for obtaining the values of the skin friction for general P_m from that for $P_m=0$ obtained by the simple solution as given by Rossow. If we fix ν and increase H or σ, m increases. This makes D_f increase for the insulating plate, and decrease for the conducting plate. On the other hand, the total resistance D_f+D_m increases, irrespective of the conductivity of the plate.

Previously (3) the author considered Rayleigh's problem for a flat plate with uniform suction or injection (transverse velocity V) distributed over its surface. It is interesting to note that D_f expressed by (5.2) is equal to the arithmetical mean of the skin frictions for the two cases of suction and injection if we put $V=2(\nu m)^{\frac{1}{2}}$ in the previous problem.

On the other hand, if we put $P_m=0$ formally in the first line of (5.3′) or try to obtain the limiting value of $D_m/(\mu\sigma UH^2(\nu t)^{\frac{1}{2}})$ as shown in Rossow's paper, we get an infinite value, as long as we take m to be finite. The induced magnetic field may be small but extends far away if R_m is small compared with R. In this case Rossow's approximate equation is valid in the vicinity of the plate (*i.e.*, within the usual viscous boundary layer) and his assumptions for the strength of electric field are justified in this layer, in view of (4.32) and (4.32′). But it is inadequate to determine the delicate behavior of the magnetic field induced by the plate outside of the viscous boundary layer. D_m takes rather small values only so long as mt is small:

$$D_m \sim 2\rho\alpha U(mt/\pi)^{\frac{1}{2}} \tag{5.8}$$

For small values of mt the values of D_f tend to

$$D_f \sim D_{f0}(1 \pm mt) \to D_{f0} \tag{5.9}$$

where D_{f0} is the value of D_f in the absence of the magnetic field:

$$D_{f0} = \rho\nu U/(\pi\nu t)^{\frac{1}{2}} \tag{5.10}$$

The velocity and magnetic fields corresponding to these expressions are easily obtained by letting $p \to \infty$ in (4.12), and (4.28)-(4.29). It is found that

$$k_1 = (p/\nu)^{\frac{1}{2}}\left[1 + \frac{\alpha^2}{2(\kappa-\nu)p} + O\left(\frac{1}{p^2}\right)\right],$$

$$k_2 = (p/\kappa)^{\frac{1}{2}}\left[1 + \frac{\alpha^2}{2(\nu-\kappa)p} + O\left(\frac{1}{p^2}\right)\right]. \tag{5.11}$$

which give

$$\frac{u}{U} = E_0(\eta) - \frac{2\alpha^2 t}{\kappa-\nu}\eta E_1(\eta) - \frac{4(\nu\kappa)^{\frac{1}{2}}\alpha^2 t}{(\kappa-\nu)^2}[E_2(\eta) - E_2(\eta_m)] + O(m^2t^2), \tag{5.12}$$

$$\frac{u}{U} = E_0(\eta) - \frac{2\alpha^2 t}{\kappa-\nu}\eta E_1(\eta) - \frac{4\kappa\alpha^2 t}{(\kappa-\nu)^2}[E_2(\eta) - E_2(\eta_m)] + O(m^2t^2), \tag{5.12'}$$

$$\frac{h}{UH} = \frac{2(\nu t)^{\frac{1}{2}}}{\kappa-\nu}[E_1(\eta) - E_1(\eta_m)] + O((mt)^{\frac{3}{2}}) \leqq 0, \tag{5.13}$$

$$\frac{h}{UH} = \frac{2(\nu t)^2}{\kappa-\nu}\left[E_1(\eta) - \left(\frac{\kappa}{\nu}\right)^{\frac{1}{2}} E_1(\eta_m)\right] + O((mt)^{\frac{3}{2}}) < 0, \tag{5.13'}$$

where

$$E_n(\eta) = i^n \operatorname{erf}_c \mu = \int_\eta^\infty E_{n-1}(\eta)d\eta \sim \begin{cases} \sim 2^{-n}\pi^{-\frac{1}{2}}\eta^{-(n+1)}e^{-\eta^2} & \text{as} \quad \eta \to \infty \\ \sim 2^{-n}/\Gamma(\frac{n}{2}+1) + O(\eta) & \text{as} \quad \eta \to 0, \end{cases} \tag{5.14}$$

$$E_0(\eta) = \operatorname{erf}_c \eta = 1 - \operatorname{erf} \eta = \frac{2}{\pi^{\frac{1}{2}}}\int_\eta^\infty e^{-\eta^2}d\eta$$

and

$$\eta = Y/(2(\nu t)^{\frac{1}{2}}), \quad \eta_m = Y/(2(\kappa t)^{\frac{1}{2}}) = (\nu/\kappa)^{\frac{1}{2}}\eta. \tag{5.15}$$

