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Theory of Airscrews.

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The paper is an attempt to elucidate the mechanism of the action of airscrew in the air, based on Prandtl's theory of aerofoil.

Fundamental Equations of Airscrew in General.

Let us make the assumptions that :—

a) the air is without viscosity or compressibility

b) the number of blades is so large that it can be regarded as practically infinite.

c) the drop of pressure in the slip stream due to its rotation is negligible.

Taking the airscrew as fixed the air is considered to be moving with velocity v, the velocity of the aircraft refered to air at infinite distance.

As the air draws near the plane of rotation of the airscrew, the velocity of air is accelerated and at the plane of rotation itself the velocity is $v+W_a$.

After passing through the plane of rotation the air is still accelerated and finally at infinite distance from the plane of rotation it has the velocity $v+W_a'$.

Corrections.

Page	Line	Misprint	Correction
374	14	vahe	<u>have</u>
378	19	$\frac{T}{\frac{1}{2}\rho v^2 \pi^2 R^2}$	$\frac{T}{\frac{1}{2}\rho v^2 \underline{\pi} R^2}$
383	6	effic ncy	<u>efficiency</u>
389	6	or	<u>of</u>
391	9	$\frac{3c_f\ Nt\omega R}{16T}$	$\frac{3c_f\ Nt\omega R}{16\underline{\pi}}$
398	2	$c\,\frac{27N\epsilon\lambda^3\omega^3R^3}{128\pi}\left(1+\frac{1}{\mu^4}\right)$	$\underline{c_f}$ $\frac{,,}{,,}$
398	10	compoment	<u>component</u>
376	T_c in Table I. and Fig. 3. should be read T_c'.		

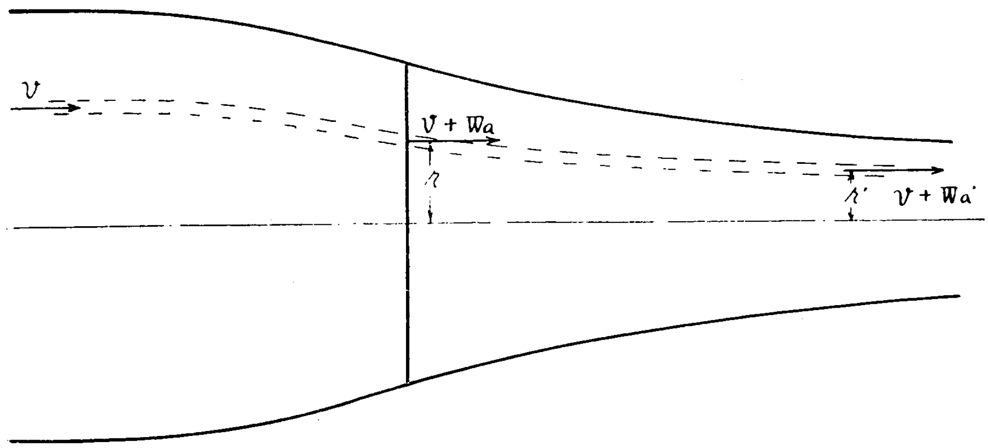


Fig. 1. Pattern of airflow.

At the same time the air acquires a rotation W_t at the plane of rotation and W_t' in the slip stream.

Take a ring area included between r and $r+dr$ at the plane of rotation, then by the circulation theorem of Kutta-Joukvski we have, if dF is the elementary force on the blade element

$$dF=\rho N\Gamma V dr$$

The direction of this force is normal to the resultant velocity V. The axial and tangential components of this force are thrust and torque forces respectively

$$dT=dF\,.\,\frac{\omega r-W_t}{V}=\rho N\Gamma(\omega r-W_t)\,dr \qquad (1)$$

$$dQ=r\,.\,dF\,.\,\frac{v+W_a}{V}=\rho N\Gamma r\,(v+W_a)\,dr \qquad (2)$$

From the consideration of momentum, thrust and torque are respectively

$$dT=2\pi\, r\,\rho\,(v+W_a)\,W_a'\,dr \qquad (3)$$

$$dQ=2\pi\, rr'\,\rho\,(v+W_a)\,W_t'\,dr \qquad (4)$$

Next equating the kinetic energy generated in the slip stream to the work done by the airscrew we have

$$\omega\, dQ = \pi\, r dr\, \rho\, (v + W_a)\, \{(W_a' + v)^2 + W_t'^2 - v^2\}$$

$$\omega\, dQ = \pi\, r\, \rho\, (v + W_a)\, (W_a'^2 + W_t'^2 + 2\, v\, W_a')\, dr \qquad (5)$$

Since compressibility is neglected the equation of continuity gives the relation

$$2\pi\, r\, (v + W_a)\, dr = 2\pi\, r'\, (v + W_a')\, dr' \qquad (6)$$

We have obtained so far 6 equations for 7 unknown quantities.

One more equation can be obtained from the following consideration.

Imagine a circle in a plane parallel to the plane of rotation of the airscrew having its centre on the axis of rotation and take circulation round it.

By virtue of the theorem of Lord Kelvin this circulation must have a constant value on each side of the plane of rotation of the airscrew. The value of the constant differs on each side.

In front of the airscrew it is obviously zero.

Now take the circle sufficiently remote from the plane of rotation where the air is not yet influenced by the airscrew.

Then the circulation is zero and consequently the circulation round any circle in front of the airscrew is zero.

Considering from the vortex theory of aerofoil the inflow velocities W_a and W_t etc. are induced by the blade and the trailing vortices.

A moment's consideration will show that the velocities induced by the blade and the trailing vortices have opposite sign in front of the airscrew and same sign behind it.

This, together with the absence of the induced tangential velocity in front of the airscrew, necessitates that the velocities induced by the blade and the trailing vortices are of the same magnitude.

Hence behind the airscrew the induced tangential velocity is twice that in the plane of rotation where the blade vortex induces no velocity.

Applying the theorem of constancy of circulation in the region behind the airscrew we have

$$\int 2\, W_t\, ds = \int W_t'\, ds'$$

$$\int_0^{2\pi} 2\, W_t\, r d\,\theta = \int_0^{2\pi} W_t'\, r' d\,\theta$$

$$\therefore \quad 2\, W_t\, r = W_t'\, r' \tag{7}$$

Thus we have 7 equations for 7 unknown quantities i. e., dT, dQ, W_a, W_t, W_a', W_t', r'/r.

In the following sections we will solve these equations for different cases of airscrew.

PART 1. AIRCRAFT AIRSCREW.

1. Solution of the Fundamental Equations.

For the aircraft airscrew the values of W_a, W_t etc. are generally small. Hence the terms higher than the second order of W_a/v, $W_t/\omega r$ etc. can safely be neglected. We have immediately from (2) and (4),

$$W_t' = \frac{N\Gamma}{2\pi\, r'} \tag{8}$$

From (7)

$$W_t = \frac{r'}{2r}\, W_t' = \frac{N\Gamma}{4\pi\, r} \tag{9}$$

From (4) and (5) we obtain

$$2\,\omega r'\, W_t' = W_a'^2 + W_t'^2 + 2v\, W_a'$$

$$W_a'\,(W_a' + 2\,v) = W_t'\,(2\,\omega r' - W_t')$$

$$W_a' = \frac{W_t'\,(2\,\omega r' - W_t')}{2\,v\left(1 + \frac{W_a'}{2\,v}\right)}$$

$$= \frac{N\omega\Gamma}{2\pi\, v}\left(1 - \frac{1}{\omega r'}\,\frac{N\Gamma}{4\pi\, r'} - \frac{1}{v}\,\frac{N\omega\Gamma}{4\pi\, v}\right) \tag{10}$$

From (1) and (3)

$$N\Gamma(\omega r - W_t) = 2\pi\, r\, (v + W_a)\, W_a'$$

$$v + W_a = \frac{N\Gamma(\omega r - W_t)}{2\pi\, r\, W_a'}$$

$$= v\left(1 - \frac{W_t}{\omega r} + \frac{W_t'}{2\omega r'} + \frac{W_a'}{2v}\right)$$

$$\therefore \quad W_a \doteqdot \frac{W_a'}{2}$$

$$= \frac{N\omega\Gamma}{4\pi\, v}\left(1 - \frac{1}{\omega r'} \cdot \frac{N\Gamma}{4\pi r'} - \frac{1}{v} \cdot \frac{N\omega\Gamma}{4\pi\, v}\right) \tag{11}$$

$v + W_a$ and $v + W_a'$ are almost independent of r, therefore integrating (8) from $r=0$ to r and r' we have

$$(v + W_a)\, r^2 = (v + W_a')\, r'^2$$

$$\frac{r'}{r} = \left(\frac{v + W_a}{v + W_a'}\right)^{\frac{1}{2}}$$

$$\doteqdot 1 - \frac{N\omega\Gamma}{8\pi\, v^2} \tag{12}$$

Putting the values of W_a and W_t above obtained we get

$$dT = \rho N\Gamma\left(\omega\, r - \frac{N\Gamma}{4\pi\, r}\right) dr \tag{13}$$

$$dQ = \rho N\Gamma r\left(v + \frac{N\omega\Gamma}{4\pi\, v}\right) dr \tag{14}$$

2. Profil Resistance.

For the real air which is viscous the frictional resistance of the blades, or more generally the profil resistance must be taken into consideration.

