No. 4.
(Published November 1922)

On the Decay of Vortical Motion in a Viscous Fluid.[1]

By

Kwan-ichi TERAZAWA, *Rigakuhakushi.*

Member of the Institute.

Introduction.

1. "Vortices in a viscous fluid" has been since long an important topic among meteorologists and those who work in aerodynamical researches, and many interesting papers relating to this subject are published. A gust in the atmosphere, smoke flowing out from a chimney, the rear stream of an aeroplane, and such phenomena observed every day in the motion of air have, almost without exception, eddy motion accompanying them. So far as we are concerned with the mean motion of the air, there are of course several problems which can be solved by applying the hydrodynamical theory of a *perfect fluid*, but if we look into the details of the phenomena concerned it will always be found that it is quite indispensable to take the *viscosity* of air into account.

In establishing the wind channel in our Institute, one of the greatest difficulties was to discover how the vortices which appeared in front of the intake of the channel could be eliminated. As these vortices had serious effects on the experiments undertaken by using this channel,

(1) This paper will, in revised form, be submitted to the publishing committee dealing with the papers to be dedicated to Professor Nagaoka to commemorate the twenty-fifth anniversary of his professorship. A part of this paper appeared in the Japanese Journal of Physics, 1 (1922), pp. 7—19.

Errata :— *The following lines are to be added to the bottom of* p. 102 of No. 4 *of this Report:*

which is evidently positive since $r > a$.

For large values of t, taking the terms of the lowest degree in $1/t$, we get

$$\frac{\partial\omega}{\partial t} = -\frac{Ca^2}{4\nu t^2}\left(1 - \frac{r^2}{2\nu t}\right), \tag{33}$$

which is negative for sufficiently large values of t.

several proposals to prevent their appearance were offered and some revision of equipments was tried. It seems, however, that it is extraordinarily difficult, or rather impossible, to attempt to obtain air streams without any trace of eddies in such mechanism as we employ in a usual wind channel. It is therefore desirable, if possible, to find to what extent the eddies affect the results of the experiments made. By applying the theory of a perfeet fluid, this problem could be solved to a first approximation, and some general features of the solution obtained. But, in effect, as the accuracy of the experiments improves, it must naturally be required, as pointed out above, to treat the problem by considering the air as a viscous fluid. In spite of the importance of such an investigation, little work has yet been accomplished, and moreover, we have no sufficient knowledge of the nature of vortices in a viscous fluid to throw light on the research of such a problem. It is thus of vital necessity, first, to study the properties of vortices.

2. In the present paper, an attempt is made to investigate the manner in which the vortices in a viscous fluid dissipate, when they are once created in some part of the fluid and are left to themselves. Owing to the lack of experimental data this work is throughout theoretical, and yet it seems to be sufficient to reveal in what manner the eddies die away, the cause of which may be considered to be mainly due to the viscosity of the fluid.

The fluid, which may be air or water, is assumed to be viscous and incompressible, and to fill up the whole space without any obstacle. The assumption of the incompressibility of the fluid may introduce some error into the result, but so far as the velocity of the fluid is small compared with that of sound, this may be considered not to be very serious.[1] If we took the compressibility of the fluid into account, the analysis would only become very complicated; and we are now obliged to leave this matter to another occasion.

---

(1) Lord. Rayleigh, Tech. Rep. of the Adv. Comm. for Aeronautics, London, (1910-11), p. 26.

As to the vortices, we consider them, again to avoid complicated calculations, as rectilinear, distributed symmetrically about an axis. Starting by solving the equations to determine the motion, we complete the solution, in a quite general form, answering to any initial distribution of eddies. Several examples are given, some of them worth notice as of practical importance.

General Solution.

3. The equation of motion of a viscous fluid, free from the action of any external force, is

$$\frac{D\mathfrak{v}}{Dt}=-\frac{1}{\rho}\operatorname{grad} p+\frac{4}{3}\nu \operatorname{grad}\operatorname{div}\mathfrak{v}-\nu\operatorname{curl}\operatorname{curl}\mathfrak{v},$$

where $\mathfrak{v}$ and p denote the volocity and the pressure respectively, ρ the density and ν the coefficient of kinematic viscosity of the fluid. The equation of continuity expressing the incompressibility of the fluid is

$$\operatorname{div}\mathfrak{v}=0.$$

According to the assumption made, the motion is symmetrical about an axis, and moreover it is the same in all planes perpendicular to the axis, *i. e.* two dimensional motion with a point of symmetry. Thus if we take the centre of symmetry as the origin of the polar coordinates (r, θ), the velocity and the pressure at any point are independent of θ. The equations of motion and of continuity in this simple case reduce to

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 u-\frac{u}{r^2}\right)^{(1)}, \qquad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{uv}{r}=\nu\left(\nabla^2 v-\frac{v}{r^2}\right), \qquad (2)$$

and

$$\frac{\partial (ru)}{\partial r}=0, \qquad (3)$$

(1) See *e.g.* G. B. Jeffery, Phil. Mag. **29** (1915), p. 452.

in which u, v are the velocity components in the r and θ directions respectively, and ∇^2 stands for the operation

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

From the equation (3) we have at once

$$u = \frac{c}{r}, \qquad (4)$$

where c may be a constant or a function of time. To make the mathematical analysis simple let us suppose that it is an absolute constant arbitrarily given.

When c is zero, the radial velocity at any point is *nil*, and consequently the motion presents itself as circulating about the origin. When c is not zero, the radial velocity at the origin becomes infinitely great, and therefore in order to render the case physically possible we may have to consider that the origin is excluded from the region under discussion by a barrier of an infinitely small extent, or more dynamically, that a source or a sink with its strength proportional to c is situated at the origin.

Putting the value of u found in (4) into the equation (2) we have

$$\frac{\partial v}{\partial t} + \frac{c}{r}\frac{\partial v}{\partial r} + \frac{cv}{r^2} = \nu\left\{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right\},$$

to determine the component v. When we find the solution of this equation, and substitute it, together with the value of u above obtained, in the equation (1), we have an equation which serves to determine the distribution of the pressure.

If we suppose that v is proportional to $e^{-\nu k^2 t}$ with respect to time, the above equation transforms to

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(1 - \frac{c}{\nu}\right)\frac{\partial v}{\partial r} + \left\{k^2 - \frac{1}{r^2}\left(1 + \frac{c}{\nu}\right)\right\}v = 0.$$

The solution of this equation is easily found to be, omitting the time factor,

$$v = r^{\frac{c}{2\nu}} J_{1+\frac{c}{2\nu}}(kr),$$

where $J_n(x)$ is the Bessel Function of the n th order.

Thus the typical solution for v is

$$v = Ae^{-\nu k^2 t} r^{\frac{c}{2\nu}} J_{1+\frac{2}{2\nu}}(kr), \tag{5}$$

A standing for any constant.

4. Let $\omega/2$ be the rectilinear vortex whose axis is perpendicular to the coordinate plane; then, as u and v are independent of θ, we have

$$\omega = \frac{1}{r} \frac{\partial (rv)}{\partial r}.$$

By using the value of v obtained in (5) it follows that

$$\omega = Ake^{-\nu k^2 t} r^n J_n(kr), \tag{6}$$

where

$$n = \frac{c}{2\nu}. \tag{7}$$

5. For the generalization of this expression so that it may represent the vortex motion subjected to any given initial distribution of ω, consider, as often we do in Fourier's analysis, the arbitrary constant A in (6) to be a function of k of the form $A(k)\,dk$ and sum up all the expressions thus formed for all admissible values of k. Thus we can write

$$\omega = r^n \int_0^\infty A(k)\, e^{-\nu k^2 t} J_n(kr)\, k\, dk.$$

Suppose the initial distribution of ω to be given in the form

$$\omega_0=f(r), \quad \text{for } t=0, \tag{8}$$

where $f(r)$ is any function of r at our disposal, then comparing this with the above, we must have

$$f(r)=r^n\int_0^\infty A(k)\,J_n(kr)\,k\,dk.$$

Solving this equation for $A(k)$, by the aid of Hankel's reciprocal relation(1), we obtain

$$A(k)=\int_0^\infty f(\alpha)\,J_n(k\alpha)\,\alpha^{1-n}\,d\alpha, \tag{9}$$

for

$$n>-1.$$

Therefore the general solution appropriate to the initial condition (8), n being as defined in (7), is

$$\omega=r^n\int_0^\infty e^{-\nu k^2 t}\,J_n(kr)\,k\,dk\int_0^\infty f(\alpha)\,J_n(k\alpha)\,\alpha^{1-n}\,d\alpha, \tag{10}$$

provided that

$$n>-1, \quad \text{or} \quad c>-2\nu.$$

The expression (10) can be partly integrated, by making use of the formula

$$\int_0^\infty e^{-\lambda k^2}\,J_n(kr)\,J_n(k\alpha)\,k\,dk=\frac{1}{2\lambda}e^{-\frac{r^2+\alpha^2}{4\lambda}}\,I_n\left(\frac{\alpha r}{2\lambda}\right),^{(2)}$$

where

$$I_n(x)=i^{-n}\,J_n(ix),$$

(1) N. Nielsen, Cylinderfunctionen § 140.

(2) N. Nielsen, *l.c.* p. 184. Most integrals occuring in this paper are effected by the aid of similar formulæ found in his book.

in the form

$$\omega=\frac{r^n}{2\nu t}e^{-\frac{r^2}{4\nu t}}\int_0^\infty f(\alpha)\,e^{-\frac{\alpha^2}{4\nu t}}I_n\left(\frac{\alpha r}{2\nu t}\right)\alpha^{1-n}\,d\alpha. \qquad (11)$$

Either of the formulæ (10) or (11) may be used for the evaluation of the distribution of vortices at any time according to the form of the given function $f(r)$. A number of special examples will be given in the following paragraphs.

The case of no dynamical effect.

6. It will be convenient to consider first the simple case in which there is no dynamical effect, *i.e.* no radial velocity, as it is in this case easier to comprehend the manner of dissipation of vortex than in the general case. The solution to this special case is as follows:

When

$$\omega_0=f(r), \quad \text{for} \quad t=0, \qquad (12)$$

then

$$\omega=\int_0^\infty e^{-\nu k^2 t}J_0(kr)\,k\,dk\int_0^\infty f(\alpha)\,J_0(k\alpha)\,\alpha\,d\alpha, \qquad (13)$$

or

$$\omega=\frac{1}{2\nu t}e^{-\frac{r^2}{4\nu t}}\int_0^\infty f(\alpha)\,e^{-\frac{\alpha^2}{4\nu t}}I_0\left(\frac{\alpha r}{2\nu t}\right)\alpha\,d\alpha, \qquad (14)$$

for any time.(1)

7. Let us take, first, the simplest example in which the initial condition is given in the from

$$\omega_0=\frac{C}{r}, \quad \text{for} \quad t=0, \qquad (15)$$

(1) It will be noticed that this simple case has the same form of solution with that in the conduction of heat in a similar case.

C being a constant. To realize this distribution physically makes a necessity of having an infinite amount of energy, but this example may serve to afford a simple idea of the decay in question. In this case, since

$$\int_0^\infty f(\alpha)\, J_0\,(k\alpha)\, \alpha\, d\alpha = C\int_0^\infty J_0\,(k\alpha)\, d\alpha = \frac{C}{k},$$

we have

$$\omega = C\int_0^\infty e^{-\nu k^2 t}\, J_0\,(k\alpha)\, dk.$$

After performing the integration, we obtain

$$\omega = C\sqrt{\frac{\pi}{4\nu t}}\, e^{-\frac{r^2}{8\nu t}}\, I_0\left(\frac{r^2}{8\nu t}\right), \tag{16}$$

for the value of ω at time t.

Fig. 1 is drawn to illustrate how fast the vortices in a viscous fluid dissipate, when they are initially distributed according as (15).