For the term containing $\exp(-k_1 Y)$ or η, ν plays the important role, and for that containing $\exp(-k_2 Y)$ or η_m, κ is important. In this sense, we may call them respectively the viscous and magnetic terms. If mt is small, the viscous term predominates in the velocity profile. For the induced magnetic field, however, this is not necessarily so. The two terms are of the same order in the vicinity of the insulating plate, and the magnetic term is rather important for large η and on the conducting plate if P_m is small, although the absolute values of h are small in this case. These are in accord with the considerations given in the previous section. We can also note that there is the induced fluid motion even in the case of non-viscous fluid if the plate is a conductor:

$$u/U \sim (4\alpha^2/\kappa)tE_2(Y/(\kappa t)^{\frac{1}{2}}), \tag{5.16'}$$

$$h/(UH) \sim -(2\kappa t)^{\frac{1}{2}}E_1(Y/(\kappa t)^{\frac{1}{2}}). \tag{5.17'}$$

(3) H. Hasimoto: J. Phys. Soc. Japan 12 (1957) 68.

in which magnetic field is seen to play more important role.

If mt is large (5.2)-(5.3) give respectively

$$D_f=\rho\alpha U/(1+(\kappa/\nu)^{\frac{1}{2}}) \quad (5.18), \qquad D_f=0, \quad (5.18')$$

$$D_m=0 \quad (5.19), \qquad D_m=\rho\alpha U=(\rho\mu)^{\frac{1}{2}}HU. \quad (5.19')$$

These may be considered to correspond to the so-called static solutson (near the plate) of (3.11) and (3.13). By letting $p\to 0$ in (4.28)-(4.29) and interpreting them we obtain as crude approximations of the fields:

$$k_{\substack{1\\2}}\sim\alpha/(\nu\kappa)^{\frac{1}{2}}+p/\alpha_1+O(p^2),\ k_{\substack{2\\1}}\sim p/\alpha+O(p^2),\ (\kappa\gtrless\nu),\ \alpha_1=2(\nu\kappa)\alpha^{\frac{1}{2}}/(\nu+\kappa)\leqq\alpha, \quad (5.20)$$

$$u\sim\frac{U}{\nu^{\frac{1}{2}}+\kappa^{\frac{1}{2}}}\left[\nu^{\frac{1}{2}}\delta+\delta_1\kappa^{\frac{1}{2}}\exp\left(-\frac{\alpha}{(\nu\kappa)^{\frac{1}{2}}}Y\right)\right], \quad (5.21) \qquad u\sim U\delta, \quad (5.21')$$

$$h\sim-\frac{\nu^{\frac{1}{2}}UH}{\alpha(\nu^{\frac{1}{2}}+\kappa^{\frac{1}{2}})}\left[\delta-\delta_1\exp\left(-\frac{\alpha}{(\nu\kappa)^{\frac{1}{2}}}Y\right)\right], \quad (5.22) \qquad h\sim-\frac{UH}{\alpha}\delta, \quad (5.22')$$

where

$$\begin{cases}\delta=0 \text{ for } (Y\gg\alpha t), & \delta=1 \text{ for } (Y\ll\alpha t),\\ \delta_1=0 \text{ for } (Y\gg\alpha_1 t), & \delta_1=1 \text{ for } (Y\ll\alpha_1 t).\end{cases}$$

The first terms of these solutions containing δ come from $\exp(-k_1Y)$ for $\nu>\kappa$,and from $\exp(-k_2Y)$ for $\nu<\kappa$. It is interesting that the roles of two terms for the functional forms change according as $\kappa>\nu$ or $\kappa<\nu$, although the distinct separation of the roles of two terms is not possible. In this case, however, u and h tend to finite values for $\alpha t>Y\to\infty$, contrary to the conditions at $y=\infty$, though they give (5.18)-(5.19) and satisfy the other boundary conditions. This is attributed to the wave nature of the field which is also dispersive. According to (5.21)-(5.22′) there must be inflection points of the velocity and the induced magnetic field profiles in the transient regions represented by discontinuities in (5.21)-(5.22′) between the static field near the plate and field at $Y>\alpha t\to\infty$. These inflection points are nothing but the wave fronts for the non-dissipative case.