This force acts in the direction of the resultant velocity and the axial and tangential component of this force modify the expressions of thrust and torque respectively.

The amount is for each blade

$$dR=\frac{1}{2}c_f\rho V^2t\,dr$$

where t=breadth of the blade

c_f=coefficient of the profil resistance of the aerofoil corresponding to the blade section or coefficient of the resistance for aspect ratio $1:\infty$.

The axial component of which is

$$dR_a=\frac{1}{2}c_f\rho V^2t\,dr\frac{v}{V}=\frac{1}{2}c_f\rho Vt\,v\,dr \tag{15}$$

The tangential component is

$$dR_t=\frac{1}{2}c_f\rho V^2t\,dr\frac{\omega r}{V}=\frac{1}{2}c_f\rho Vt\,\omega r\,dr \tag{16}$$

And the resultant velocity is

$$V=\sqrt{(v+W_a)^2+(\omega r-W_t)^2}=\sqrt{\omega^2r^2+v^2} \tag{17}$$

Therefore the expressions of the thrust and torque become, neglecting the small quantities of second order

$$dT=\left[\rho N\Gamma\left(\omega r-\frac{N\Gamma}{4\pi r}\right)-\frac{1}{2}c_f\rho Nvt\sqrt{v^2+\omega^2r^2}\right]dr \tag{18}$$

$$dQ=\left[\rho N\Gamma r\left(v+\frac{N\omega\Gamma}{4\pi v}\right)+\frac{1}{2}c_f\rho Nt\,\omega r^2\sqrt{v^2+\omega^2r^2}\right]dr \tag{19}$$

Thus the assumption a) at the beginning of the present paper is amended. Consequently the only assumptions in the equation (18) and (19) are the multiplicity of the blades and the neglect of the drop of pressure in the slip stream which is very small.

3. Most Efficient Value of Circulation.

In the expressions of dT and dQ the terms due to the profil resistance are very small.

Therefore the circulation for maximum efficiency when the profil resistance is neglected will not differ appreciably from the corresponding one when it is taken into consideration.

Let $\varDelta W$ be the work lost in transforming the torque into the thrust when the profil resistance was neglected.

Then

$$\varDelta W=\int_{r=0}^{r=R}(\omega\, dQ-v\, dT)$$

$$=\int_0^R\left[\rho N\Gamma\,\omega r\,\frac{N\omega\Gamma}{4\pi\, v}+\rho N\Gamma\, v\,\frac{N\Gamma}{4\pi\, v}\right]dr \qquad (20)$$

The work supplied is

$$\omega\, Q=\int_0^R\left[\rho N\Gamma\omega r\left(v+\frac{N\omega\,\Gamma}{4\pi\, v}\right)\right]dr.$$

For simplisity's sake suppose that the second term i. e. the axial inflow velocity is independent of the radius.(1)

Then we have

$$\omega Q=\int_0^R\rho N\Gamma\,\omega r\,(1+\mu)\, v\, dr \qquad (21)$$

where $(1+\mu)$ is a number slightly greater than 1.

By the calculas of variation the required circulation is obtained when

(1) That the supposition is justifiable will be seen from the following circumstances:—

(a) the axial inflow velocity $\dfrac{N\omega\Gamma}{4\pi\, v}$ is generally below 10 percent of v.

(b) when the tangential inflow velocity, which is very small, is neglected the circulation giving the maximum efficiency is $\Gamma=$const. i. e. axial inflow velocity$=$const.

$$\delta(\Delta W - \lambda' \omega Q) = 0$$

where λ' is Lagrange's factor

Therefore

$$\rho N \Gamma \omega r \frac{N\omega \Gamma}{2\pi v} + \rho N v \frac{N \Gamma}{2\pi r} - \lambda' \rho N \omega r (1+\mu) v = 0$$

$$\Gamma = \frac{2\pi \lambda'}{N} \cdot \frac{v^2 (1+\mu) \omega r^2}{v^2 + \omega^2 r^2} \tag{22}'$$

or putting $\lambda = \lambda'(1+\mu)$ we have

$$\Gamma = \frac{2\pi \lambda}{N} \cdot \frac{v^2 \omega r^2}{v^2 + \omega^2 r^2} \tag{22}$$

The substitution of Γ in (18) and (19) gives the following values for thrust and torque

$$T = T' \left[1 - \frac{c_f N \epsilon \sigma_1}{4\pi \lambda \varphi \psi_1}\right] \tag{23}$$

$$Q = Q' \left[1 + \frac{c_f N \epsilon \sigma_2}{8\pi \lambda \varphi \psi_2}\right] \tag{24}$$

where

$$T' = \rho \pi R^2 v^2 \lambda \varphi \psi_1$$

$$Q' = \rho \pi R^2 \frac{v^2}{\omega} \lambda \varphi \psi_2$$

$$\psi_1 = 1 - \frac{\lambda}{2} \left(\frac{1}{\varphi} \frac{z^2}{1+z^2} - 1 \right)$$

$$\psi_2 = 1 + \frac{\lambda}{2} \left(2 - \frac{1}{\varphi} \frac{z^2}{1+z^2} \right)$$

$$\varphi = 1 - \frac{1}{z^2} \log_e (1+z^2)$$

$$z = \frac{\omega R}{v}$$

$$\epsilon = \frac{t}{R} \text{ (mean)}$$

$$\sigma_1 = \sqrt{1+z^2} + \frac{1}{z} \log_e (z + \sqrt{1+z^2})$$

$$\sigma_2 = \sqrt{(1+z^2)^3} - \frac{1}{2} \sigma_1$$

The maximum efficiency becomes

$$\eta_{max.} = \frac{vT}{\omega Q} = \frac{\psi_1}{\psi_2} \cdot \frac{1 - \dfrac{c_f N \epsilon \sigma_1}{4\pi \lambda \varphi \psi_1}}{1 + \dfrac{c_f N \epsilon \sigma_2}{8\pi \lambda \varphi \psi_2}} \tag{25}$$

The value of the constant λ can be obtained by solving (23) or (24). In the expression of T as the first term is predominent we have approximately

$$T \doteqdot \rho\pi R^2 v^2 \lambda\varphi.$$

$$\therefore \quad \lambda = \frac{T}{\rho\pi R^2 v^2 \varphi} = \frac{T_c}{2\varphi} \tag{26}$$

where $\qquad T_c = \dfrac{T}{\frac{1}{2} \rho v^2 \pi R^2}$ = thrust coefficient.

4. Effective and Apparent Angle of Incidence.

Owing to the existence of the inflow velocities, both axial and tangential, the angle of incidence is not $i = \theta - tg^{-1} \dfrac{v}{\omega r}$ but a smaller angle α which we will see below.

Let W_{an} and W_{tn} be the components of W_a and W_t normal to the resultant velocity V.

Then we have

$$W_{an} = W_a \frac{\omega r}{V} = \frac{N\omega\Gamma}{4\pi v} \cdot \frac{\omega r}{V}$$

$$W_{tn} = W_t \frac{v}{V} = \frac{N\Gamma}{4\pi r} \cdot \frac{v}{V}$$

$$\therefore \quad W_{ai} + W_{tn} = \frac{N\Gamma V}{4\pi\, vr}$$

$$\therefore \qquad i - \alpha = \frac{N\Gamma V}{4\pi\, vr} \Big/ V = \frac{N\Gamma}{4\pi\, vr} \text{ radians}$$

$$= \frac{N\Gamma}{4\pi\, vr} \times 57.4 \text{ degrees.} \qquad (27)$$

Generally, the lift coefficient is a linear function of α.

$$c_a = c_1\, \alpha + c_2 \qquad (28)$$

Now

$$\Gamma = \frac{1}{2}\, c_a\, t\, V \qquad (29)$$

From (27), (28) and (29) we have

$$c_a = \frac{c_{ai}}{1 + \dfrac{NtV}{8\pi\, vr} \times 57.4\, c_1} \qquad (30)$$

In the above expression c_a is effective and c_{ai} is apparent lift coefficient.