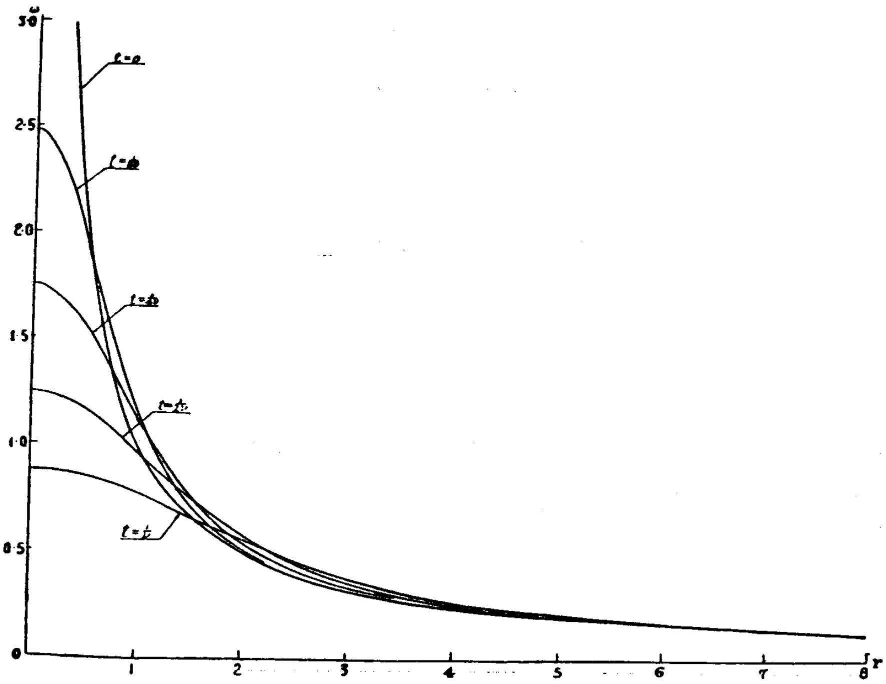

Fig. 1.

The ordinate is taken as the value of ω, for the times $t=0$, $1/8\nu$, $1/4\nu$, $1/2\nu$, $1/\nu$, and the abscissa the value of the distance r from the origin, both in arbitrary units. The constant C is put$=1$.

At the centre we have

$$\omega = C\sqrt{\frac{\pi}{4\nu t}},$$

for finite values of t. The strength of the vortex at the centre decreases at a decreasing rate as time increases; this will be seen from the figure.

For large values of r, taking the first term of the asymptotic expansion of the I_0-function, we have approximately

$$\omega = \frac{C}{r}.$$

Thus at a great distance from the centre the distribution of vortical motion does not change much from the initially given state. This is also shown in the same figure.

8. Next suppose that the initial condition is given by

$$\omega_0 = Ce^{-\frac{r^2}{a^2}}, \quad \text{for} \quad t=0, \tag{17}$$

a being a constant. This distribution of vortices is such that by taking a very small it approaches to the case of concentrated vortices, of finite amount, at points very near to the origin, and may be looked upon as the isolated vortex usually observed in nature. By integrating the expression (17) multiplied by $2\pi r\,dr$ over the whole coordinate plane, it will be easily seen that the total amount of vortices is $\pi a^2 C$, and the parameter a may be considered as the effective radius of the vortex initially given.

In this case use is made of the formula (14) to evaluate the distribution of ω at any subsequent time. Thus

$$\omega = \frac{C}{2\nu t} e^{-\frac{r^2}{4\nu t}} \int_0^\infty e^{-\left(\frac{1}{a^2}+\frac{1}{4\nu t}\right)\alpha^2} I_0\left(\frac{\alpha r}{2\nu t}\right) \alpha\, d\alpha.$$

This gives us, on integrating, simply

$$\omega=\frac{Ca^2}{a^2+4\nu t}e^{-\frac{r^2}{a^2+4\nu t}}. \tag{18}$$

In Fig. 2 we take $a=1$ for an example. The curves in it represent ω at $t=0$, $1=8\nu$, $1/4\nu$, $1/2\nu$. We can grasp from the figure that when initially a given amount of vorticity is concentrated at a point, it decreases rapidly in strength at that point, and at the same time spreads out in all directions.

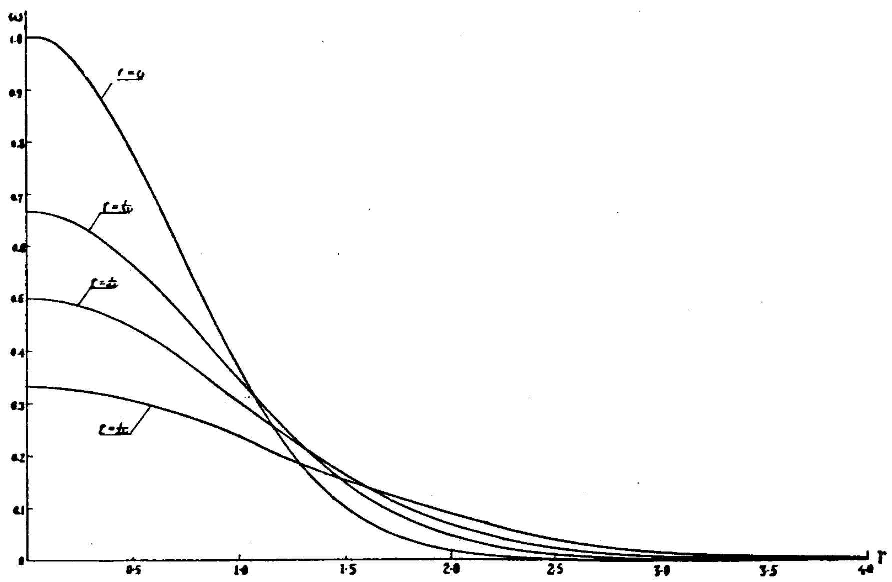

Fig. 2.

At the centre we have

$$\omega=\frac{Ca^2}{a^2+4\nu t}. \tag{19}$$

The rate at which the vortex at the centre decays is much larger than that in the last example.

The time τ at which the strength of vortex at the centre becomes $1/2$ of the original value is

$$\tau=\frac{a^2}{4\nu},$$

proportional to the square of the effective radius of the initial vortex.

For air, if we assume $\nu=0.13$ c. g. s., when $a=1$,

$$\tau=2 \ sec.$$

For water, if we take $\nu=0.01$ c. g. s.

$$\tau=25 \ sec.$$

In the atmosphere ν increases with the height, and accordingly we may, by applying the result of this example, suppose that the vortical motion at a high altitude dissipates more quickly than that near the ground. Thus, at a high altitude the state of motion of air may well be considered to be comparatively uniform, and one may find there a *safe region for flying*.

9. Let the initial condition be

$$\omega_0 = Cre^{-\frac{r^2}{a^2}}, \quad \text{for} \quad t=0. \tag{20}$$

This distribution is such that there is no vorticity at the centre, and a maximum intensity appears at a little distance off the centre. In this case, by using the formula (14),

$$\omega = \frac{C}{2\nu t} e^{-\frac{r^2}{4\nu t}} \int_0^\infty e^{-\left(\frac{1}{a^2}+\frac{1}{4\nu t}\right)\alpha^2} I_0\left(\frac{\alpha r}{2\nu t}\right)\alpha^2 \, d\alpha.$$

A little calculation gives us the result :

$$\omega = \frac{Ca^3\sqrt{\pi\nu t}}{(a^2+4\nu t)^{3/2}} e^{-\frac{r^2}{8\nu t}\frac{a^2+8\nu t}{a^2+4\nu t}} \cdot \Bigg[I_0\left(\frac{a^2 r^2}{8\nu t\,(a^2+4\nu t)}\right)$$
$$+\frac{a^2 r^2}{8\nu t\,(a^2+4\nu t)}\left\{ I_0\left(\frac{a^2 r^2}{8\nu t\,(a^2+4\nu t)}\right) + I_1\left(\frac{a^2 r^2}{8\nu t\,(a^2+4\nu t)}\right)\right\}\Bigg]. \tag{21}$$

At the centre we have simply

$$\omega = \frac{Ca^3\sqrt{\pi\nu t}}{(a^2+5\nu t)^{3/2}}.$$

It is interesting to observe that, calculating the time rate of ω, near the centre, where there was no vortex initially, a vortical motion begins suddenly to set in and continues to grow until the time $t=a^2/8\nu$ is reached, when it has a maximum value, and after that time it subsides slowly at an increasing rate. Fig. 3 shows the approximate features of the distribution of vortices for the times $t=0$, $1/8\nu$, $1/4\nu$, $1/2\nu$, when $a=1$.

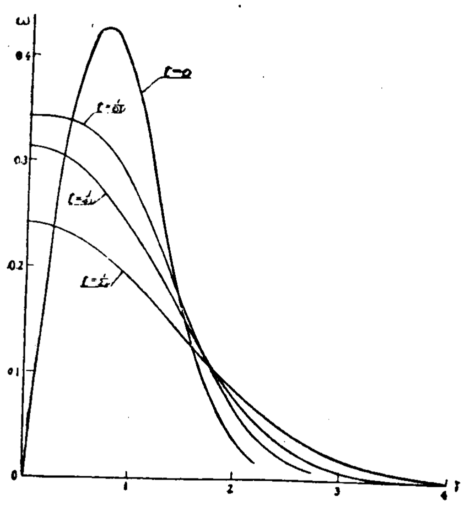

Fig. 3.

From this example it can be seen that when a group of rectilinear parallel vortices is distributed forming a hollow cylindrical shape, it will rapidly collapse towards the centre, leaving no free part in the interior, and producing, after a while, a maximum strength at that point, after which it will gradually dissipate. A similar example will be given again presently in a later paragraph.

10. So far the examples considered are exclusively such that the initial distribution of vortices spreads over the whole space, though their effective part is gathered into a finite region, with decreasing strength as the distance therefrom increases. Now let us take an example in which the initial vortices are limited to the finite region of a circular cylinder, and have a constant intensity, without which there is no vortex motion ; *i. e.*

$$\begin{aligned} \omega_0 &= C, \quad \text{for} \quad r<a, \\ \text{,,} &= 0, \quad \text{,,} \quad r>a, \end{aligned} \qquad (22)$$

when $t=0$, a being the radius of the cylinder.

In this case we have to calculate

$$\omega = C\int_0^\infty e^{-\nu k^2 t} J_0(kr)\, k\, dk \int_0^a J_0(k\alpha)\, \alpha\, k\alpha,$$

to obtain the value of ω for any time and at any point. The process of integration in this case may be worth sketching in a few lines.