In order to clarify these features we want to investigate the special case of $P_m=1$ precisely in the next section. This case seems to be excluded in (5.12)-(5.13) and the above formulae. However, it can be deduced by a limiting process and allows closed solutions for arbitrary y and t.

We also note that we can obtain the same kind of quasi-stationary solutions as (5.18)-(5.22) for the case of finite conductivity of the plate. Letting $p\to 0$ in (4.24) and (4.25) taking into account (4.23), (5.20) and (3.16) we obtain

$$D_f=\rho\alpha U/G, \quad (5.23)$$

$$D_m=(\sigma_0/\sigma)MD_f \quad (5.24)$$

where

$$M=\alpha L/(\nu\kappa)^{\frac{1}{2}}=\mu^{\frac{1}{2}}HL/(\rho\nu\kappa)^{\frac{1}{2}},\quad G=1+1/P_m^{\frac{1}{2}}+(\sigma_0/\sigma)M.$$

M is the Hartman number of the plate, referred to its half thicknes L. These correspond to the field quantities and magnetic field in the plate for $mt\to\infty$:

$$u/U\sim[\{1+(\sigma_0/\sigma)M\}\delta+(1/P_m)^{\frac{1}{2}}\delta_1\exp(-MY/L)]/G, \quad (5.25)$$

$$h/U \sim -(\rho/\mu)^{\frac{1}{2}}[\{1+(\sigma_0/\sigma)M\}\delta-\delta_1 \exp(-MY/L)]/G, \tag{5.26}$$

$$E/(\mu HU) \sim -[1+(\sigma_0/\sigma)M]\delta/G = -u\begin{pmatrix}Y=\infty\\ ct>Y\end{pmatrix}\Big/U, \tag{5.27}$$

$$h_0/U = -(\sigma_0/\sigma)Hy/(G(\nu\kappa)^{\frac{1}{2}}) \quad (t \gg \mu_0\sigma_0L^2). \tag{5.28}$$

We can see that the effect of finite conductivity is reflected in the parameter (σ_0/σ) M and interpolates the two extreme cases. The induced electric field behind of the wave front is uniform and constant, which is a feature of the steady two-dimensional field. Also note that we get for $Y \to \infty$

$$\mu h^2/2 \sim \rho u^2/2 \ (\alpha t \gg Y \to \infty), \tag{5.29}$$

i.e. the equipartition between the kinetic energy of the disturbed fluid and the magnetic energy of the induced magnetic field, after the dissipation of the high frequency components. This is a typical feature of Alfvén wave.*

These simple kinds of fields are to be expected as the asymptotic behavior of the fully developed boundary layer for a flat plate in steady motion in the presence of a uniform magnetic field, although the transition region between the outer fieldamay be quite complicated.

6. Special Case of $P_m=1$.

Putting $P_m=1$, *i.e.* $\nu=\kappa$ in (4.12) we obtain

$$k_{\frac{1}{2}} = \frac{1}{\nu^{\frac{1}{2}}}(p+m^{\frac{1}{2}} \pm m^{\frac{1}{2}}) \tag{6.1}$$

where

$$m = \alpha^2/(4\nu)$$

Introducing these expressions into (4.28)-(4.29) we get

$$u = \frac{1}{2}(u_1+u_2) \tag{6.2}$$

$$h = \frac{\rho^{\frac{1}{2}}}{2\mu^{\frac{1}{2}}}(u_1-u_2) \tag{6.3}$$

where

$$u_j^* = U^* \exp(-k_jY) \quad (6.4) \qquad u_j^* = (\nu/(p+m))^{\frac{1}{2}}k_lU^* \exp(-k_jY) \quad (6.4')$$

$$j=1, 2, \qquad l=1,2 \neq j.$$

For the case of constant U, u_j are given by

$$u_1 = \frac{U}{2}\left[\mathrm{erf}_c\,\eta_+ + \exp\left(-\frac{\alpha Y}{\nu}\right)\mathrm{erf}_c\,\eta_-\right] \quad u_2 = \frac{U}{2}\left[\mathrm{erf}_c\,\eta_- + \exp\left(\frac{\alpha Y}{\nu}\right)\mathrm{erf}_c\,\eta_+\right], \tag{6.5}$$