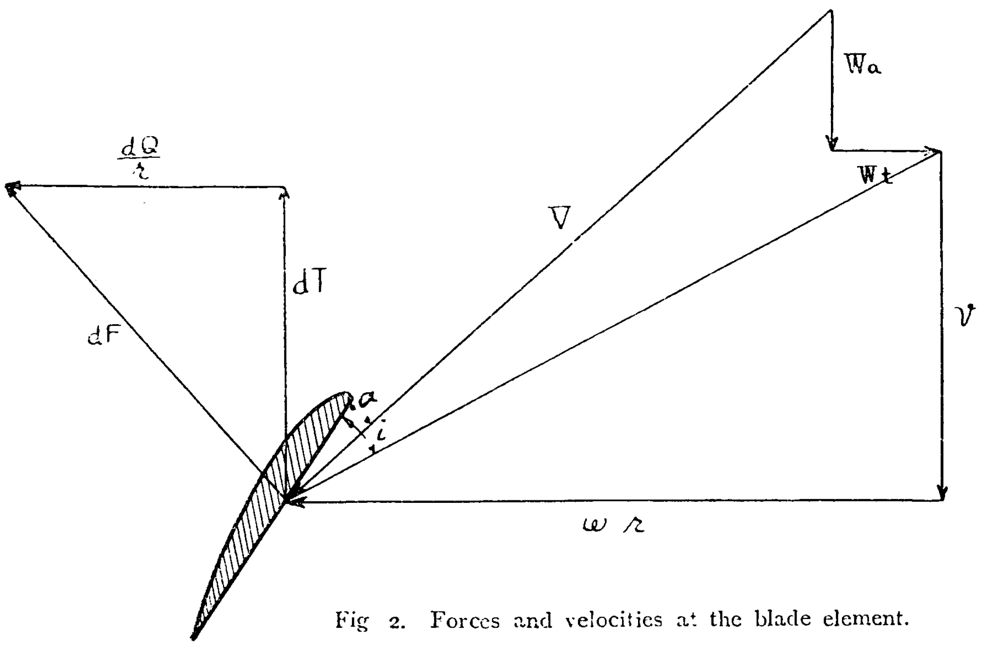


Fig 2. Forces and velocities at the blade element.

5. Theory of Tandem Airscrew.

To simplify the problem let us make the following assumptions.

a) The front airscrew of the tandem arrangement is not influenced by the presence of the rear one.

b) The rear airscrew is in the steady slip stream of the front one.

c) The contraction of the slip stream of the front airscrew is neglected.

The assumption a) and b) are always justifiable when two airscrews are sufficiently separated. It can be seen that the assumption c) has only a very small influence on the result.

Let the quantities with dash relate to the front airscrew and the quantities without dash to the rear one.

Then the element of the blade situated distant r from the axis is equivalent to the corresponding element of the airscrew with the translational velocity of

$$v+\frac{N'\omega'\Gamma'}{2\pi v}$$

and angular velocity of

$$\omega\pm\frac{N'\Gamma'}{2\pi r^2}$$

The meaning of the sign is as follows :

(−) when two airscrews are rotating in the same direction.

(+) when they are rotating in the opposite direction.

Therefore the thrust and torque are respectively

$$dT=\left[\rho N\Gamma\left(\omega r\pm\frac{N'\Gamma'}{2\pi r}-\frac{N\Gamma}{4\pi r}\right)\right.$$

$$\left.-\frac{1}{2}c_f\rho Ntv\sqrt{\omega^2r^2+v^2}\right]dr \qquad (31)$$

$$dQ=\left[\rho N\Gamma r\left(v+\frac{N'\omega'\Gamma'}{2\pi v}+\frac{N\omega\Gamma}{4\pi v}\right)\right.$$

$$\left.+\frac{1}{2}c_f\rho Nt\,\omega r^2\sqrt{\omega^2r^2+v^2}\right]dr \tag{32}$$

Let us seek the most efficient distribution of circulation by the same way as was seen before.

After calculation we have the following result.

$$\Gamma=\frac{2\pi r}{N}\cdot\frac{\lambda v^2\omega r}{v^2+\omega^2r^2}+\frac{N'\Gamma'}{N}\cdot\frac{v^2\mp\omega\omega' r^2}{v^2+w^2r^2}. \tag{33}$$

Supposing the front airscrew is working at the maximum efficiency we have

$$\Gamma'=\frac{2\pi r}{N'}\cdot\frac{\lambda' v^2\omega' r}{v^2+\omega'^2r^2}$$

For further simplification let us suppose

$$\omega'=\omega$$

Then when the two airscrews are rotating in opposite direction we have

$$\Gamma=\frac{2\pi r}{N}\cdot\frac{\lambda v^2\omega r}{v^2+\omega^2r^2}\left\{1+\frac{v^2-\omega^2r^2}{v^2+\omega^2r^2}\cdot\frac{\lambda'}{\lambda}\right\} \tag{34}$$

For the airscrew of the aircraft v is small compared with ωr. Therefore writing $\frac{\omega^2r^2-v^2}{\omega^2r^2+v^2}\doteqdot 1$ we yet obtain the circulation giving very nearly maximum efficiency.

Then we have

$$\Gamma=\frac{2\pi r}{N}\cdot\frac{\lambda v^2\omega r}{v^2+\omega^2r^2}\left\{1-\frac{\lambda'}{\lambda}\right\} \tag{35}$$

When the direction of the rotation is the same we have

$$\Gamma=\frac{2\pi r}{N}\cdot\frac{\lambda v^2\omega r}{v^2+\omega^2r^2}\left\{1+\frac{\lambda'}{\lambda}\right\} \tag{36}$$

Substituting the value of circulation above obtained in the expressions of thrust and torque we have

$$T=T_0\left[1-\frac{c_f N\epsilon\sigma_1}{4\pi\lambda_0\varphi\psi_1}\right] \tag{37}$$

$$Q=Q_0\left[1+\frac{c_f N\epsilon\sigma_2}{8\pi\lambda_0\varphi\psi_2}\right] \tag{38}$$

The maximum efficiency becomes

$$\eta_{max.}=\frac{\psi_1}{\psi_2}\cdot\frac{1-\dfrac{c_f N\epsilon\sigma_1}{4\pi\lambda_0\varphi\psi_1}}{1+\dfrac{c_f N\epsilon\sigma_2}{8\pi\lambda_0\varphi\psi_2}} \tag{39}$$

where

$$T_0=\rho\pi R^2 v^2\lambda_0\varphi\psi_1$$

$$Q_0=\rho\pi R^2\frac{v^3}{\omega}\lambda_0\varphi\psi_2$$

$$\psi_1=1-\left(\frac{\lambda_0}{2}\mp\lambda'\right)\left(\frac{1}{\varphi}\,\frac{z^2}{1+z^2}-1\right)$$

$$\psi_2=1+\left(\frac{\lambda_0}{2}+\lambda'\right)\left(2-\frac{1}{\varphi}\,\frac{z^2}{1+z^2}\right)$$

$$\lambda_0\equiv\lambda'\mp\lambda=\frac{T_r}{2\varphi},\qquad \lambda'=\frac{T_c'}{2\varphi'}$$

$$\varphi=1-\frac{1}{z^2}\log_e(1+z^2)$$

$$\varphi'=1-\frac{1}{z'^2}\log_e(1+z'^2)$$

$$\sigma_1=\sqrt{1+z^2}+\frac{1}{z}\log_e\left(z+\sqrt{1+z^2}\right)$$

$$\sigma_2=\sqrt{(1+z^2)^3}-\frac{1}{2}\sigma_1$$

$$z=\frac{\omega R}{v}$$

$$z'=\frac{\omega R'}{v}$$

$$\epsilon=\frac{t}{R}\ \text{(mean)}.$$

Let us compare the maximum efficiency of a combination of an airscrew-engine when it is solitary and when it is in the rear of another airscrew under the supposition that the velocity of advance is same in two cases. We have obtained in Section 4 for maximum efficiency of a solitary airscrew

$$\eta_{max.}'=\frac{\psi_1'}{\psi_2'}\cdot\frac{1-\dfrac{c_f N\epsilon\sigma_1}{4\pi\lambda_1\varphi\psi_1'}}{1+\dfrac{c_f N\epsilon\sigma_2}{8\pi\lambda_1\varphi\psi_2'}} \tag{40}$$