Since

$$\int_0^a J_0(k\alpha)\, \alpha\, d\alpha = \frac{a}{k} J_1(ka),$$

the above expression becomes

$$\omega = Ca\int_0^\infty e^{-\nu k^2 t} J_0(kr)\, J_1(ka)\, dk.$$

Now by the aid of Neumann's addition theorem for $J_0(kR)$, it is easy to show that

$$J_0(kr)\, J_1(ka) = \frac{1}{\pi}\int_0^\pi J_1(kR)\, \frac{a - r\cos\theta}{R}\, d\theta,$$

where

$$R = \sqrt{a^2 - 2ar\cos\theta - r^2}.$$

Therefore

$$\omega = \frac{Ca}{\pi}\int_0^\pi \frac{a - r\cos\theta}{R}\, d\theta \int_0^\infty e^{-\nu k^2 t} J_1(kR)\, dk.$$

On carrying out the second integration, we get

$$\omega = \frac{Ca}{\pi}\int_0^\pi \left\{1 - e^{-\frac{R^2}{4\nu t}}\right\} \frac{a - r\cos\theta}{R^2}\, d\theta. \tag{23}$$

It can easily be found, by making use of the consine-series for $(a - r\cos\theta)/R^2$, that

$$\int_0^\pi \frac{a-r\cos\theta}{R^2}d\theta=\frac{\pi}{a}, \qquad r<a,$$

$$\text{,,} \qquad =0, \qquad r>a,$$

and, of Hansen's integral formula for $J_m(x)$, that

$$\int_0^\pi \frac{a-r\cos\theta}{R^2}e^{-\frac{R^2}{4\nu t}}d\theta=\frac{\pi}{a}e^{-\frac{a^2+r^2}{4\nu t}}\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^m I_m\left(\frac{ar}{2\nu t}\right), \quad r<a,$$

$$=-\frac{\pi}{a}e^{-\frac{a^2+r^2}{4\nu t}}\sum_{m=1}^{\infty}\left(\frac{a}{r}\right)^m I_m\left(\frac{ar}{2\nu t}\right), \quad r>a.$$

Substituting these values of integrals in (23), we obtain

$$\left.\begin{aligned}\omega&=C-Ce^{-\frac{a^2+r^2}{4\nu t}}\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^m I_m\left(\frac{ar}{2\nu t}\right), \qquad r<a,\\ \text{,,}&=Ce^{-\frac{a^2+r^2}{4\nu t}}\sum_{m=1}^{\infty}\left(\frac{a}{r}\right)^m I_m\left(\frac{ar}{2\nu t}\right), \qquad r>a.\end{aligned}\right\} \tag{24}$$

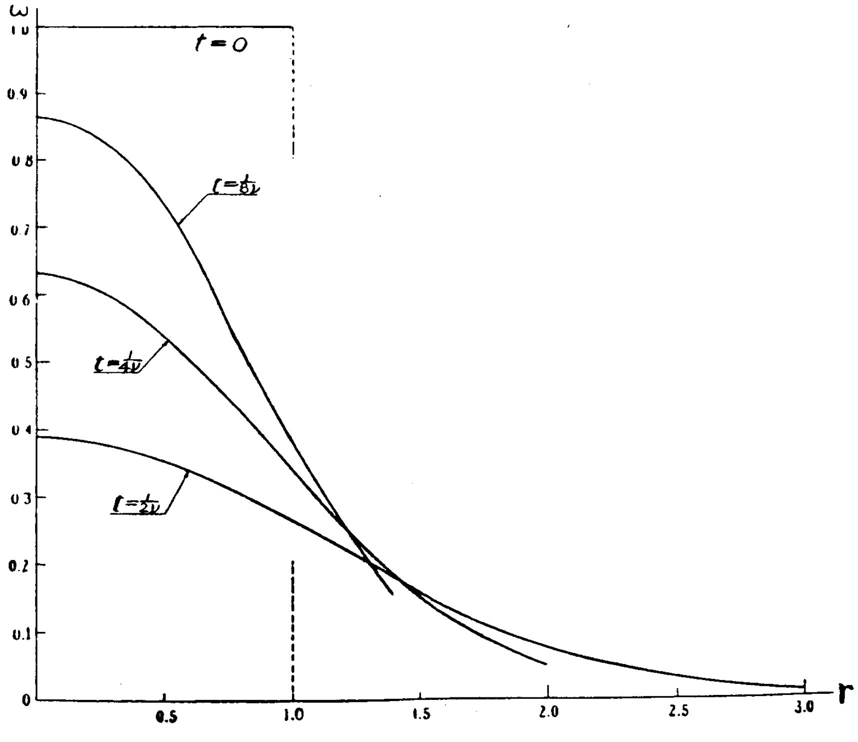

Fig. 4.

Numerical calculations have been worked out for the case $a=1$ in the above expressions. The results of these are shown in Fig. 4. The curves in it represent ω at $t=0$, $1/8\nu$, $1/4\nu$, $1/2\nu$.

At the centre of the initially disturbed area we have, by putting $r=0$ in the upper formula of (24), that

$$\frac{\omega_0-\omega}{\omega_0}=e^{-\frac{a^2}{4\nu t}}. \tag{25}$$

If we look upon the quantity $(\omega_0-\omega)/\omega_0$ as the measure of decay of vortex, it increases, at the centre, rapidly with time. The time τ' at which the *amount of decaye* at the centre becomes e^{-1} of the original strength of the vortex is

$$\tau'=\frac{a^2}{4\nu}. \tag{26}$$

again proportional to the square of the radius of initial vortex.

For air, if we take $\nu=0.13$ c. g. s., we get, when $a=1$,

$$\tau'=2 \ sec.$$

nearly. For water if we take $\nu=0.01$ c. g. s., then

$$\tau'=25 \ sec.$$

11. To see what takes place at the periphery of the circle, along which initially a discontinuous distribution of vortices exists, will be of interest. Both the formulæ (24) converge to the same expression

$$\omega=\frac{C}{2}\left\{1-e^{-\frac{a^2}{2\nu t}}I_0\left(\frac{a^2}{2\nu t}\right)\right\}^{(1)}, \tag{27}$$

which may be obtained, except when $t=0$, by putting $r=a$ in either of them. Thus we have at the point just inside the circle

$$\frac{\omega_0-\omega}{\omega_0}=\frac{1}{2}\left\{1+e^{-\frac{a^2}{2\nu t}}I_0\left(\frac{a^2}{2\nu t}\right)\right\}, \tag{28}$$

(1) When we put $t=0$ in this expression, it does not afford the initially prescribed condition, but the mean of the values at points just inside and just outside the circle, as usually the case in the expansion ef discontinuous function. This remark applies also to (29) and (30).

as the measure of decay.

For small values of t, taking the first term of the asymptotic expansion for the I_0-function into account, we get

$$\frac{\omega_0-\omega}{\omega_0}=\frac{1}{2}\left\{1+\left(\frac{\nu t}{\pi a^2}\right)^{\frac{1}{2}}\right\}. \tag{29}$$

For large values of t, we obtain approximately

$$\frac{\omega_0-\omega}{\omega_0}=\frac{1}{2}\left\{1+e^{-\frac{a^2}{2\nu t}}\right\}. \tag{30}$$

From (27) or (29), it appears that, for small values of t, the strength of the vortex at the point very near to the periphery of the circle is nearly $\omega_0/2$, never exceeding this value. Thus at the point just inside the cylinder the strength of the vortex will decrease suddenly to half of its initial value and the vorticity will be communicated to the neighbouring part of the fluid which was initially just outside the cylinder.

Taking the ratios of (27) to (29) and to (30) it will be seen at once that, for small values of t, the measure of decay at the centre is much less than that at the periphery, and ultimately, as time advances, they tend to the same limiting value, when the vortices will altogether dissipate.

12. The rate at which the vortex grows up at the points initially undisturbed ($r>a$) is obtained by differentiating the lower formula of (24) with respect to t. Thus

$$\frac{\partial\omega}{\partial t}=\frac{C}{4\nu t^2}e^{-\frac{a^2+r^2}{4\nu t}}\sum_{m=1}^{\infty}\left(\frac{a}{r}\right)^m\left\{(a^2+r^2)\,I_m\left(\frac{ar}{2\nu t}\right)-2ar\,I'_m\left(\frac{ar}{2\nu t}\right)\right\}, \tag{31}$$

the dash denoting the differentiation with respect to the argument.

For small values of t, taking the first term of the asymptotic expansions of I_m and I'_m into account we find

$$\frac{\partial\omega}{\partial t}=\frac{C(r-a)}{4}e^{-\frac{(r-a)^2}{4\nu t}}\left(\frac{a}{\pi\nu r t^3}\right)^{\frac{1}{2}}, \tag{32}$$

which is evidently positive since $r>a$.

For large values of t, taking the terms of the lowest degree in $1/t$, we get

$$\frac{\partial\omega}{\partial t}=-\frac{Ca^2}{4\nu t^2}\left(1-\frac{r^2}{2\nu t}\right), \tag{33}$$

which is negative for sufficiently large values of t.

In this way, at any point initially free from vortical motion, the vortex develops at first, at a considerable rate, to a certain amount, attaining its maximum strength, and then ultimately dies away at an evanescent rate.

The courses of ω and of $\partial\omega/\partial t$ for the point $r=2a$ are exhibited in Fig. 5, in which the abscissa is the value of $\nu t/a^2$; the scale of the ordinate for each of them is quite different, that for ω being much enlarged. From the figure can be seen what we have stated in the above.

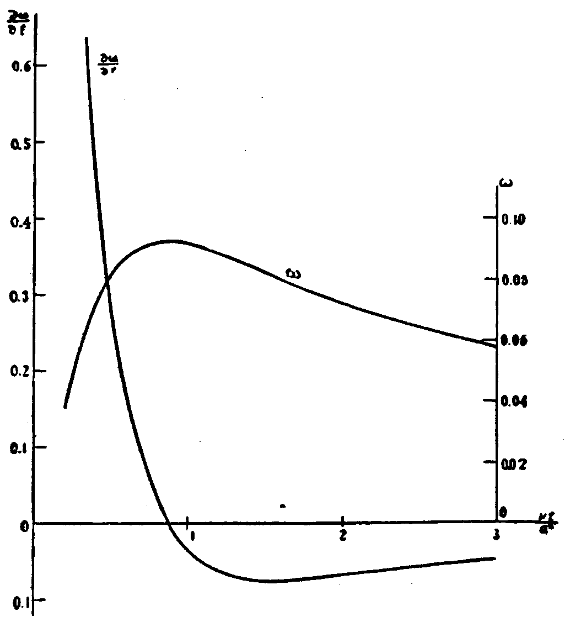

Fig. 5.

13. *Equalizer in a wind Channel.* The result of the example in the preceeding three paragraphs may find its application in determining the position of the equalizer in a wind channel. Usually in a wind channel an equalizar of honeycomb shape, made of some sheets of metal, is placed ahead of the experimental post to which the model is attached. Suppose that the equalizer is made of metal sheets of 1 mm. thickness and that a uniform air stream passing this forms, at the point just leaving it, an eddy of 1 mm. in its diameter. This eddy will drift along the general current decreasing its strength in its course. The amount of its decay can be approximately calculated by using the equations in the preceeding paragraphs, provided that the dynamical effects are considered to be negligible. Thus in the formula (26) if we put a=0.05 cm., we shall have τ'=0.005 sec.; in words: the time in which the strength of the eddy becomes $1-e^{-1}$, or nearly 63% of its initial strength, is 0.005 sec. During this time the eddy will be carried away down the stream as far as accords with the velocity of the general current. The distance covered in this way by the vortex in the interval 0.005 sec. is as follows:

velocity of wind in m/sec.	distance covered in cm.
10	5
20	10
30	15
60	30
100	50
150	75

These figures may be utilised to determine approximately the relative position of the model and the equalizer, when the velocity of the general current is given, according to the required degree of accuracy of the experimennts. Conversely, in a wind channel where the relative position of them is fixed, the accuracy of the experiments may be prejudged in connection with the velocity of the stream. But, from the fromula (26), the time τ' is proportional to the square of the thickness of the sheets, and, as we usually empoloy thinner sheets than those given in the above example, the figures in the second column in practical cases will become far smaller; and thus the effect of the vortical motion due to the equalizer will not be very serious in usual wind channels.

14. Lastly we consider the case in which the vortices initially distribute themselves along the circumference of a circle of radius a with an infinitely small thickness δa, forming a sort of cylindrical vortex sheet; to be precise:—

$$
\begin{aligned}
\omega_0 &= 0, &&\text{for} && r<a,\\
\text{,,} &= C, && \text{,,} && a<r<a+\delta a, \qquad\qquad (34)\\
\text{,,} &= 0, && \text{,,} && a+\delta a<r,
\end{aligned}
$$

when $t=0$.