$$= u_2 \exp(-\alpha Y/\nu),$$

---

* We also noie that the non-homogeneous parts of the stationary solution for u and h satisfy a simple relation

$$\mu h_{n.h}^2 = P_m \cdot \rho U_{n.h}^2 (= \rho U^2 \exp(-2LY/L)/G^2)$$

This is a typical feature of the stationary solution of (3.11) and (3.13)

$$u_1 = U\,\mathrm{erf}_c\,\eta_+, \qquad u_2 = U\,\mathrm{erf}_c\,\eta_-, \tag{6.5'}$$

or

$$\left.\begin{matrix} u/U \\ -\mu^{\frac{1}{2}}h/\rho^{\frac{1}{2}} \end{matrix}\right\} = \frac{1}{2}\begin{Bmatrix}\cosh \\ \sinh\end{Bmatrix}\frac{\alpha}{2\nu}Y\left\{e^{\frac{\alpha}{2\nu}Y}\,\mathrm{erf}_c\,\eta_+ + e^{-\frac{\alpha}{2\nu}Y}\,\mathrm{erf}_c\,\eta_-\right\} \tag{6.6}$$

for the first case; where

$$\eta_\pm = \frac{1}{2(\nu t)^{\frac{1}{2}}}(Y \pm \alpha t) \tag{6.7}$$

$\mathrm{erf}_c\,\eta_-$ corresponcs to the hydromagnetic wave advancing with velocity α perpendicular to the plate in the non-dissipative case, although in this case its wave front is obscured by the presence of dissipation. $\mathrm{erf}_c\,\eta_+$ may be considered to be a kind of reaction advancing in the opposite direction.

We also note that u_2 and u_1 for the case of insulating plate are completely equivalent to the expressions for the velocity profiles due to uniform injection and suction distributed on the surface of a flat plate, if we put injection velocities V in that case equal to $\pm\alpha$.

Fig. 2–Fig. 7 show the values of u and h for our two typical cases. These are complicated curves with one or two inflection points which move far away with increasing $\alpha^2 t/\nu$. $-h$ has a maximum at $Y = Y_0$ in the fluid for $\sigma_0/\sigma = 0$, (inversionpoint of current $-\partial h/\partial y = j \gtreqless 0$ according as $|Y| \lesseqgtr Y_0$) since there is no current in the plate. This is also seen in (5.26), (5.13) and is evident from the general consideration of total current. We also note that the bohaviour of u and h for $Y > Y_0$ is similar to those in the case of $\sigma_0/\sigma = \infty$ in $Y > 0$.

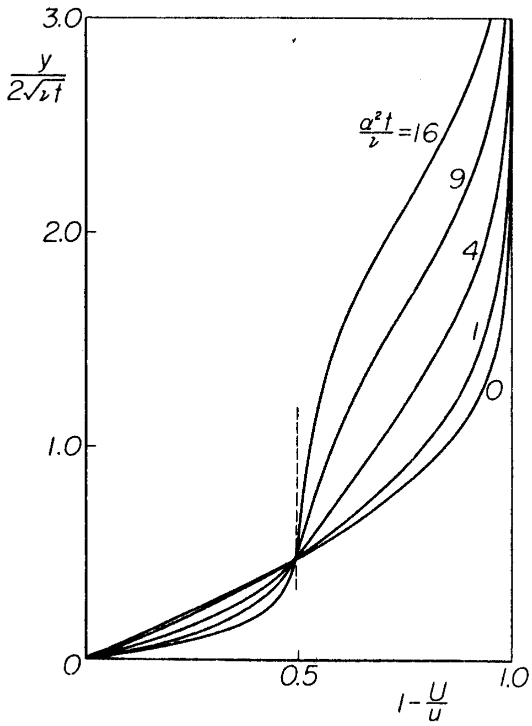

FIGURE 2. Velocity distribution above an insulating plate for $P_m = 1$.