Therefore if ν is the ratio of the maximum efficiencies we have

$$\nu=\frac{\eta_{max.}}{\eta_{max.}'}=\frac{\psi_1}{\psi_2}\Bigg/\frac{\psi_1'}{\psi_2'}\cdot\frac{1-\dfrac{c_f N\epsilon\sigma_1}{4\pi\lambda\varphi\psi_1}}{1+\dfrac{c_f N\epsilon\sigma_2}{8\pi\lambda\varphi\psi_2}}\Bigg/\frac{1-\dfrac{c_f N\epsilon\sigma_1}{4\pi\lambda_1\varphi\psi'}}{1+\dfrac{c_f N\epsilon\sigma_2}{8\pi\lambda_1\varphi\psi_2'}}$$

In the above expression $\dfrac{1-\dfrac{c_f N\epsilon\sigma_1}{4\pi\lambda\varphi\psi_1}}{1+\dfrac{c_f N\epsilon\sigma_2}{8\pi\lambda\varphi\psi_2}}$ and $\dfrac{1-\dfrac{c_f N\epsilon\sigma_1}{4\pi\lambda_1\varphi\psi_1'}}{1+\dfrac{c_f N\epsilon\sigma_2}{8\pi\lambda_1\varphi\psi_2'}}$

are very nearly equal and as the engine and the velocity of advance of the aircraft are supposed identical in both cases we vahe

$$\nu\doteq\frac{\psi_1}{\psi_2}\Bigg/\frac{\psi_1'}{\psi_2'}$$

while we have

$$\frac{\psi_1}{\psi_2}=\frac{1-\left(\frac{\lambda_0}{2}\mp\lambda'\right)\left(\frac{1}{\varphi}\,\frac{z^2}{1+z^2}-1\right)}{1+\left(\frac{\lambda_0}{2}+\lambda'\right)\left(2-\frac{1}{\varphi}\,\frac{z^2}{1+z^2}\right)}$$

Neglecting the second order of $\left(\frac{\lambda_0}{2}+\lambda'\right)\left(2-\frac{1}{\varphi}\,\frac{z^2}{1+z^2}\right)$ which is small we obtain

$$\frac{\psi_1}{\psi_2}=1-\left(\frac{\lambda_0}{2}\mp\lambda'\right)\left(\frac{1}{\varphi}\,\frac{z^2}{1+z^2}-1\right)-\left(\frac{\lambda_0}{2}+\lambda'\right)\left(2-\frac{1}{\varphi}\,\frac{z^2}{1+z^2}\right)$$

$$=1-\frac{\lambda_0}{2}-\lambda'\left\{\left(2-\frac{1}{\varphi}\,\frac{z^2}{1+z^2}\right)\mp\left(\frac{1}{\varphi}\,\frac{z^2}{1+z^2}-1\right)\right\}$$

Similarly

$$\frac{\psi_1'}{\psi_2'}=\frac{1-\frac{\lambda_1}{2}\left(\frac{1}{\varphi}\,\frac{z^2}{1+z^2}-1\right)}{1+\frac{\lambda_1}{2}\left(2-\frac{1}{\varphi}\,\frac{z^2}{1+z^2}\right)}$$

$$\doteqdot 1-\frac{\lambda_1}{2}$$

Therefore

$$\nu=\frac{1-\frac{\lambda_0}{2}-\lambda'\left\{\left(2-\frac{1}{\varphi}\,\frac{z^2}{1+z^2}\right)\mp\left(\frac{1}{\varphi}\,\frac{z^2}{1+z^2}-1\right)\right\}}{1-\frac{\lambda_1}{2}}$$

Now

$$\lambda_0\doteqdot\frac{T_c}{2\varphi}=\frac{\nu\,T_c'}{2\varphi}=\nu\lambda_1$$

Therefore

$$\nu=\frac{1-\frac{\nu\lambda_1}{2}-\lambda'\left\{\left(2-\frac{1}{\varphi}\,\frac{z^2}{1+z^2}\right)\mp\left(\frac{1}{\varphi}\,\frac{z^2}{1+z^2}-1\right)\right\}}{1-\frac{\lambda_1}{2}}$$

$$\therefore \quad \nu = 1 - \lambda' \left\{ \left(2 - \frac{1}{\varphi} \frac{z^2}{1+z^2} \right) \mp \left(\frac{1}{\varphi} \frac{z^2}{1+z^2} - 1 \right) \right\}$$

If ν_1 is the ratio of efficiency when the direction of rotation is same and ν_2 when opposite we have

$$\nu_1 = 1 - \lambda' = 1 - \frac{T_c'}{2\varphi} \qquad (41)$$

$$\nu_2 = 1 - \lambda' \left(3 - \frac{2}{\varphi} \frac{z^2}{1+z^2} \right)$$

$$= 1 - \frac{T_c'}{2\varphi} \left(3 - \frac{2}{\varphi} \frac{z^2}{1+z^2} \right) \qquad (42)$$

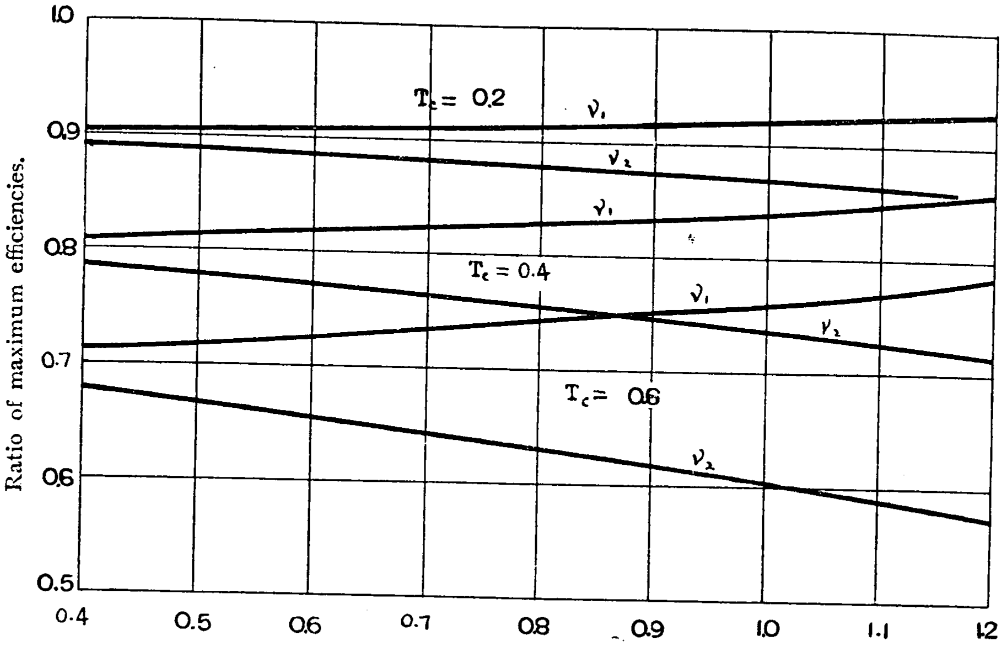
We see that the maximum efficiency falls off when an airscrew is in the rear of another one.

In Table 1. and Figure 3. the values of ν_1 and ν_2 are calculated.

The variable z is transformed into v/nD which is more commonly used and is equal to π/z.

Table 1. Values of ν_1 and ν_2.

v/nD	$T_c=0.2$		$T_c=0.4$		$T_c=0.6$	
	ν_1	ν	ν_1	ν_2	ν_1	ν_2
0.4	0.893	0.905	0.786	0.810	0.678	0.714
0.5	0.890	0.906	0.779	0.812	0.669	0.719
0.6	0.886	0.909	0.772	0.818	0.658	0.726
0.7	0.882	0.911	0.764	0.823	0.646	0.735
0.8	0.878	0.915	0.756	0.829	0.634	0.744
0.9	0.873	0.917	0.746	0.835	0.619	0.752
1.0	0.868	0.920	0.736	0.839	0.605	0.759
1.1	0.863	0.923	0.725	0.846	0.588	0.769
1.2	0.857	0.930	0.714	0.857	0.571	0.786



V/nD Fig. 3.