In this case making use of equation (13), we have

$$\omega = C\int_0^\infty e^{-\nu k^2 t} J_0(kr)\, k\, dk \int_a^{a+\delta a} J_0(k\alpha)\, \alpha\, d\alpha.$$

Since

$$\int J_0(k\alpha)\,\alpha\, d\alpha = \frac{1}{k^2} J_1(k\alpha)\, k\alpha,$$

if we neglect $(\delta a)^2$ and higher terms in it, we get, on writing A for $C\delta a$,

$$\omega = Aa\int_0^\infty e^{-\nu k^2 t} J_0(ka)\, J_0(kr)\, k\, dk.$$

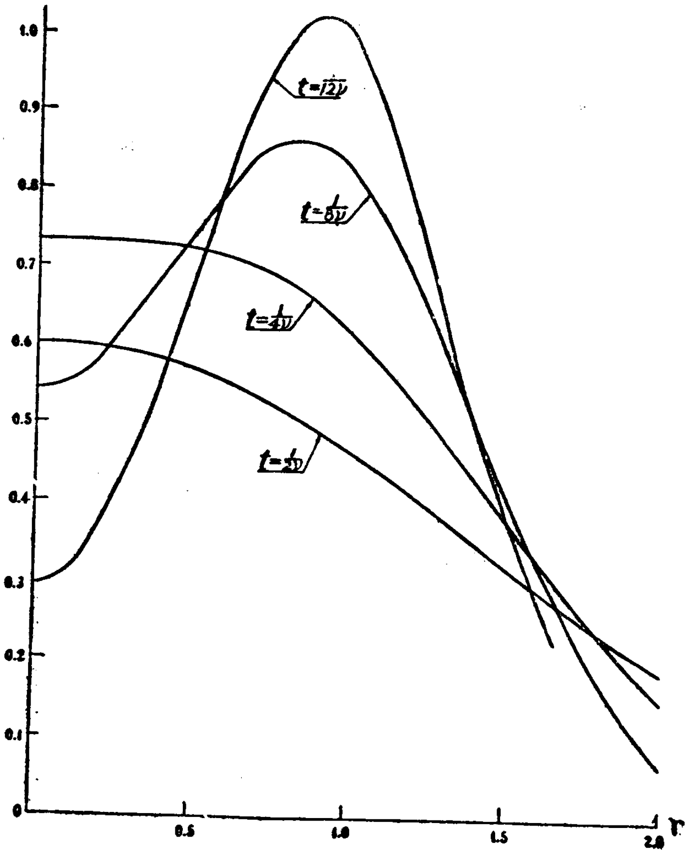

Fig. 6.

On carrying out the integration, it follows that

$$\omega=\frac{Aa}{2\nu t}e^{-\frac{a^2+r^2}{4\nu t}}I_0\left(\frac{ar}{2\nu t}\right). \tag{35}$$

The same result may of course be arrived at, to the same order of approximation, on starting from the general equation (14).

In Fig. 6, the approximate curves of ω for the times $t=1/12\nu$, $1/8\nu$, $1/4\nu$, $1/2\nu$, taking $a=1$, $A=1$, are shown.

The position of maximum intensity of ω may be found by putting the right hand side of the equation

$$\frac{\partial\omega}{\partial r}=\frac{Aa}{4\nu^2t^2}\left\{a\,I_1\left(\frac{ar}{2\nu t}\right)-r\,I_0\left(\frac{ar}{2\nu t}\right)\right\}e^{-\frac{a^2+r^2}{4\nu t}},$$

equal to zero. As we have always

$$I_0>I_1,$$

for all real values of the argument, the value of r giving the maximum position of ω must be less than a; and obviously $r=0$ makes the above expression vanish, affording an extremum value of ω. More detailed investigation will show that for small values of t a maximum of ω appears at the point whose distance from the centre is very little less than a, and a minimum at the centre, and that as time advances the maximum point travels towards the centre, coalescing at last with the minimum point $r=0$, when the centre is transformed to the point of maximum strength of the vortex. In this way, if, in a viscous fluid, an isolated cylindrical vortex sheet is once created, its effective radius becomes smaller and smaller, tending to collapse towards the centre, and then it dissipates in such a manner as we saw in many examples in preceeding paragraphs.

15. It will be interesting to consider the case in which two *coaxial* cylindrical vortex sheets such as discussed in the preceeding paragraph exist at the same time. Indeed if these were not coaxial it would be more interesting and important in practice, but, in the present investigation, such a general case is unfortunately excluded and must be left to be studied on another occasion.

In this case the initial conditions are:

$$\begin{aligned} \omega_0 &= 0, && \text{for} && r<b, \\ ,, &= C', && ,, && b<r<b+\delta b, \\ ,, &= 0, && ,, && b+\delta b<r<a, \\ ,, &= C, && ,, && a<r<a+\delta a, \\ ,, &= 0, && ,, && a+\delta a<r, \end{aligned} \tag{36}$$

when $t=0$.

On writing A for $C\delta a$ and B for $C'\delta b$, we shall have

$$\omega = \frac{1}{2\nu t} e^{-\frac{r^2}{4\nu t}} \left\{ Aa I_0\left(\frac{ar}{2\nu t}\right) e^{-\frac{a^2}{4\nu t}} + Bb\, I_0\left(\frac{br}{2\nu t}\right) e^{-\frac{b^2}{4\nu t}} \right\} \tag{37}$$

up to the second order in δa and δb.

The question is what will be the mutual interference of these two systems of vortices when they are created at the same time and left to themselves. Anyhow, as we know from the preceeding examples, the vortices spread out through the whole space distributing themselves in a proper way; but where will be the centre, if any, of the joint existence? The centre, or the point at which the strength of the resulting vortex is maximum will be found as usual by obtaining $\partial\omega/\partial r$ and putting the result equal to zero. Thus

$$Aa\left\{ aI_1\left(\frac{ar}{2\nu t}\right) - rI_0\left(\frac{ar}{2\nu t}\right)\right\} e^{-\frac{a^2}{4\nu t}} + Bb\left\{ b\, I_1\left(\frac{br}{2\nu t}\right) - rI_0\left(\frac{ar}{2\nu t}\right)\right\} e^{-\frac{b^2}{4\nu t}} = 0.$$

It can be seen without laborious calculation that when

(i) A and B are of the same sign,

there is one value of r such that $b<r<a$ for which $\partial\omega/\partial r$ vanishes, at least for small values of t, and for the values of r greater than a it is constantly negative. On the contrary, when

(ii) A and B are of different sign,
there can be one value of r in the region $r>a$ which makes $\partial\omega/\partial r$ vanish, and between b and a it is of a definite sign.

When $r=0$, $\partial\omega/\partial r$ vanishes for all values of t. Although this affords a maximum or a minimum value of ω, this point, for the present, does not play an important rôle and may be dispensed with, as the existence of extremum values of ω at the origin may be ascribed to

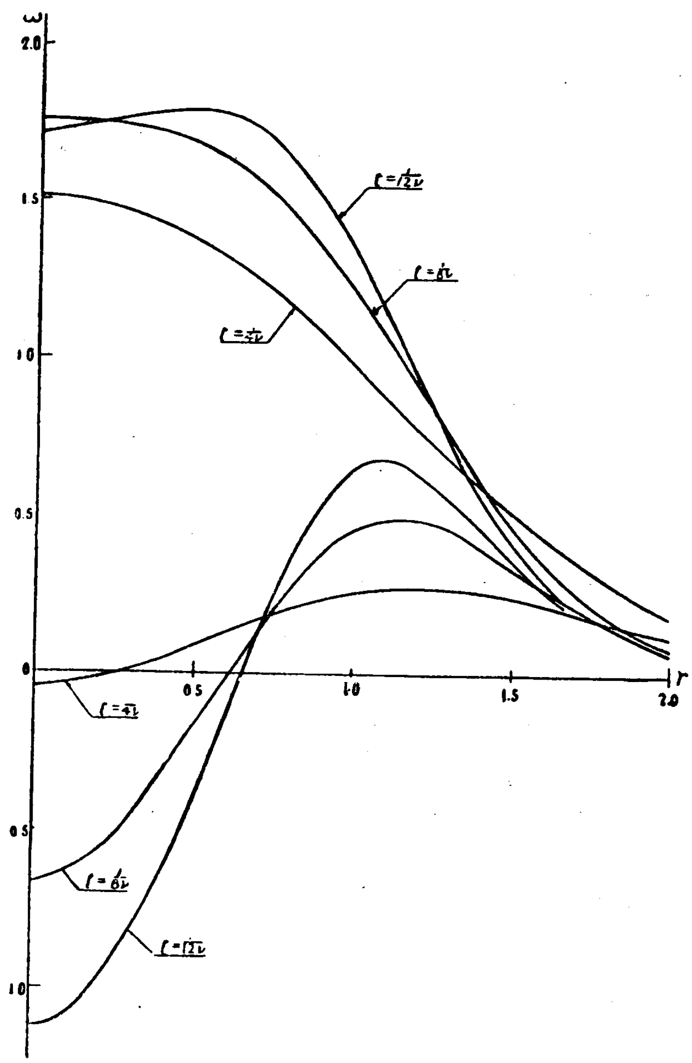

Fig. 7.

the property of one hollow cylindrical vortex sheet but not of the joint action of two systems of such vortex sheets.

From the above analysis an interesting result may be enunciated; viz., when two coaxial cylindrical vortex sheets are created at the same time they tend to approach each other, coalescing if the sens of rotation of the two is the same, and, on the contrary, flying apart if the sense of rotation is different. These properties may be observed in Fig. 7, in which the resultants of two vortical systems are approximately exhibited. In this figure we take as the one vortex system: radius $a=1$ and intensity $A=1$, and as the other: $b=1/2$, $B=\pm 1$. The upper three curves are for the case $A=1$, $B=1$, *i.e.* A and B are of the same sign, at the times $t=1/12\nu$, $1/8\nu$, $1/4\nu$; and the lower three are for the case $A=1$, $B=-1$, *i.e.* they are of opposite sign, at the same times as the above.

16. *Okada's law in meteorology.* According to Dr. S. Fujiwhara[1], if two vortical motions exist in the atmosphere at the same time, they tend to approach one another, when they have the same vortical sense, and they tend to repulse each other when they are of the opposite sense. He says that this law was deduced many years ago by Dr. T. Okada in the Central Meteorological Office in Tokio from a number of observations of cyclones and is called by his name among some meteorologists in Japan. This law, established statistically, is not only important in practical meteorology, but is very interesting from the theoretical point of view. In effect, it is a well known fact that if two rectilinear vortices exist in a *perfect fluid* they rotate round each other with the centre at the mean point of their distance when the sense is the same, and travel along a certain direction through the fluid without

---

(1) 氣象集誌第四十一年第六號，渦動報告第二報，岡田ノ法則及其擴張. Also see: the same author, On the Growth and Decay of Vortical Motion. Bul. Meteorological Observatory. Tokyo. **3**, No. 4.

changing their mutual distance when the sense is opposite. Long ago, in his splendid work on the theory of motion of the atmosphere, the late Professor J. Kitao[1] obtained a series of remarkable theorems some of which may be looked upon, at least in some cases, as the general basis of Okada's law. He considers the atmosphere to be, so to say, a *semiviscous fluid* or in other words, subjected to the frictional force due to the rotation of the earth and transmitting the force vertically with more or less decrease as the height from the surface of the earth increases. He says that two cyclones in the atmosphere approach each other rotating about their middle centre in the cyclonic sense, while two anticyclones separate themselves rotating about their middle centre in the anticyclonic sense. Since, in Japan, cyclones are usually observed and one may well deduce statistically the law concerning the coexistence of two cyclones. The first part of Okada's law affirms the first part of Kitao's theorem, but the second part does not. On the other hand, Kitao further says that when a cyclone and an anticyclone occur at the same time they approach each other or separate, describing spiral orbits in the sense of cyclonic or anticyclonic, according as the strength of the cyclone surpasses that of the anticyclone or not. Thus the second part of Okada's law does in one case confirm this statement, and does not in another. Kitao's atmosphere, however, is of a particular nature as stated in the above, and the viscosity of air in its ordinary meaning is not taken into account. Consequently it cannot well be said that his theorem, though elegant in its form, governs all the phenomena of the real atmosphere. At any rate, if the law discovered through meteorological observations by Dr. Okada is true, it is desirable to establish the theoretical basis on which it stands.