The qualitative behavior is the same as that expected in the previous sections. If we fix Y and make $t-Y/\alpha$ large $\mathrm{erf}_c\,\eta_-$ tends to 2 and we get the quasi-

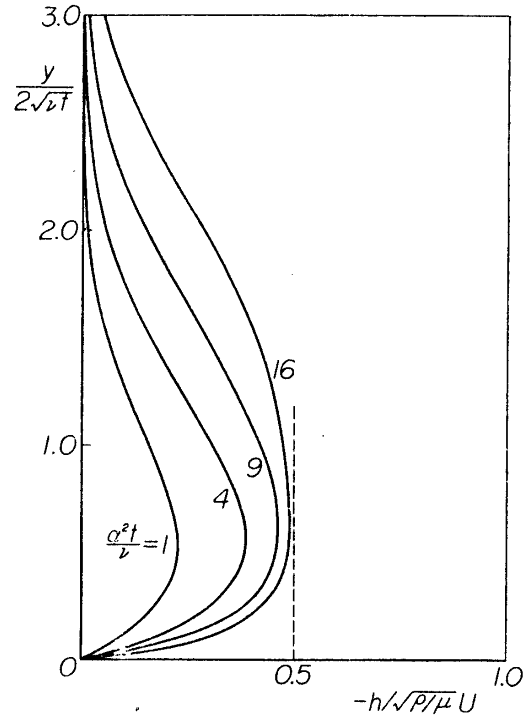

FIGURE 3. Induced magnetic field above an insulating plate for $P_m=1$.

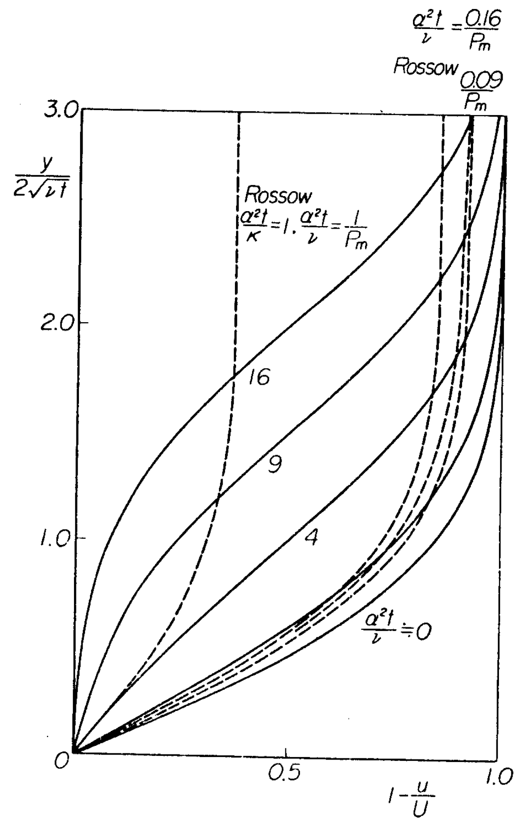

FIGURE 4. Velocity distribution above a perfect conducting plate for $P_m=1$.

stationary fields corresponding to (5.21) and (5.22). The curves near the plate for $\alpha^2 t/\nu = 9$, 16 have already settled down.

On the other hand if we fix t and make $\alpha Y/\nu$ large

$$u_1 \ll u_2 \tag{6.8}$$

and there is the equipartition of energy as shown in (5.29) even for finite t (see Fig. 6, 7). It is surprising that the equipartition of induced energy density is attained for a wide range of t and Y (especially for $\sigma_0/\sigma = \infty$) in the case of $P_m = 1$.

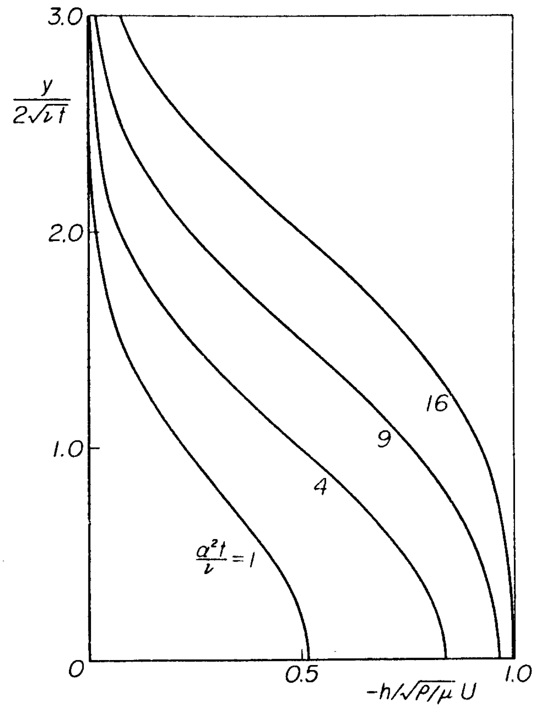

FIGURE 5. Induced magnetic field above a perfect conducting plate for $P_m = 1$.