From these we see that in the tandem arrangement the maximum efficiency of the rear one is always smaller than the front one.

The amount the efficiency falls off is greater for the smaller values of v/nD and if the two engines are rotating in the same direction the drop of efficiency is still greater, especially for large values of v/nD contrary to the former case.

In both cases for large values of T_c the amount of drop is great.

From these we draw the following conclusions :

1° The direction of the rotation of the two airscrews must be opposite.

2° Efficiency of the tandem arrangement is the greater, the greater the value of v/nD and the smaller the value of T_c, or in plain words the faster the aircraft for given values of n and D.

6. Theory of Contra Propeller.

Let us consider two airscrews placed very closely to each other and rotating in the opposite direction. Suppose that they are identical.

Then at the plane of rotation the axial inflow velocity must be $\frac{N\omega\Gamma}{2\pi v}$ and the tangential inflow velocity must be zero.

Therefore the elementary thrust and torque are for each of them

$$dT=\left[\rho N\Gamma\omega r-\frac{1}{2}c_f\rho N tv\sqrt{v^2+\omega^2 r^2}\right]dr \tag{45}$$

$$dQ=\left[\rho N\Gamma r\left(v+\frac{N\omega\Gamma}{2\pi v}\right)+\frac{1}{2}c_f\rho N t\omega r^2\sqrt{v^2+\omega^2 r^2}\right]dr \tag{46}$$

We obtain as in the preceeding Sections the circulation corresponding to the maximum efficiency

$$\Gamma=\frac{\lambda\pi v^2}{N\omega}\cdot$$

Putting this value of circulation in the expressions of dT and dQ we have

$$T=\frac{1}{2}\rho v^2 R^2\left\{\pi\lambda-\frac{1}{2}c_f N\epsilon\sigma_1\right\} \tag{47}$$

$$Q=\frac{1}{2}\rho\frac{v^3}{\omega}R^2\left\{\pi\lambda\left(1+\frac{\lambda}{2}\right)+\frac{1}{4}c_f N\epsilon\sigma_2\right\} \tag{48}$$

And maximum efficiency

$$\eta_{max.}=\frac{vT}{\omega Q}=\frac{\pi\lambda-\frac{1}{2}c_f N\epsilon\sigma_1}{\pi\lambda\left(1+\frac{\lambda}{2}\right)+\frac{1}{4}c_f N\epsilon\sigma_2} \tag{49}$$

In the expression of thrust the term due to the profil resistance is exceedingly small. Therefore we have

$$T\doteq\int_0^R\rho N\Gamma\omega r\,dr=\frac{1}{2}\rho N\Gamma\omega R^2=\frac{1}{2}\rho v^2 R^2\pi\lambda$$

$$\therefore\quad\lambda=\frac{T}{\frac{1}{2}\rho v^2\pi^2 R^2}=T_c \tag{50}$$

The maximum efficiency becomes

$$\eta_{max.}=\frac{\pi T_c-\frac{1}{2}c_f N\epsilon\sigma_1}{\pi T_c\left(1+\frac{T_c}{2}\right)+\frac{1}{4}c_f N\epsilon\sigma^c} \tag{51}$$

As a special case, suppose that the rear airscrew is standstill and it is so arranged as to nullify the rotation in the slip stream.

Then we have an arrangement known as "contra propeller".

The thrust of the rear airscrew is small compared with that of the front one. For it can be regarded as an airscrew of very slow rotation. The thrust can be positive or negative according to the case.

Now consider the front airscrew. As the inflow velocity caused by the rear stationary airscrew is very small the thrust and the torque are

$$dT=\left[\rho N\Gamma\omega r-\frac{1}{2}c_f\rho Nvt\sqrt{v^2+\omega^2 r^2}\right]dr \tag{52}$$

$$dQ=\left[\rho N\Gamma r\left(v+\frac{N\omega\Gamma}{4\pi v}\right)+\frac{1}{2}c_f\rho Nt\,\omega r^2\sqrt{v^2+\omega^2 r^2}\right]dr \tag{53}$$

The circulation for maximum efficiency and the value of the maximum efficiency are

$$\Gamma=\frac{2\pi\lambda v^2}{N\omega} \tag{54}$$

$$\eta_{max.}=\frac{2\pi\lambda-\frac{1}{2}c_f N\epsilon\sigma_1}{2\pi\lambda\left(1+\frac{\lambda}{2}\right)+\frac{1}{4}c_f N\epsilon\sigma_2} \tag{55}$$

as $\lambda\doteqdot\frac{T_c}{2}$ we have

$$\eta_{max.}=\frac{\pi T_c-\frac{1}{2}c_f N\epsilon\sigma_1}{\pi T_c\left(1+\frac{T_c}{4}\right)+\frac{1}{4}c_f N\epsilon\sigma_2} \tag{56}$$

For the comparsion with a solitary airscrew the neglect of the term due to the profil resistance in both cases will not introduce much divergence.

For solitary airscrew neglecting the profil resistance we have

$$\eta_{max.(1)} \doteqdot \frac{1-\dfrac{T_c}{4\varphi}\left(\dfrac{1}{\varphi}\dfrac{z^2}{1+z^2}-1\right)}{1+\dfrac{T_c}{4\varphi}\left(2-\dfrac{1}{\varphi}\dfrac{z^2}{1+z^2}\right)} \doteqdot 1-\frac{T_c}{4\varphi}^{(1)} \tag{57}$$

and for airscrew with contra propeller

$$\eta_{max.(2)} = \frac{1}{1+\dfrac{T_c}{4}} \doteqdot 1-\frac{T_c}{4} \tag{58}$$

Similarly for coaxial airscrew

$$\eta_{max.(3)} = \frac{1}{1+\dfrac{T_c}{2}} \doteqdot 1-\frac{T_c}{2} \tag{59}$$

Whence we know that the contra propeller is always superior to the solitary airscrew and the coaxial airscrew is superior when

$$\varphi < \frac{1}{2}.$$

For coaxial airscrew it is more reasonable to compare with the solitary airscrew having twice the value of thrust coefficient.

Then we are to compare

$$\eta_{max.(1)} = 1-\frac{T_c}{2\varphi} \tag{60}$$

and

$$\eta_{max.(3)} = 1-\frac{T_c}{2} \tag{61}$$

Therefore also the coaxial airscrew is superior to solitary one.

In Table 2, 3 and Figure 4, 5 the ratios of efficiencies are given as function of v/nD.

We conclude from these that the contra propeller is not so efficient for aircraft (with small T_c) as for ship (with comparatively large T_c).

---

(1) neglecting the higher orders of $\dfrac{T_c}{4\varphi}\left(2-\dfrac{1}{\varphi}\dfrac{z^2}{1+z^2}\right)$.

Table 2. Increase of maximum efficiency of airscrew with contra propeller.

v/nD	$T_c=0.2$	$T_c=0.4$	$T_c=0.6$
0.4	1.005	1.008	1.013
0.5	1.006	1.012	1.018
0.6	1.007	1.015	1.025
0.7	1.009	1.020	1.032
0.8	1.011	1.024	1.040
0.9	1.014	1.030	1.050
1.0	1.017	1.036	1.060
1.1	1.020	1.043	1.070
1.2	1.024	1.050	1.082

Table 3. Increase of maximum efficiency of coaxial airscrew.

v/nD	$T_c=0.2$	$T_c=0.4$	$T_c=0.6$
0.4	1.007	1.018	1.033
0.5	1.011	1.026	1.046
0.6	1.016	1.036	1.063
0.7	1.020	1.047	1.083
0.8	1.025	1.058	1.103
0.9	1.031	1.072	1.130
1.0	1.037	1.086	1.156
1.1	1.043	1.100	1.190
1.2	1.050	1.118	1.225