We have obtained, in the preceding article, a similar result

(1) Beiträge zur Theorie der Bewegung der Erdatmosphäre und der Wirbelstürme. Journal of the College of Science, Imperial University Tokyo. **2**, part 5 (1889), p. 330—.

to Okada's law in connection with two systems of vortices, considering the air as a viscous fluid. The analytical result obtained, however, concerns only the case in which two vortical systems are of *coaxial* cylindrical sheets, but not two systems of random distribution. Therefore it cannot be regarded as a theoretical interpretation of the law in hand, but it may well serve as a demonstration of a special case. A satisfactory treatment in the general case will be undertaken on another opportunity. At present, the author is content with only slightly touching on the above law and a recollection of the said theorem, both enunciated by Japanese meteorologists.

Case with dynamical effects.

17. Hitherto we have been exclusively concerned with the simple case in which there are no dynamical effects, and many examples with interesting and important result have been illustrated. In the following we shall give a few examples of the general case in which dynamical effects give rise to results. Here by dynamical effects is meant the existence of the radial velocity u, obtained in § 3 as a possible solution of the general equations of motion. The investigation of this case may not play an important part in finding simple properties of the vortices themselves as in the case previously considered, yet we may have interesting results from examining special examples in this general case. The solution to this case is given in (10) or (11) answering to the initial distribution of vortices such as expressed in (8), the radial velocity u being prescribed as in (4).

18. Let us take, first, the example which has been considered in § 7 with the initial distribution of vortices as

$$\omega_0=\frac{C}{r}, \quad \text{for} \quad t=0, \qquad (38)$$

but with the radial velocity

$$u=\frac{2\nu}{r}. \tag{39}$$

In this case we have $n=1$, n being defined in (7), and

$$\omega=Cr\int_0^\infty e^{-\nu k^2 t}J_1(kr)\,k\,dk.$$

Performing the integration we shall obtain the expression

$$\omega=C\sqrt{\frac{\pi}{\nu t}}\,\frac{r^2}{8\nu t}e^{-\frac{r^2}{8\nu t}}\left\{I_0\left(\frac{r^2}{2\nu t}\right)-I_1\left(\frac{r^2}{2\nu t}\right)\right\}, \tag{40}$$

as the distribution of ω at any time.

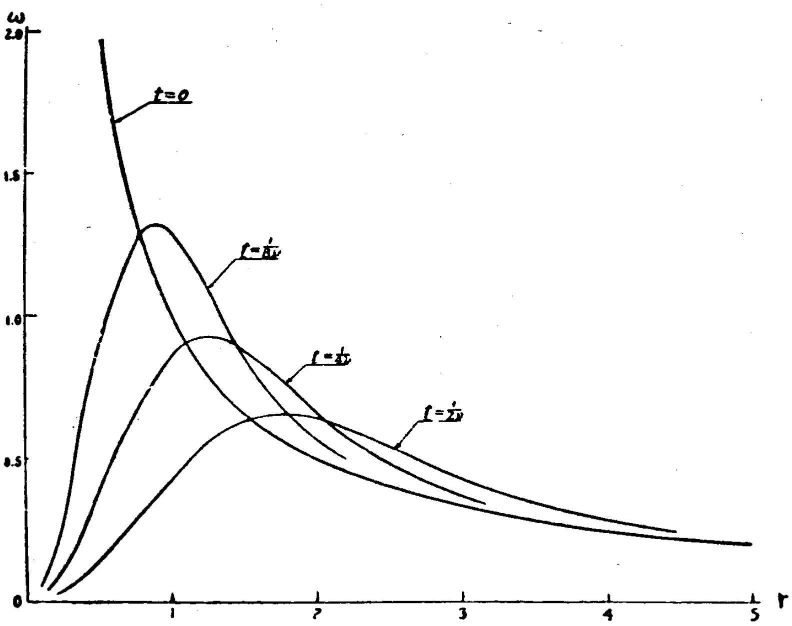

Fig. 8.

Fig. 8 is the general representation of the distribution of vortices. The curves in it are ω for the times 0, $1/8\nu$, $1/4\nu$, $1/2\nu$, the abscissa being r as before.

As we see from the figure, the intensity of the vortices near the point $r=0$, where it was initially greatest, is evanescently small, while there appears a maximum intensity at the point a little distance off. This maximum travels outwards as time advances, which may be ascribed, on comparing the above figure with Fig. 1, to the existence of the radial velocity.

The very point $r=0$, however, must be put aside out of the consideration, since the analytical method employed in obtaining the general solntion in § 5 is not valid, in general at this point, unless the order of the Bessel Function involved is zero, which is the case of no dynamical effect. This remark will apply to all the following paragraphs.

The maximum point of ω at any instant can be found by differentiating the expression for ω with respect to r and putting it equal to zero. In this way we have the equation

$$\frac{r^2}{4\nu t}\left\{I_0\left(\frac{r^2}{8\nu t}\right)-I_1\left(\frac{r^2}{8\nu t}\right)\right\}-I_0\left(\frac{r^2}{8\nu t}\right)=0, \tag{41}$$

to determine the value of r for which the intensity of ω is maximum at time t. To solve this equation may not be very difficult, but it will not reward the labour, since the example in this paragraph is not, as stated in § 7, very important in its practical application.

The velocity U with which the maximum point travels can be obtained by finding the value of $\partial r/\partial t$ from the equation (41). A little calculation will give us simply that

$$U=\frac{r_1}{2t}=\frac{\nu}{ut},$$

where r_1 is the root of the equation (41). From this we see that the most intense part, in this special example, of the vortical motion is not composed of the same fluid.

19. Next consider the initial distribution of ω such that

$$\omega_0 = Ce^{-\frac{r^2}{a^2}}, \quad \text{for} \quad t=0. \tag{42}$$

This example has been discussed already in § 8 in the case of no dynamical effect. Let us now superpose the radial velocity

$$u = \frac{2\nu}{r}, \tag{43}$$

on the above vortical motion.

Making use of the formula (11) we have to calculate

$$\omega = \frac{Cr}{2\nu t} e^{-\frac{r^2}{4\nu t}} \int_0^\infty e^{-\left(\frac{1}{a^2}+\frac{1}{4\nu t}\right)\alpha^2} I_1\left(\frac{\alpha r}{2\nu t}\right) d\alpha.$$

The evaluation of this integral can be effected in terms of Bessel's modified function of the order $1/2$ and finally we shall arrive simply at

$$\omega = C\left\{e^{-\frac{r^2}{a^2+4\nu t}} - e^{-\frac{r^2}{4\nu t}}\right\}. \tag{44}$$

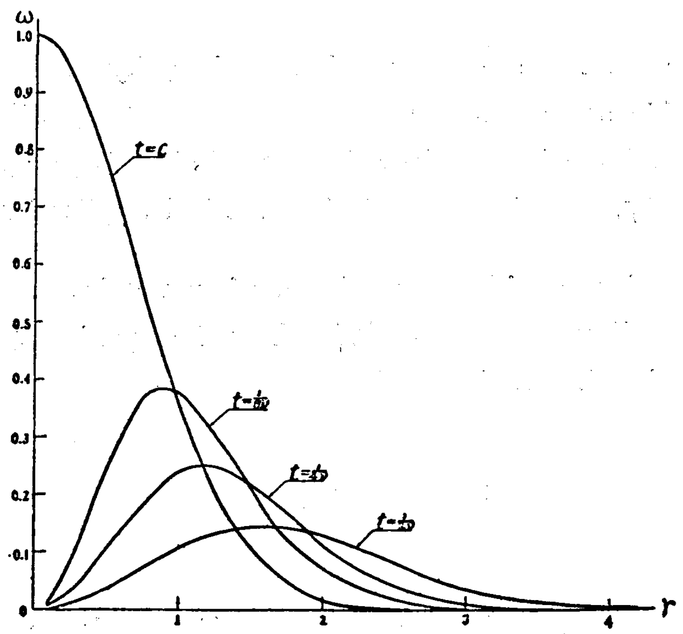

Fig. 9.

Fig. 9 shows the approximate courses of ω at the times 0, $1/8\nu$,

$1/4\nu$, $1/2\nu$, taking $a=1$ as before.

Again we have the point of maximum intensity for each instant, and this point travels, as in the last example, outward from the centre. The position of this maximum point is easily found from

$$r_1^2=4\nu t\left(1+\frac{4\nu t}{a^2}\right)log\left(1+\frac{a^2}{4\nu t}\right), \tag{45}$$

for any value of t. The velocity U with which the point of maximum intensity travels is

$$U=\frac{r_1}{2t}\left(1+\frac{4\nu t}{a^2+4\nu t}\right)-u.$$

It can be computed that this velocity is less than that of the fluid particles for the finite values of t, and as time increases indefinitely the former converges to the latter, both being of course inflnitely small.

The maximum intensity of the vortex for any time t is obtained by substituting the value of r, found from (45), in (44), and if we write ω_m for it, we have

$$\omega_m=\frac{Ca^2}{a^2+4\nu t}\left(1+\frac{a^2}{4\nu t}\right)^{-\frac{4\nu t}{a^2}}. \tag{46}$$

By comparison of this value with that of (19) we see that the maximum value of ω found here is always less than the corresponding value in the case of no dynamical effect.

The type of the initial distribution of the vortices considered in this article, as stated already in § 8, is of common occurrence in nature and it is worth while to pay special notice to it. From the above we see that when there is a source of a certain strength, as in (43), at the vortical centre, the most intense part of the vortical motion travels, as might be imagined, outward with a velocity decreasing with time but always less than that of the fluid due to the source, and that the decay of the vortices occurs more quickly than when there is no source at the centre.

20. Take the initial distribution of vortices such as

$$\omega_0 = Cr^2 e^{-\frac{r^2}{a^2}}, \quad \text{for} \quad t=0, \tag{47}$$

with the radial velocity

$$u = \frac{2\nu}{r}. \tag{48}$$

This example may be considered as a special case of the initial distribution such as

$$\omega_0 = Cr^{2m} e^{-\frac{r^2}{a^2}}, \tag{49}$$

when $m=1$. In this latter case we have to calculate

$$\omega = \frac{Cr^m}{2\nu t} e^{-\frac{r^2}{4\nu t}} \int_0^\infty e^{-\left(\frac{1}{a^2}+\frac{1}{4\nu t}\right)\alpha^2} I_m\left(\frac{\alpha r}{2\nu t}\right) \alpha^{m+1} d\alpha.$$

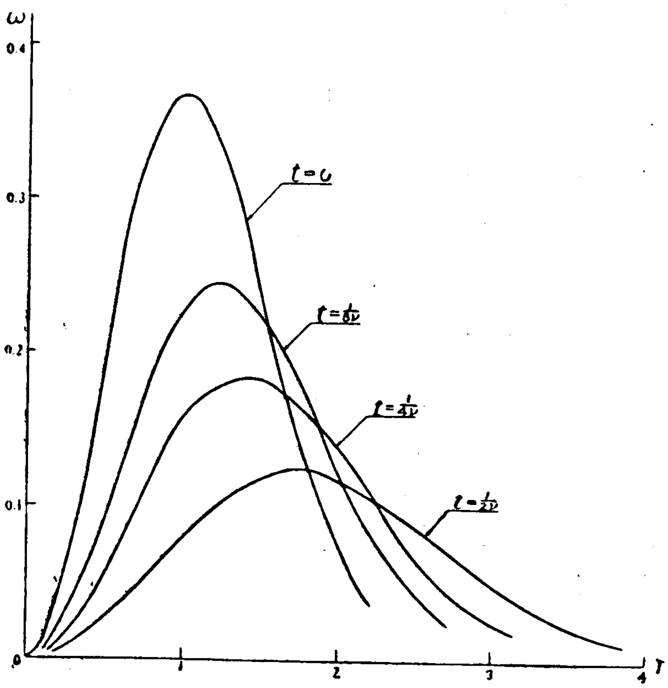

Fig. 10.