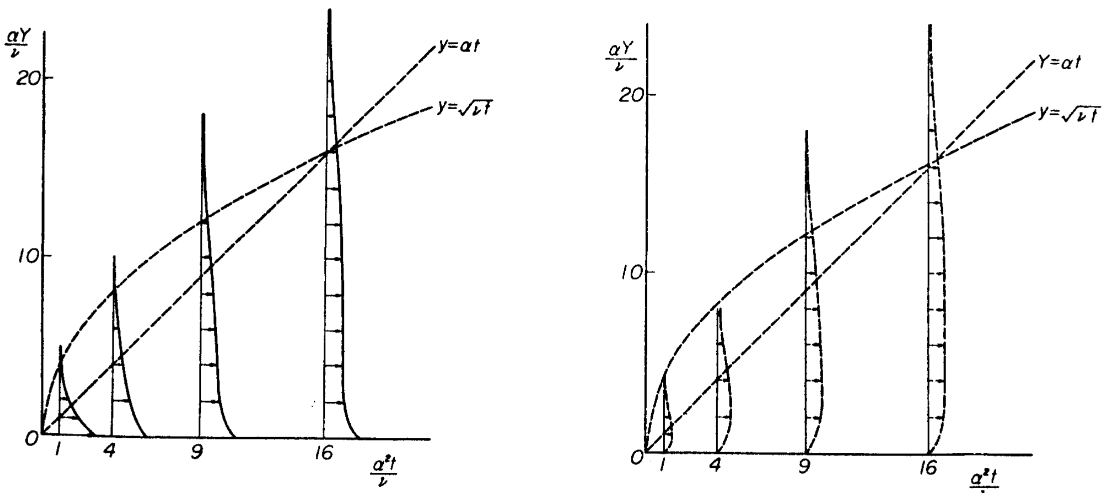

FIGURE 6. and FIGURE 6′. Equipartition of induced velocity and magnetic field, ——: the values of u/U, -----: the values of $h/\sqrt{\rho/\mu}U$.

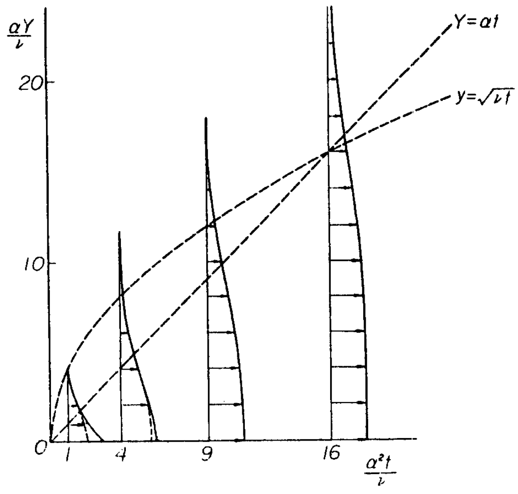

FIGURE 7. Equipartition of induced velocity and magnetic field, ——: the values of u/U, -----: the values of $h/\sqrt{\rho/\mu}U$.

7. THICK WALL OF FINITE CONDUCTIVITY

Recently, our problem has been discussed extensively by many authors for various cases (e.g. Chang & Yen [5], Ludford [6], Hide & Roberts [7], References in 4).

The influence of finite wall conductivity is discussed by Bryson and Rośsocisczewski [9]. They treated two types of wall situation; A) a semi-infinite wall of electrical conductivity σ_0 in $y<0$ and B) a thin plane wall with the same fluid in $y<0$ as in $y>0$. The latter case corresponds to one of our special case $\beta=0$ in the previous sections. In addition, they consider the motion of fluid induced by the discharge of a current parallel to the surface and show that the solution of this problem is derived from that of Rayleigh's problem by simply interchang ing ν and κ, and u and h.

They formulate Rayleigh's problem for A) for arbitrary wall conductivity, but no explicit solutions are obtained for general values of σ_0.