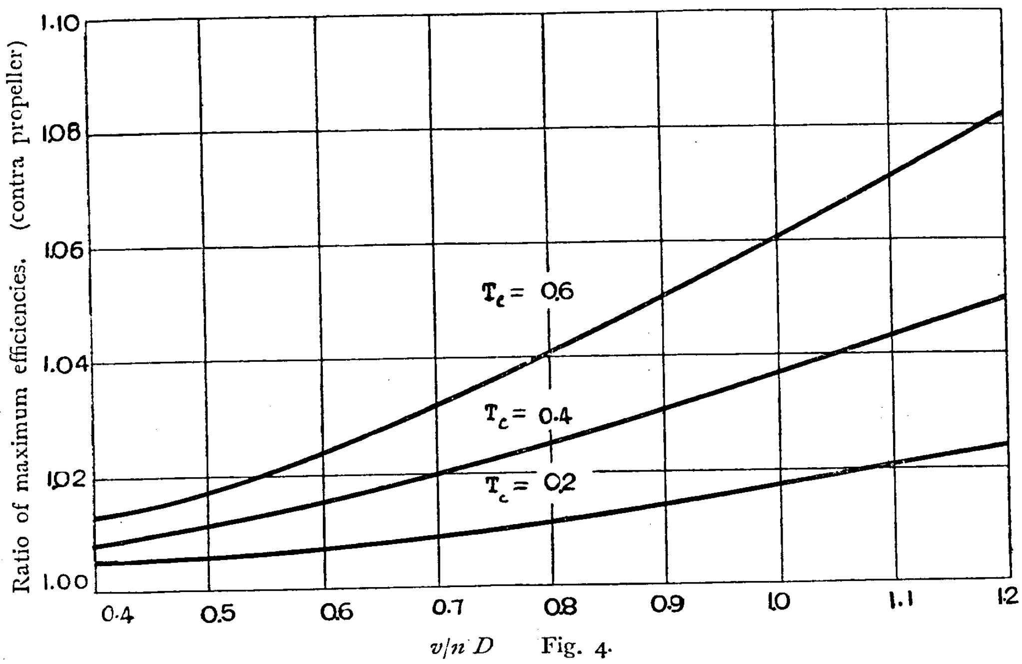

1.10
1.08
1.06
1.04
1.02
1.00
Ratio of maximum efficiencies. (contra propeller)
T_c = 0.6
T_c = 0.4
T_c = 0.2
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
v/n D

Fig. 4.

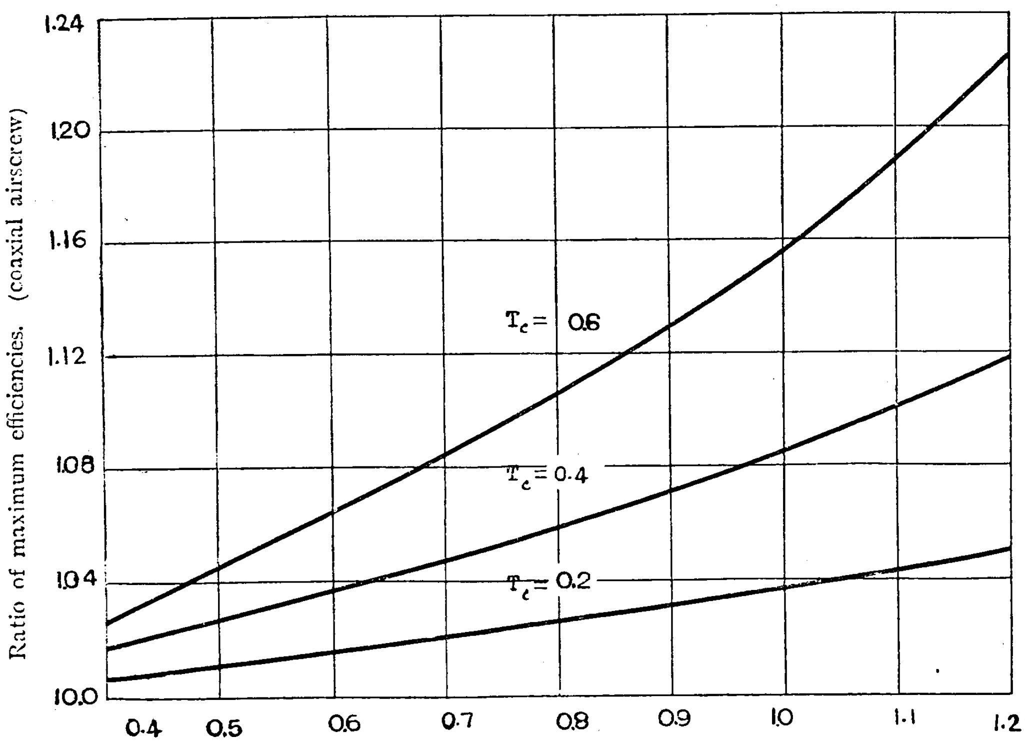

1.24
1.20
1.16
1.12
1.08
1.04
10.0
Ratio of maximum efficiencies. (coaxial airscrew)
T_c = 0.6
T_c = 0.4
T_c = 0.2
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
v/n D

Fig. 5.

7. Extension of the Above Theory to the Case in which the Number of Blades is small.

When the number of blades is small the circulation drops off near the tip of the blade and the requirement for the maximum efficiency in Section 4. can not be realised. Whence the drop of the maximum effic ncy results.

Therefore the airscrew with small number of blades can be regarded as equivalent to an airscrew with infinite number of blades with its radius reduced to a certain extent.

Let R' be the radius of this equivalent airscrew and suppose

$$R'=R/k$$

Then the expressions of thrust and torque are respectively[1]

$$T=T'\left\{-\frac{c_f N \epsilon\sigma_1 k^2}{4\pi\lambda\varphi\psi_1}\right\} \tag{62}$$

$$Q=Q'\left\{+\frac{c_f N \epsilon\sigma_2 k^2}{8\pi\lambda\varphi\psi_2}\right\} \tag{63}$$

and the maximum efficiency

$$\eta_{max.}=\frac{\psi_1}{\psi_2}\cdot\frac{1-\dfrac{c_f N \epsilon\sigma_1 k^2}{4\pi\lambda\varphi\psi_1}}{1+\dfrac{c_f N \epsilon\sigma_2 k^2}{8\pi\lambda\varphi\psi_2}} \tag{64}$$

where

$$T'=\frac{1}{k^2}\rho\pi R^2 v^2\lambda\varphi\psi_1$$

$$Q'=\frac{1}{k^2}\rho\pi R^2\frac{v^3}{\omega}\lambda\varphi\psi_2$$

$$\psi_1=1-\frac{\lambda}{2}\left(\frac{1}{\varphi}\frac{z_1^2}{1+z_1^2}-1\right)$$

$$\psi_2=1+\frac{\lambda}{2}\left(2-\frac{1}{\varphi}\frac{z_1^2}{1+z_1^2}\right)$$

---

(1) The profil resistance is not influenced by the number of blades.

$$\varphi = 1 - \frac{1}{z_1^2} \log_e (1 + z_1^2)$$

$$\lambda \doteqdot \frac{k^2 T_c}{2\varphi}$$

$$z_1 = \frac{z}{k} = \frac{1}{k} \cdot \frac{\omega R}{v}$$

$$\sigma_1 = \sqrt{1 + z^2} + \frac{1}{z} \log_e (z + \sqrt{1 + z^2})$$

$$\sigma_2 = \sqrt{(1 + z^2)^3} - \frac{\sigma_1}{2}$$

The value of k depends upon the value of z and N.

The value of k can be obtained either by the analysis of the experiment or from the theoretical consideration.

To obtain the value of k from the experiment, airscrews must be designed for given values of T_c and z to develop the maximum efficiency.

The value of R' must be then calculated from the condition that the maximum efficiency is equal to the calculated one of airscrew with radius R'.

This calculation must be repeated for each value of T_c and z.

From theoretical consideration[1], Prandtl gave as the value of k

$$k = \frac{1}{1 - \frac{1.386}{N\sqrt{1 + z^2}}} \qquad (65)$$

This was obtained from the following consideration.

The distribution of the circulation of the multiplane which gives uniform down-wash was calculated, and the circulation near the tip of the blade was assimulated ; with this the value of k was calculated. This seems to give very good approximation.

---

(1) Betz: Schraubenpropeller mit geringsten Energieverlust, Zusatz.
Nach. von der K. Ges. d. Wissensch. zu Göttingen, Math-Phys. Kl. (1919) 193.

According to him the circulation above mentioned is

$$\Phi=\frac{2}{\pi} cos^{-1} e^{-\frac{R-r}{r}\cdot\frac{N}{2}\sqrt{1+z^2}} \tag{66}$$

In the above expression the circulation remote from the tip was taken as unity.