On carrying out the integration it follows that

$$\omega = Cr^{2m}\left(\frac{a^2}{a^2+4\nu t}\right)^{m+1} e^{-\frac{r^2}{a^2+4\nu t}}. \tag{50}$$

Putting $m=1$ in this expression we get

$$\omega = Cr^2\left(\frac{a^2}{a^2+4\nu t}\right)^2 e^{-\frac{r^2}{a^2+4\nu t}}, \tag{51}$$

as the expression for the vortical distribution answering to the prescribed conditions (47) and (48).

The approximate courses of the distribution at the times 0, $1/8\nu$, $1/4\nu$, $1/2\nu$ are drawn in Fig. 10, taking $a=1$.

The point at which the vortical motion is most intense can be obtained in the usual way and is found to be determined by

$$r_1^2 = a^2 + 4\nu t,$$

for any value of t. The traveling velocity U of this maximum point becomes

$$U = \frac{2\nu}{r} = u,$$

being equal to that of the fluid due to the source at the centre. This property, as may be easily calculated, is common to every vortical distribution corresponding to the condition (49) with any admissible value of m. Another example of this property will follow immediately.

21. Next consider

$$\omega_0 = Cre^{-\frac{r^2}{a^2}}, \quad \text{for} \quad t=0, \tag{52}$$

with the radial velocity

$$u = \frac{\nu}{r}. \tag{53}$$

This initial distribution of vortices has been discussed previously in

§9, but without the dynamical effect. The present case is also a special one of (49) with the value of $m=1/2$, and consequently the expression for ω at any time is obtained by putting $m=1/2$ in the formula (50); thus

$$\omega = Cr\left(\frac{a^2}{a^2+4\nu t}\right)^{\frac{3}{2}} e^{-\frac{r^2}{a^2+4\nu t}}. \tag{54}$$

Fig. 11 represents the distribution of ω for the times 0, $1/8\nu$, $1/4\nu$, $1/2\nu$ in the case $a=1$. Comparison of this figure with Fig. 3, which corresponds to the same initial condition but without the radial velocity, will serve to see the effect of the existence of the source at the centre.

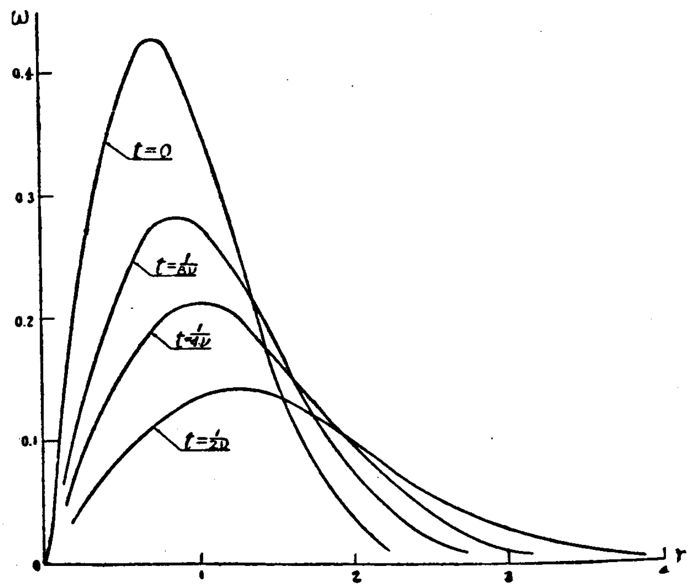

Fig. 11.

The point at which the vortical motion is most intense is found from

$$r_1^2 = \frac{1}{2}\left(a^2+4\nu t\right),$$

and the velocity of travel of this maximum is

$$U = \frac{\nu}{r} = u,$$

again equal to that of the fluid due to the source.

22. Again consider the same initial distribution of vortices as in the last article, *i.e.*,

$$\omega_0 = Cre^{-\frac{r^2}{a^2}}, \quad \text{for} \quad t=0, \tag{55}$$

but with different radial velocity

$$u = \frac{4\nu}{r}. \tag{56}$$

In this case, employing the formula (11), we have

$$\omega = \frac{Cr^2}{2\nu t} e^{-\frac{r^2}{4\nu t}} \int_0^\infty e^{-\left(\frac{1}{a^2}+\frac{1}{4\nu t}\right)\alpha^2} I_2\left(\frac{ar}{2\nu t}\right) d\alpha,$$

and performing the integration we obtain

$$\omega = \frac{Ca\sqrt{\pi}}{2} \frac{r^2}{\sqrt{\nu t(a^2+4\nu t)}} e^{-\frac{r^2}{8\nu t}\left(1+\frac{4\nu t}{a^2+4\nu t}\right)} I_1\left(\frac{a^2 r^2}{8\nu t(a^2+4\nu t)}\right). \tag{57}$$

Fig. 12 exhibits the approximate vortical distribution for the times 0, $1/8\nu$, $2/4\nu$, $1/2\nu$, taking again $a=1$ and $C=1$.

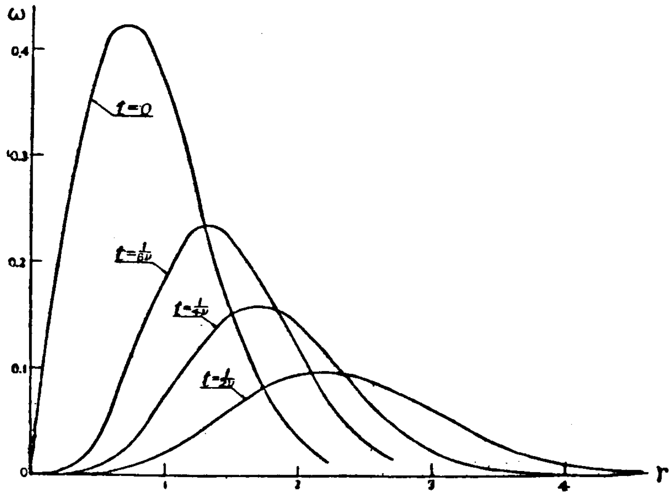

Fig. 12.

The maximum point of the vortical distribution will be found by solving the equation

$$\left(1+\frac{8\nu t}{a^2}\right)I_1(R^2)=I_0(R^2),$$

where

$$R^2=\frac{a^2r^2}{8\nu t\,(a^2+4\nu t)}.$$

The velocity U of travel of the maximum point cannot be obtained as in the former examples, but we may estimate from the figure that the velocity U is less that that of the fluid due to the source. The initial distribution of vortices in this example is the same as that in the last paragraph, but the radial velocity u in each of them is different, that for the former being four times as large as that for the latter. Thus the strength of the source makes, as might be expected, its effect on the travelling velocity of the most intense part of the vortical motion.

23. So far, we have been considering only the case in which the origin is a source. In this closing paragraph let us take the other case, having a sink at the origin, though the result of this is not very interesting.

Assume the same initial condition as that in the two preceeding articles, *i.e.*

$$\omega_0=Cre^{-\frac{r^2}{a^2}},\quad \text{for}\quad t=0, \tag{58}$$

but with the negative radial velocity

$$u=-\frac{\nu}{r}. \tag{59}$$

In this case we have to find the value of

$$\omega=\frac{C}{2\nu t\sqrt{r}}e^{-\frac{r^2}{4\nu t}}\int_0^\infty e^{-\left(\frac{1}{a^2}+\frac{1}{4\nu t}\right)\alpha^2}I_{-\frac{1}{2}}\left(\frac{\alpha r}{2\nu t}\right)\alpha^{\frac{5}{2}}d\alpha.$$

By expressing the functien $I_{-\frac{1}{2}}$ in terms of hyperblic functions, we get finally that

$$\omega = C\frac{2\nu t}{r}\left(\frac{a^2}{a^2+4\nu t}\right)^{\frac{3}{2}}\left\{1+\frac{a^2r^2}{2\nu t(a^2+4\nu t)}\right\}e^{-\frac{r^2}{a^2+4\nu t}}. \qquad (60)$$

The approximate course of the distribution of vortical motion is shown in Fig. 13. The curves in it are for ω at times 0, $1/8\nu$, $1/4\nu$, $1/2\nu$, assuming $a=1$.

As will be seen from the figure, when there is a sink at the centre of initial distribution the vortical motion will tend to accumlate near that point. Other examples with a negative radial velocity has been tried, but the results are not, generally, much different from that of the present example.

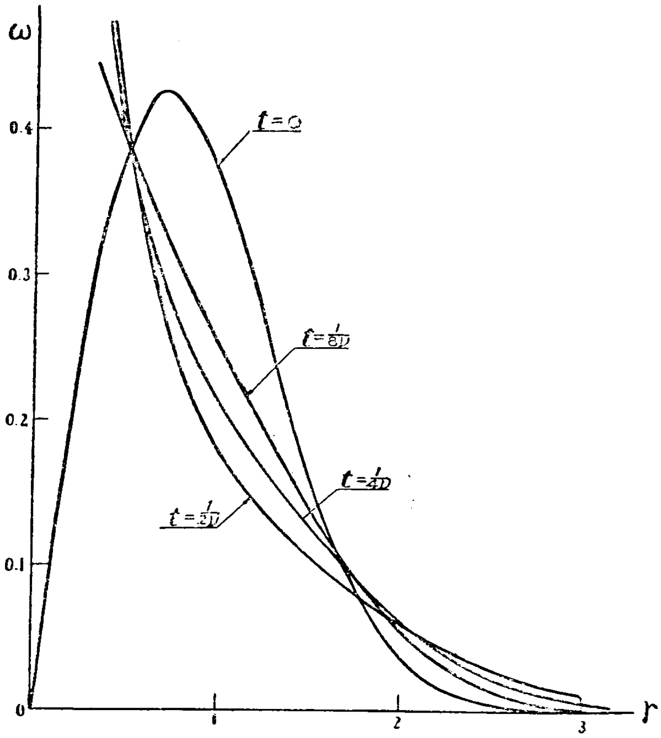

Fig. 13.

Examples in which the maximum point travels towards the centre have been sought, as one might imagine it would involve the case of a sink at the centre, but, so far, no interesting results have been obtained.

The only defect of the analytical method used in the present paper is, as mentioned already, that the condition $n > -1$ (eq. 9) must be satisfied, viz. a sink of an arbitrary strength cannot be assumed, contrary to the case of a source.

Tables for plotting the curves.

Table 1, (for Fig. 1.)