Recently the present author [4] obtained a general solution in the Laplace transform, which comprises all cases treated in the previous sections and the case A). It is obtained in the following manners.

We assume that the domain $y<0$ is a homogeneous conductor and

$$j=0 \quad \text{at} \quad y=-\infty \quad \text{for} \quad t<\infty, \tag{7.1}$$

which yields

$$E=-\mu HU \quad \text{at} \quad y=-\infty, t<\infty \tag{7.2}$$

from (3.14).

Then we have only to solve Eq. (3.9)-(3.15) with the initial and the boundary conditions similar to (3.17)-(3.19′):

* They consider the fluid and the wall to be moving with the same speed U until at time $t=0$, and the wall is stopped at $t=0$. This case can be easily reduced to the case treated here by replacing their u by $U-u$.

$$u=h=0 \quad \text{for} \quad t=0,\, y>0 \quad \text{and} \quad \infty>t>0,\, y=\infty, \tag{7.3}$$

$$u=U(t),\ h=h_0,\ \frac{1}{\sigma}\frac{\partial h}{\partial y}=\frac{1}{\sigma_0}\frac{\partial h_0}{\partial y} \ \text{for } t>0,\ y=0 \tag{7.4}$$

and (7.1).

The Laplace transform of the solution is given by

$$h^*=-HU^*\sum_{i=1}^{2}\tilde{A}_i e^{-k_iY},\quad h_0^*=-HU^*\tilde{\alpha}\exp k_0Y, \tag{7.5}$$

$$u^*=U^*\sum_{i=1}^{2}\frac{1}{k_i}(p-\kappa k_i^2)\tilde{A}_i e^{-k_iY} \tag{7.6}$$

where Y is the distance from the surface of the wall, *i.e.*

$$Y=y \tag{7.7}$$

in this section. $\tilde{A}_i$ and $\tilde{\alpha}$ are determined from (7.4) (the other conditions have been satisfied already) *i.e.*

$$\sum_{i=1}^{2}\frac{1}{k_i}(p-\kappa k_i^2)\tilde{A}_i=1, \tag{7.8}$$

$$\tilde{A}_1+\tilde{A}_2=\tilde{\alpha} \tag{7.9}$$

$$k_1\tilde{A}_1+k_2\tilde{A}_2=-\frac{\sigma}{\sigma_0}k_0 \tag{7.10}$$

which yield

$$\tilde{A}_1=\frac{\nu^{\frac{1}{2}}(p^{\frac{1}{2}}+\beta\kappa^{\frac{1}{2}}k_2)}{(\nu-\kappa)p_-^{\frac{1}{2}}(p^{\frac{1}{2}}+\beta p_+^{\frac{1}{2}})},\quad \tilde{A}_2=\frac{\nu^{\frac{1}{2}}(p^{\frac{1}{2}}+\beta\kappa^{\frac{1}{2}}k_1)}{(\kappa-\nu)p_-^{\frac{1}{2}}(p^{\frac{1}{2}}+\beta p_+^{\frac{1}{2}})}, \tag{7.11}$$

$$\alpha\tilde{\alpha}=\gamma m^{\frac{1}{2}}/(p^{\frac{1}{2}}+\beta p_+^{\frac{1}{2}}), \tag{7.12}$$

where

$$\beta=(\sigma_0/\sigma)(\kappa_0/\kappa)^{\frac{1}{2}}=(\sigma_0\mu/\sigma\mu_0)^{\frac{1}{2}}.$$

It is easily seen that the solution (7.5)–(7.11) also solves the case in § 1–6 in the upper half plane, if we replace $\exp k_0Y$ by $\sinh k_0y/\sinh k_0L$ in (7.5) and β in (7.11) by $(\sigma_0\mu/\sigma\mu_0)^{\frac{1}{2}}\tanh k_0L$ (Compare with (4.9)–(4.23). A_i and $\tilde{A}_i$ are related by $A_i=(p-\kappa k_i^2)\tilde{A}_i/k_i$).