Therefore the circulation for maximum efficiency becomes [1]

$$\Gamma=\frac{2\pi r}{N}\cdot\frac{\lambda v^2\,\omega r}{v^2+\omega^2 r^2}\Phi \tag{67}$$

8. Comparison of the Theoretical Consideration with the Results of the Experiments.

As first example, let us calculate the value of maximum efficiency of the airscrews tested at various laboratories.

The value of the coefficient of profil resistance is difficult to know in each case.

In the next example the value of c_f was taken as 0.016 which is the mean value of c_f for the angles of incidence practically used.

The value of c_f was then so adjusted as to bring the calculated and experimental efficiencies into coincidence.

Table 4.

Airscrew	N	T_c	η *max.* (calculated)	η *max.* (measured)	c_f
Durand No. 3	2	0.208	0.82	0.81	0.018
Durand No. 32	2	0.465	0.73	0.71	0.017
Durand No. 11	2	0.61	0.715	0.70	0.017
"A" N. P. L.	4	0.51	0.73	0.705	0.020

Assumed value of c_f=0.016 The value of η *max.* was calculated from (64).

(1) For near the tip where $\omega^2 r^2$ is very large compared with v^2, the expression

$$\frac{2\pi r}{N}\,\frac{\lambda v^2 \omega r}{v^2+\omega^2 r^2}$$

is nearly constant.

The values of c_f thus obtained are given in the last column of the Table 4.

The values obtained are all reasonable ranging from 0.017 to 0.02.

The next example is a comparison of the distribution of thrust and torque obtained from the measurement of the pressure distribution on the airscrew blade with those calculated from the theory.

The experiment used is that executed at the National Physical Laboratory (Report and Memoranda of Advisory Committee for Aeronautics. No. 681 March 1921.)

In the first calculation, the drop of the circulation near the tip of the blade is neglected, and in the second (circulation above calculated) $\times\Phi$ was taken as circulation.

The real circulation will lie somewhere between these two.

The calculations are given in Table 5. and in Figure 6, 7 the values of dT/dr and dQ/dr are compared graphically.

We see that the coincidence is fairly good except the region near the tip of the blade.

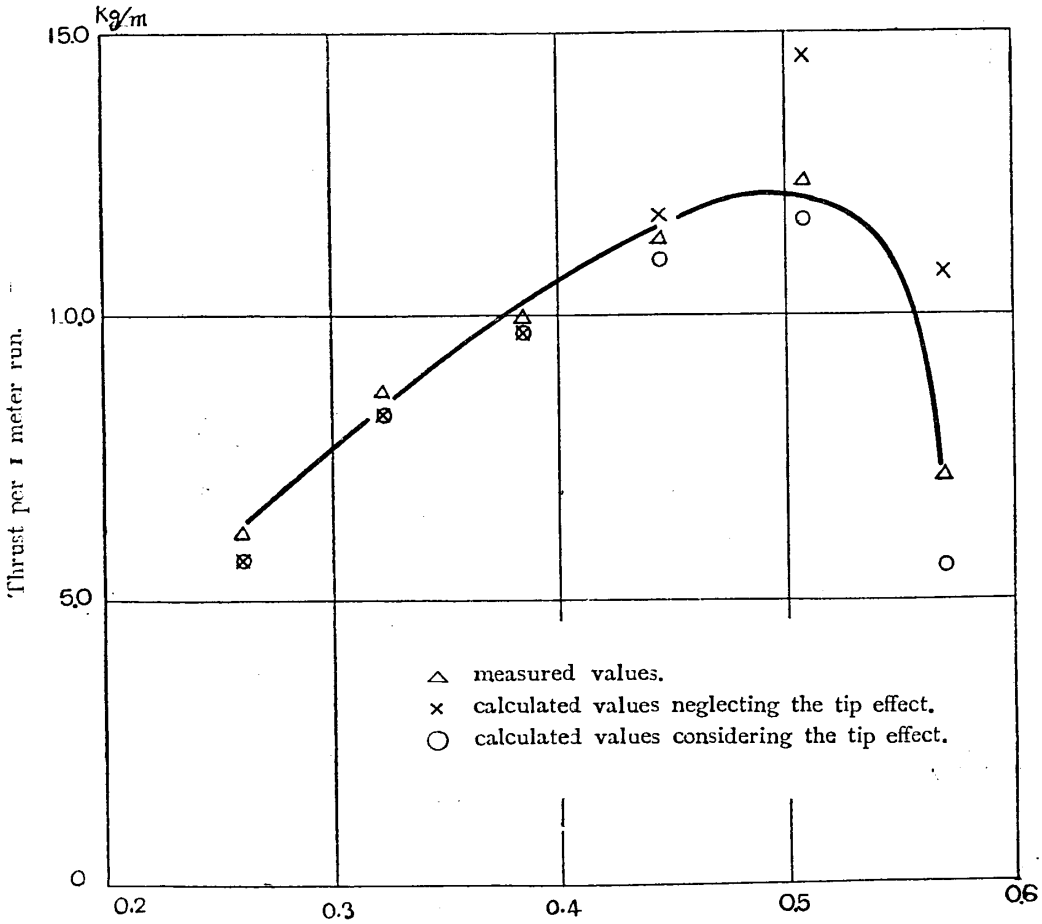
Table 5. Comparison of the values of dT/dr and dQ/dr from the theory and experiment.

$v=9.1$ meters/sec. $\omega=100$ radians/sec.

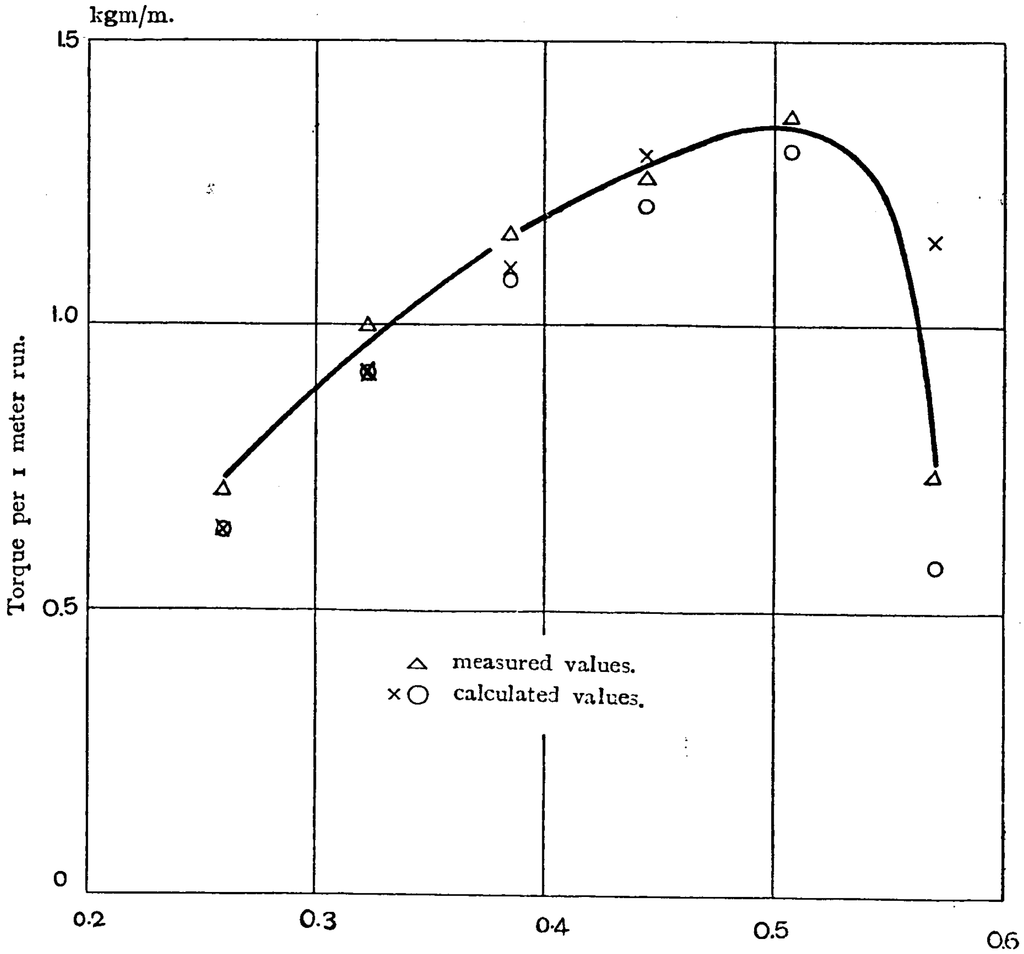
Section	r	t	θ°	$tg^{-1}\frac{v}{\omega r}$	i	Φ	c_{ai}	c_a	Γ	$\frac{N\Gamma}{4\pi r}$	$\frac{N\omega\Gamma}{4\pi v}$
A	0.57	0.0473	12.5	9.00	3.5	0.506	0.70	0.563	0.772	0.215	1.35
B	0.508	0.0662	14.05	10.2	3.85	0.80	0.86	0.643	1.16	0.359	2.01
C	0.445	0.0808	15.10	11.5	3.46	0.93	0.81	0.573	1.07	0.376	1.84
D	0.384	0.0915	17.10	13.3	3.8	0.93	0.85	0.577	1.05	0.434	1.83
E	0.322	0.0960	20.40	15.8	4.6	1.00	0.97	0.644	1.04	0.514	1.82
F	0.259	0.0945	23.90	19.3	4.6	1.00	1.04	0.692	0.90	0.552	1.57