$t=0$		$t=1/8\nu$		$t=1/4\nu$		$t=1/2\nu$		$t=1/\nu$	
r	ω	r	ω	r	ω	r	ω	r	ω
0.30	3.3333	0.000	2.4808	0.000	1.7542	0.000	1.2404	0.000	0.8771
0.35	2.8571	0.224	2.3612	0.316	1.6696	0.448	1.1806	0.632	0.8348
0.40	2.5000	0.316	2.2503	0.447	1.5912	0.632	1.1252	0.894	0.7956
0.45	2.2222	0.447	2.0515	0.632	1.4506	0.894	1.0258	1.265	0.7253
0.50	2.0000	0.632	1.7301	0.894	1.2234	1.264	0.8651	1.549	0.6117
0.55	1.8182	0.775	1.4868	1.095	1.0513	1.550	0.7434	2.191	0.5257
0.60	1.6667	0.894	1.3003	1.265	0.9195	1.788	0.6502	2.530	0.4597
0.70	1.4286	1.000	1.1555	1.414	0.8171	2.000	0.5778	2.828	0.4085
0.80	1.2500	1.095	1.0414	1.549	0.7364	2.190	0.5207	3.098	0.3682
0.90	1.1111	1.183	0.9503	1.673	0.6720	2.366	0.4752	3.347	0.3360
1.0	1.0000	1.265	0.8765	1.789	0.6198	2.530	0.4383	3.578	0.3099
1.2	0.8333	1.342	0.8159	1.897	0.5769	2.684	0.4029	3.795	0.2885
1.4	0.7143	1.414	0.7654	2.000	0.5412	2.828	0.3827	4.000	0.2706
1.6	0.6250	1.549	0.6863	2.191	0.4853	3.098	0.3432	4.382	0.2426
1.8	0.5556	1.612	0.6547	2.280	0.4629	3.224	0.3274	4.561	0.2315
2.0	0.5000	1.732	0.6029	2.449	0.4263	3.464	0.3015	4.899	0.2132
2.5	0.4000	1.844	0.5617	2.608	0.3972	3.688	0.2809	5.215	0.1986
3.0	0.3333	1.949	0.5281	2.757	0.3734	3.898	0.2641	5.514	0.1867
3.5	0.2857	2.049	0.5002	2.898	0.3537	4.098	0.2501	5.797	0.1768
4.0	0.2500	2.145	0.4760	3.033	0.3366	4.290	0.2380	6.066	0.1683
5.0	0.2000	2.236	0.4555	3.162	0.3221	4.472	0.2278	6.325	0.1610
6.0	0.1667	2.449	0.4136	3.464	0.2925	4.898	0.2018	6.928	0.1462
7.0	0.1429								

Table 2, (for Fig. 2).

r	$t=0$	$t=1/8\nu$	$t=1/4\ \nu$	$t=1/2\nu$
0.0	1.0000	0.6667	0.5000	0.3333
0.1	0.9901	0.6621	0.4975	0.3322
0.2	0.9608	0.6491	0.4901	0.3289
0.4	0.8521	0.5990	0.4615	0.3160
0.6	0.6977	0.5244	0.4176	0.2956
0.8	0.5273	$0.439\bar{5}$	0.3631	0.2693
1.0	0.3679	0.3421	0.3033	0.2389
1.2	0.2369	0.2553	0.2434	0.2063
1.5	0.1054	0.1488	0.1623	$0.157\bar{5}$
1.8	0.0392	0.0769	0.0989	0.1132
2.2	0.0079	$0.026\bar{5}$	$0.044\bar{5}$	0.0664
2.6	0.0013	0.0074	0.0170	0.0350
3.0	0.0001	0.0017	0.0056	0.0166
3.6		0.0001	0.0008	0.0044
4.2			0.0001	0.0009
4.8				0.0002

Table 3. (for Fig. 3).

$t=0$		$t=1/8\nu$		$t=1/4\nu$		$t=1/2\nu$	
r	ω	r	ω	r	ω	r	ω
0.000	0.0000	0.000	0.3411	0.000	0.3133	0.000	0.2412
0.050	0.0488	0.122	0.3411	0.200	0.3101	0.346	0.2341
0.100	0.0990	0.173	0.3410	0.283	0.3070	0.490	0.2271
0.200	0.1922	$0.24\bar{5}$	0.3407	0.400	0.3007	0.693	0.2137
0.283	0.2611	0.300	0.3402	0.490	0.2943	0.849	0.2010
0.316	0.2861	0.346	0.3396	0.566	0.2879	0.980	0.1889
0.447	0.3661	0.387	0.3388	0.632	0.2819	1.095	$0.177\bar{5}$
0.632	0.4240	0.548	$0.332\bar{5}$	0.894	0.2500	1.549	0.1290
$0.77\bar{5}$	0.4251	$0.77\bar{5}$	0.3120	$1.26\bar{5}$	0.1921	2.191	$0.066\bar{5}$
0.894	0.4019	0.949	0.2855	1.549	0.1439	2.683	0.0334
1.000	0.3679	1.095	0.2566	1.789	0.1059	3.098	$0.016\bar{5}$
1.095	0.3297	$1.22\bar{5}$	0.2275	2.000	0.0769	3.464	0.0080
1.183	0.2918	1.342	0.1997	2.191	0.0553	$3.79\bar{5}$	0.0039
$1.26\bar{5}$	0.2554	1.449	0.1739	2.366	0.0394	4.099	0.0018
1.342	0.2218	1.549	$0.150\bar{5}$	2.530	0.0279	4.382	0.0009
1.414	0.1914	1.643	0.1295	2.683	0.0197	4.648	0.0004
1.549	0.1405	1.732	0.1110	2.848	0.0138	4.899	0.0002
1.673	0.1018	1.857	0.0874	3.033	0.0081	5.254	0.0001
1.844	0.0615	$1.97\bar{5}$	0.0684	$3.22\bar{5}$	0.0030		
2.000	0.0366	2.086	0.0533	3.406	0.0027		
2.121	0.0236	2.191	0.0413	3.578	0.0015		
2.236	0.0151	2.291	$0.031\bar{5}$	3.742	0.0009		
		2.387	0.0246	3.899	0.0005		
		2.510	0.0172	4.099	0.0002		
		2.627	0.0121	4.290	0.0001		

Table 4, (for Fig. 4).

r	$t=0$	$t=1/8\nu$	$t=1/4\nu$	$1=1/2\nu$
0.0	1	0.8647	0.6321	0.3935
0.2	1	0.8430	0.6176	0.3875
0.4	1	0.7785	0.5756	0.3699
0.6	1	0.6745	0.5111	0.3424
0.8	1	0.5411	0.4316	0.3072
1.0	—	0.3965	0.3457	0.2671
1.2	0	0.2620	0.2620	0.2250
1.4	0	0.1544	0.1874	0.1836
1.6	0		0.1262	0.1450
2.0	0		0.0524	0.0819
2.4	0			0.0404
3.0	0			0.0108

Table 5, (for Fig. 5).

$\frac{\nu t}{a^2}$	ω	$\frac{\nu t}{a^2}$	$\frac{4}{\nu}\frac{\partial\omega}{\partial t}$
0.200	0.0352	0.333	$+0.640\bar{5}$
0.250	0.0472	0.500	0.2962
0.333	0.0628	0.625	0.1454
0.500	0.0819	0.714	0.0738
0.625	0.0886	0.833	$+0.011\bar{5}$
0.714	0.0910	1.000	−0.0389
0.833	0.0922	1.250	0.0708
1.000	0.0915	1.667	0.0793
1.250	0.0879	2.500	0.0614
1.667	0.0785	5.000	−0.0252
2.500	0.0651		
3.000	0.0583		

Table 6, (for Fig. 6).

$t=1/12\nu$		$t=1/8\nu$		$t=1/4\nu$	
r	ω	r	ω	r	ω
0.000	0.2987	0.00	0.5413	0.0	0.7358
0.100	0.3166	0.10	0.5521	0.1	0.7357
0.200	0.3730	0.20	0.5829	0.2	$0.735\overline{5}$
0.300	0.4537	0.30	0.6302	0.3	0.7343
0.400	0.5636	0.40	0.6948	0.4	0.7314
0.500	0.6887	0.50	$0.748\overline{5}$	0.5	$0.725\overline{5}$
0.600	0.8144	0.60	$0.803\overline{5}$	0.6	0.7154
0.700	0.9233	0.70	0.8446	0.7	0.7002
0.800	0.9982	0.80	0.8650	0.8	0.6976
0.900	1.0260	0.85	0.8659	0.9	0.6512
1.000	1.0000	0.90	0.8600	1.0	0.6170
1.333	0.5898	1.00	0.8280	1.2	0.5316
1.500	0.3830	1.20	0.6927	1.4	0.4309
1.667	0.2227	1.40	0.5020	1.6	0.3269
		1.75	0.2023	1.8	0.2313
		2.00	0.0776	2.0	0.1523

Table 7, (for Fig. 7).

r	npper three curves			lower three curve		
	$t=1/12\nu$	$t=1/8\nu$	$t=1/4\nu$	$t=1/12\nu$	$t=1/8\nu$	$t=1/4\nu$
0.000	1.7158	1.7544	1.5146	−1.1183	−0.6717	−0.0430
0.100	1.7229	1.7530	1.5087	1.0898	0.6489	0.0372
0.200	1.7455	1.7480	1.4912	0.9996	0.5821	−0.0203
0.300	1.7659	1.7367	1.4622	0.8585	0.4763	+0.0065
0.400	1.7858	1.7224	1.4218	0.6585	0.3327	0 0491
0.500	1.7910	1.6800	1.3705	0.4136	0.1830	0.0804
0.600	1.7718	1.6264	1.2979	−0.1431	−0.0195	0.1330
0.700	1.7203	1.5519	1.2376	+0.1262	+0.1374	0.1628
0.800	1.6317	1.4553	1.1766	0.3647	0.2748	0.2186
0.900	1.5052	1.3376	1.0714	0.5467	0.3824	0.2310
1.000	1.3444	1.2022	0.9798	0.6557	0.4538	0.2543
1.200	—	0.9002	0.7887	—	0.4851	0.2744
1.333	0.6671	—	—	0.5125	—	—
1.400	—	0.6021	0.6012	—	0.4019	0.2604
1.500	0.4120	—	—	0.3540	—	—
1.600	—	—	0.4322	—	—	0.2215
1.667	0.2320	—	—	+0.2134	—	—
1.750	—	0.2219	—	—	0.1827	—
1.800	—	—	0.2920	—	—	0.1706
2.000	—	0.822	0.1848	—	+0.0730	+0.1198

Table 8, (for Fig. 8).

$t=0$		$t=1/8\nu$		$t=1/4\nu$		$t=1/2\nu$	
r	ω	r	ω	r	ω	r	ω
0.50	2.0000	0.100	0.0494	0.141	0.0349	0.200	0.0247
0.60	1.6667	0.141	0.0973	0.200	0.0688	0.283	0.0487
0.70	1.4286	0.200	0.1889	0.283	0.1336	0.400	0.094$\bar{5}$
0.80	1.2500	0.24$\bar{5}$	0.2750	0.346	0.194$\bar{5}$	0.490	0.1375
0.90	1.1111	0.283	0.3560	0.400	0.2517	0.566	0.1780
1.00	1.0000	0.316	0.4319	0.447	0.3055	0.632	0.2160
1.15	0.8696	0.447	0.7468	0.632	0.5281	0.894	0.3734
1.30	0.7692	0.682	1.1243	0.894	0.7950	1.26$\bar{5}$	0.5621
1.45	0.6897	0.77$\bar{5}$	1.2849	1.095	0.9086	1.549	0.6424
1.60	0.6250	0.894	1.3221	1.26$\bar{5}$	0.9348	1.789	0.6610
1.80	0.5556	1.000	1.2927	1.414	0.9141	2.000	0.6463
2.00	0.5000	1.095	1.2304	1.549	0.8700	2.191	0.6152
2.20	0.4545	1.183	1.1549	1.673	0.8167	2.366	0.577$\bar{5}$
2.50	0.4000	1.26$\bar{5}$	1.0772	1.789	0.7617	2.530	0.5386
2.80	0.3571	1.342	1.0030	1.897	0.7092	2.683	0.501$\bar{5}$
3.20	0.3125	1.414	0.9349	2.000	0.6610	2.828	0.4674
3.60	0.2778	1.549	0.8199	2.191	0.5797	3.098	0.4099
4.00	0.2500	1.673	0.7309	2.366	0.5168	3.347	0.3654
4.50	0.2222	1.844	0.6341	2.608	0.4484	3.688	0.3171
		2.000	0.5665	2.828	0.4006	4.000	0.2833
		2.121	0.5234	3.000	0.3701	4.243	0.2617
		2.236	0.4905	3.162	0.3468	4.472	0.2453

Table 9, (for Fig. 9).

r	$t=0$	$t=1/8\nu$	$t=1/4\nu$	$t=1/2\nu$
0.0	1.0000	—	—	—
0.1	0.9901	0.0130	0.0050	0.0017
0.2	0.9608	0.0506	0.0194	0.0066
0.4	0.8521	0 1724	0.0710	0.0249
0.6	0.6977	0.2999	0.1376	0.0517
0.8	0.5273	0.3812	0.1989	0.0817
1.0	0.3679	0.3782	0.2387	0.1102
1.2	0.2369	0.3268	0.2498	0.1320
1.5	0.1054	0.2120	0.2193	0.1477
1.8	0.0392	0.1138	0.1587	0.1417
2.2	0.0079	0.0396	0.0810	0.1103
2.6	0.0013	0.0111	0.0328	0.0710
3.0	0.0001	0.0025	0.0110	0.0387
3.6		0.0002	0.0015	0 0118
4.2			0.0002	0.0026
4.8				0.0005

Table 10, (for Fig. 10).