The skin frictional drag D_f^* and the magnetic resistance D_m^* per unit area of the plate are

$$D_f^*=-\rho\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}=\rho\nu U^*\sum_{i=1}^{2}(p-\kappa k_i^2)\tilde{A}_i=\frac{\rho\nu^{\frac{1}{2}}p^{\frac{1}{2}}[\beta p^{\frac{1}{2}}+p_+^{\frac{1}{2}}]}{p^{\frac{1}{2}}+\beta p_+^{\frac{1}{2}}}U^*, \tag{7.13}$$

$$D_m^*=-\mu Hh_{y=0}=\mu H^2U^*\tilde{\alpha}=\rho\alpha U^*\frac{\beta m^{\frac{1}{2}}}{p^{\frac{1}{2}}+\beta p_+^{\frac{1}{2}}}, \tag{7.14}$$

For two special cases i) $\beta=0$ *i.e.* the case of an insulator wall or a ferromagnetic wall* ii) $\beta=\infty$ *i.e.* the case of a perfect conductor wall (7.13) and (7.14) are perfectly in accordance with (4.30)–(4.31′).

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* The discussion in 4) should be corrected.

In Rayleigh's problem $U^*=U/p$, (7.13) and (7.14) are expressed in closed forms as follows:

a) $\beta=1$

$$D_f^*=\rho\nu^{\frac{1}{2}}p^{\frac{1}{2}}U^*, \qquad D_m^*=\frac{\rho\alpha U^*}{m^{\frac{1}{2}}}(p_+^{\frac{1}{2}}-p^{\frac{1}{2}})$$

i.e.

$$D_f=\rho U\left(\frac{\nu}{\pi t}\right)^{\frac{1}{2}}, \qquad D_m=\rho\alpha U\left[\frac{e^{-mt}}{(\pi mt)^{\frac{1}{2}}}+\mathrm{erf}\,(mt)^{\frac{1}{2}}-\frac{1}{(\pi mt)^{\frac{1}{2}}}\right]. \tag{7.15}$$

b) $\beta>1$

$$D_f=\rho U(\nu m)^{\frac{1}{2}}\left[\frac{e^{-mt}}{(\pi mt)^{\frac{1}{2}}}+\frac{e^{-\beta^2T^2}}{(\beta^2-1)^{\frac{1}{2}}}\{w(\beta T)-w(T)\}\right],$$

$$D_m=\rho\alpha U\left[\mathrm{erf}\,(mt)^{\frac{1}{2}}-\frac{e^{-\beta^2T^2}}{(\beta^2-1)^{\frac{1}{2}}}\{w(\beta T)-w(T)\}\right], \tag{7.16}$$

where

$$T^2=mt/|\beta^2-1|, \tag{7.17}$$

and

$$w(\xi)=\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\xi}\exp\xi^2 d\xi. \tag{7.18}$$

c) $\beta<1$

$$D_f=\rho U(\nu m)^{\frac{1}{2}}\left[\frac{e^{-mt}}{(\pi mt)^{\frac{1}{2}}}-\frac{e^{\beta^2T^2}}{(1-\beta^2)^{\frac{1}{2}}}\{\mathrm{erf}\,(\beta T)-\mathrm{erf}\,T\}\right],$$

$$D_m=\rho\alpha U\left[\mathrm{erf}\,(mt)^{\frac{1}{2}}+\frac{e^{\beta^2T^2}}{(1-\beta^2)^{\frac{1}{2}}}\{\mathrm{erf}\,(\beta T)-\mathrm{erf}\,T\}\right]. \tag{7.19}$$

The formula

$$L^{-1}\left[\frac{(p+a)^{\frac{1}{2}}}{p+b}\right]=\frac{e^{-at}}{(\pi t)^{\frac{1}{2}}}+(a-b)^{\frac{1}{2}}e^{-bt}\,\mathrm{erf}\,[(a-b)^{\frac{1}{2}}t^{\frac{1}{2}}] \tag{7.20}$$

has been used in the invesrse transform [8] of D_f^* and D_m^*.

For large values of mt (7.13)-(7.19) yield for $\beta\neq 0$

$$D_f^*\sim\frac{\rho(\nu p)^{\frac{1}{2}}}{\beta}U^*, \qquad D_m^*\sim\rho\alpha U^* \tag{7.21}$$

i.e.

$$D_f\sim\frac{\rho U}{\beta}\left(\frac{\nu}{\pi t}\right)^{\frac{1}{2}}, \qquad D_m\sim\rho\alpha U \tag{7.22}$$

for Rayleigh's problem.

These formulae are useful in interpolating the special cases of $\beta=0$ and $\beta=\infty$.

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