$t=0$		$t=1/8\nu$		$t=1/4\nu$		$t=1/2\nu$	
r	ω	r	ω	r	ω	r	ω
0.000	0.0000	0.122	0.0066	0.141	0.0050	0.173	0.0033
0.100	0.0099	$0.24\bar{5}$	0.0256	0.283	0.0192	0.346	0.0128
0.200	0.0384	0.346	0.0492	0.400	0.0369	0.490	0.0246
0.283	0.0738	0.387	0.0603	0.447	0.0452	0.548	0.0302
0.316	$0.090\bar{5}$	0.548	0.1092	0.632	0.0819	$0.77\bar{5}$	0.0546
0.447	0.1637	$0.77\bar{5}$	0.1788	0.894	0.1341	1.095	0.0894
0.632	0.2681	0.949	0.2195	1.095	0.1696	1.342	0.1098
$0.77\bar{5}$	0.3293	1.095	0.2396	$1.26\bar{5}$	0.1798	1.549	0.1198
0.894	$0.359\bar{5}$	$1.22\bar{5}$	0.2453	1.414	0.1839	1732	0.1226
1.000	0.3679	1.342	0.2410	1.549	0.1807	1.897	$0.120\bar{5}$
1.095	0.3614	1.449	0.2302	1.673	0.1726	2.049	0.1151
1.183	0.3452	1.549	0.2154	1.789	0.1615	2.191	0.1077
$1.26\bar{5}$	0.3230	1.643	0.1984	1.897	0.1488	2.324	0.0992
1.342	0.2975	1.732	0.1804	2.000	0.1353	2.449	0.0902
1.414	0.2707	1.897	0.1451	2.191	0.1089	2.683	0.0726
1.549	0.2177	2.049	0.1135	2.366	0.0851	2.898	0.0568
1.673	0.1703	2.258	0.0756	2.608	0.0567	3.193	0.0378
1.844	$0.113\bar{5}$	2.449	0.0488	2.828	0.0361	3.464	0.0244
2.000	0.0733	2.598	0.0333	3.000	0.0250	3.674	0.0167
2.121	0.0500	2.739	0.0225	3.162	0.0168	3.873	0.0112
2.236	0.0337						

Table 11, (for Fig. 11).

$t=0$		$t=1/8\nu$		$t=1/4\nu$		$t=1/2\nu$	
r	ω	r	ω	r	ω	r	ω
0.000	0.0000	0·122	0.0660	0.141	0.0495	0.173	0.0330
0.050	0.0488	0.245	0.1281	0.283	0.0961	0.346	0.0641
0.100	0.0990	0.346	0.1741	0.400	0.1305	0.490	0.0870
0.200	0.1922	0.387	0.1908	0.447	0.1431	0.548	0.0954
0.283	0.2611	0.548	0.2441	0.632	0.1831	0.77$\bar{5}$	0.1220
0.316	0.2861	0.77$\bar{5}$	0.2826	0.894	0.2120	1.095	0.1413
0.447	0.3661	0.949	0.2834	1.095	0.2126	1.342	0.1417
0.632	0.4239	1.095	0.2673	1.26$\bar{5}$	0.209$\bar{5}$	1.549	0.1336
0.77$\bar{5}$	0.4251	1.22$\bar{5}$	0.2453	1.414	0.1839	1.732	0.1226
0.894	0.4019	1.342	0.2198	1.549	0.1649	1.897	0.1099
1.000	0.3679	1.449	0.1985	1.673	0.1459	2.049	0.0993
1.095	0.3297	1.549	0.1703	1.789	0.1277	2.191	0.0851
1.183	0.2918	1,643	0.1478	1.897	0.1109	2.324	0.0739
1.26$\bar{5}$	0.2554	1.732	0.1276	2.000	0.0952	2.449	0.0638
1.342	0.2218	1.897	0.0937	2.191	0.0703	2.683	0.0468
1.414	0.1914	2.049	0.0678	2.366	0.0508	2.898	0.0339
1.549	0.1405	2.258	0.0410	2.608	0.0308	3.193	0.0205
1.673	0.1018	2.449	0.0244	2.828	0.0183	3.464	0.0122
1.844	0.0615	2.598	0.0157	3.000	0.0118	3.674	0.0079
2.000	0.0366	2.739	0.0100	3.162	0.0075	3.873	0.0050
2.121	0.0236						
1.236	0.0151						

Table 12, (for Fig. 12).

$t=0$		$t=1/8\nu$		$t=1/4\nu$		$t=1/2\nu$	
r	ω	r	ω	r	ω	r	ω
0.000	0.0000	0.122	0.0002	0.200	0.0003	0.346	0.0004
0.050	0.0488	0.173	0.0006	0.283	0.0009	0.490	0.0016
0.100	0.0990	0.245	0.0023	0.400	0.0036	0.693	0.0057
0.200	0.1922	0.300	0.0049	0.490	0.0075	0.849	0.0116
0.283	0.2611	0.346	0.0084	0.566	0.0126	0.980	0.0186
0.316	0.2861	0.387	0.0126	0.632	0.0186	1.095	0.0264
0.447	0.3661	0.548	0.0414	0.894	0.0553	1.549	0.0642
0.632	0.4239	0.775	0.1126	1.265	0.1232	2.191	0.0959
0.775	0.4251	0.949	0.1740	1.549	0.1560	2.683	0.0814
0.894	0.4019	1.095	0.2146	1.789	0.1575	3.098	0.0551
1.000	0.3679	1.225	0.2348	2.000	0.1411	3.464	0.0331
1.095	0.3297	1.342	0.2388	2.191	0.1175	3.795	0.0185
1.183	0.2918	1.449	0.2316	2.366	0.0933	4.099	0.0098
1.265	0.2554	1.549	0.2172	2.530	0.0716	4.382	0.0051
1.342	0.2218	1.643	0.1989	2.683	0.0537	4.648	0.0025
1.414	0.1914	1.732	0.1789	2.848	0.0395	4.899	0.0013
1.549	0.1405	1.857	0.1489	3.033	0.0244	5.254	0.0004
1.673	0.1018	1.975	0.1213	3.225	0.0147	5.586	0.0001
1.844	0.0615	2.086	0.0974	3.406	0.0087		
2.000	0.0366	2.191	0.0773	3.578	0.0051		
2.121	0.0236	2.291	0.0608	3.742	0.0030		
2.236	0.0151	2.387	0.0475	3.899	0.0017		
		2.510	0.0340	4.099	0.0008		
		2.627	0.0241	4.290	0.0004		
		2.739	0.0170	4.472	0.0002		

Table 13, (for Fig. 13).

$t=0$		$t=1/8\nu$		$t=1/4\nu$		$t=1/2\nu$	
r	ω	r	ω	r	ω	r	ω
0.000	0.0000	0.346	0.4787	0.316	0.5849	0.346	0.5551
0.050	0.0488	0.458	0.4027	0.447	0.4292	0.458	0.4190
0.100	0.0990	0.648	0.3364	0.632	0.3204	0.648	0.2943
0.200	0.1922	0.794	0.3019	$0.77\bar{5}$	0.2705	0.794	0.2378
0.283	0.2611	0.917	0.2748	0.894	$0.238\bar{5}$	0.917	0.2031
0.316	0.2861	1.010	0.2539	1.000	0.2144	1.010	0.1817
0.447	0.3661	1.123	0.2282	1.095	0.1948	1.123	0.1600
0.632	0.4240	$1.22\bar{5}$	0.2044	1.183	0.1781	$1.22\bar{5}$	0.1430
$0.77\bar{5}$	0.4251	$1.28\bar{5}$	0.1904	$1.26\bar{5}$	0.1633	$1.28\bar{5}$	0.1340
0.894	0.4019	1.386	0.1671	1.342	0.1500	1.386	0.1245
1.000	0.3679	1.449	0.1528	1.414	0.1380	1.449	0.1121
1.095	0.3297	1.549	0.1312	1.549	0.1169	1.549	$0.100\bar{5}$
1.183	0.2918	1.643	0.1123	1.673	0.0990	1.643	$0.090\bar{5}$
$1.26\bar{5}$	0.2554	1.732	0.0957	1.789	0.0838	1.732	0.0818
1.342	0.2218	1.849	0.0762	1.897	0.0708	1.849	0.0712
1.414	0.1914	$1.97\bar{5}$	0.0583	2.000	0.0598	$197\bar{5}$	0.0611
1.549	0.1405	2.100	0.0437	2.121	0.0483	2.100	0.0520
1.673	0.1018	2.191	0.0408	2.236	0.0389	2.191	0.0461
1.844	0.0615	2.324	0.0246	2.449	0.0252	2.324	0.0383
2.000	0.0366	2.449	0.0173	2.646	0.0161	2.449	0.0319
2.121	0.0236	2.683	0.0084	2.828	0.0103	2.683	0.0221
2.236	0.0151	2.898	0.0041	3.000	0.0065	2.898	0.0153
2.449	0.0061	3.000	0.0028	3.162	0.0041	3.000	0.0128
2.646	0.0024						
2.828	0.0009						
3.000	0·0004						

CONTENTS.

第四號
大正十一年十一月發行

抄　　錄

粘性流體の渦の老衰に就いて

所員 理學博士 寺　澤　寬　一

此論文は空氣とか水とかの樣な粘性流體の中に渦が一度生じた時に,其渦の强さが時と共にどう變化するものであるかと云ふ問題の解決を與へたものであります.

計算を簡單にする爲めに,流體を壓縮し得ざるものと考へ,渦を直線的平行分布をして居つて而も或る軸に關して對照を持つて居るものとしました. 其軸に垂直な平面を座標面に取り,是が軸と交る點から其面上の任意の點までの距離を r で,時を t で表はします. 渦の强さ ω が $t=0$ と云ふ時刻に $f(r)$ で表はされる樣な勝手な有樣に分布してあるとすれば,其後任意な時間 t を經過したときに渦の分布は

$$\omega=\frac{r^{c/2\nu}}{2\nu t}e^{-\frac{r^2}{4\nu t}}\int_0^\infty f(\alpha)\,e^{-\frac{\alpha^2}{4\nu t}}I_{\frac{c}{2\nu}}\left(\frac{\alpha r}{2\nu t}\right)\alpha^{1-\frac{c}{2\nu}}d\alpha,$$

て解ることになります. ν は流體の動粘性係數で,c は r の方向の速度 u が

$$u=\frac{c}{r}$$

で與へられる樣な常數であります. $u=0$ と云ふ特別な場合は重要でありますが,此は上の一般の式に $c=0$ と置いて得られます.

例として十幾つかを計算して圖解してあります. 其中で風洞の整流格子の位置を定むることに應用のできることや,地表に近い所よりも高層の氣流が比較的整つて居るので高層飛行が安全である理由や,氣象學上の岡田氏法則の解釋を試みたことなどは主な應用であります. 是等の應用が寧ろ此論文の主眼とする所であります. そこで一般の議論は別として,例の方に重きを置かるる讀者の便を計つて前頁に目次を添へてあります. も少し詳しい抄錄は航空研究所雜錄第二號第三號に出してあります.

(大正十一年九月)