Table 5. (continued)

Section	calculated ($\Phi=1$)		calculated		measured	
	dT/dr	dQ/dr	dT/dr	dQ/dr	dT/dr	dQ/dr
A	10.95	1.15	5.54	0.582	7.2	0.74
B	14.6	1.64	11.7	1.31	12.4	1.37
C	11.8	1.30	11.0	1.21	11.4	1.26
D	9.97	1.10	9.78	1.08	9.9	1.16
E	8.24	0.914	8.24	0.914	8.7	1.00
F	5.72	0.634	5.72	0.634	6.2	0.71



Radius. Fig. 6.



Radius. Fig. 7.

PART. 2. AIRSCREWS AT A FIXED POINT. (HELICOPTER AIRSCREW)

1. Solution of the Fundamental Equations.

In this case the airscrew is stationary i. e. $v=0$ and the equations become

$$dT=\rho\, N\Gamma(\omega r-W_t)\, dr \tag{68}$$

$$dQ=\rho\, N\Gamma r\, W_a\, dr \tag{69}$$

$$dT=2\pi\, r\, \rho\, W_a\, W_a'\, dr \tag{70}$$

$$dQ = 2\pi\, rr'\, \rho\, W_a\, W_t'\, dr \tag{71}$$

$$2\pi\, r\, W_a\, dr = 2\pi\, r'\, W_a'\, dr' \tag{72}$$

$$dQ = \pi\, r\, \rho\, W_a\, (W_a'^2 + W_t'^2)\, dr \tag{73}$$

$$2 W_t\, r = W_t'\, r' \tag{74}$$

Neglecting the higher orders of $W_t/\omega r$ and supposing W_a, W_a' to be independent or r, r' in (72), we arrive at the following solution after the similar calculations as those in Section I. of the first part

$$\left.\begin{aligned} & W_a = \frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}, \quad W_a' = 2 W_a = \frac{\sqrt{N\omega\Gamma}}{\sqrt{\pi}} \\ & W_t = \frac{N\Gamma}{4\pi\, r}, \qquad W_t' = \frac{N\Gamma}{2\pi\, r'} \\ & r/r' \doteqdot \sqrt{2} \end{aligned}\right\} \tag{75}$$

Therefore the axial and tangential components of the frictical force are respectively for each blade

$$dR_a = \frac{1}{2}\, c_f\, \rho\, Vt\, \frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\, dr \doteqdot \frac{1}{2}\, c_f\, \rho\, \omega r\, t\, \frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\, dr \tag{76}$$

$$dR_t = \frac{1}{2}\, c_f\, \rho\, Vt\, \omega r\, dr \doteqdot \frac{1}{2}\, c_f\, \rho\, \omega^2 r^2\, t\, dr \tag{77}$$

Hence the expressions of thrust and torque become neglecting the small quantities of the second order

$$dT = \left[\rho N\Gamma \left(\omega r - \frac{N\Gamma}{4\pi\, r}\right) - \frac{1}{2}\, c_f\, \rho\, Nt\, \omega r\, \frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\right] dr \tag{78}$$

$$dQ = \left[\rho N\Gamma \frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}} + \frac{1}{2}\, c_f\, \rho Nt\, \omega^2\, r^3\right] dr \tag{79}$$

2. Most Efficient Value of Circulation.

As W_t is very small compared with ωr except for small values of r and this part of the blade is occupied by boss having no contribution to thrust, dT can be written:

$$dT=\left[\rho N\Gamma\omega r-\frac{1}{2}c_f\rho Nt\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\omega r\right]dr \tag{80}$$

The part of thrust and torque due to frictional resistance are small and their influence is negligible on dT and dQ whether the distribution of circulation is one or the other.

Bearing these circumstances in mind the required circulation is such that

$$\left.\begin{aligned} T&=\int_0^R(\rho N\Gamma\omega r-\text{term independent of }\Gamma)\,dr\\ &=\text{constant}\\ \omega Q=P&=\omega\int_0^R\left(\rho N\Gamma r\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\right.\\ &\qquad\left.+\text{term independent of }\Gamma\right)dr=\text{minimum}\end{aligned}\right\} \tag{81}$$

By calculus of variation

$$\delta(\omega Q-\lambda T)=0$$

$$\rho N\omega r-\lambda\frac{3}{2}\rho N\omega r\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}=0$$

$$\Gamma=\frac{16\pi}{9\lambda^2\omega N} \tag{82}$$

where λ is Lagrange's factor.

Putting the value of Γ in (78) and (79) we have

$$T=\rho N\frac{16\pi}{9\lambda^2\omega N}\omega\frac{R^2}{2}-\frac{1}{2}c_f\rho Nt\frac{\sqrt{N\omega}}{2\sqrt{\pi}}\frac{4\sqrt{\pi}}{3\lambda\sqrt{N\omega}}\omega\frac{R^2}{2}$$

$$=\frac{8\pi\rho R^2}{9\lambda^2}-c_f\frac{\rho Nt\omega R^2}{6\lambda}$$

$$=\frac{8\pi\rho R^2}{9\lambda^2}\left(1-c_f\frac{3N\epsilon\omega R\lambda}{16\pi}\right) \tag{83}$$

$$Q=\rho N\frac{64\pi^{\frac{3}{2}}}{27\lambda^3N^{\frac{3}{2}}\omega^{\frac{3}{2}}}\cdot\frac{\sqrt{N\omega}}{2\sqrt{\pi}}\cdot\frac{R^2}{2}+\frac{1}{2}c_f\rho Nt\omega^2\frac{R^4}{4}$$

$$=\frac{16\pi\rho R^2}{27\lambda^3\omega}\left(1+c_f\frac{27Nt\omega^3R^3\lambda^3}{128\pi}\right) \tag{84}$$

$$\frac{T}{P}=\frac{T}{\omega Q}=\frac{3\lambda}{2}\cdot\frac{1-c_f\dfrac{3N\epsilon\omega R\lambda}{16\pi}}{1+c_f\dfrac{27N\epsilon\omega^3R^3\lambda^3}{128\pi}} \tag{85}$$

The value of λ can be obtained from (83) solving this as an equation of λ.

$$\frac{1}{\lambda^2}-\frac{3c_fNt\omega R}{16T}\cdot\frac{1}{\lambda}-\frac{9T}{8\pi\rho R^2}=0.$$

Or putting $T_c'=T/\pi R^2$

$$\frac{1}{\lambda^2}-\frac{3c_fNt\omega R}{16\pi}\cdot\frac{1}{\lambda}-\frac{9T_c'}{8\rho}=0 \tag{86}$$

If we neglect the frictional resistance we have

$$\lambda=\frac{2\sqrt{2\rho}}{3\sqrt{T_c'}} \tag{87}$$

and T/P becomes

$$T/P=\frac{3\lambda}{2}=\frac{\sqrt{2\rho}}{\sqrt{T_c'}}$$

or $$T=(2\pi\rho)^{\frac{1}{3}}(PR)^{\frac{2}{3}} \tag{88}$$

This is the theoretical maximum value of thrust obtainable with given power and diameter of the airscrew under the suppositions that

a) the number of blades is infinite

b) the rotation of the slip stream is negligible

c) the frictional resistance of air is negligible.

3. Effective Angle of Incidence of Blade Element.

The effective angle of incidence is

$$\alpha=\theta-tg^{-1}\frac{W_a}{\omega r-W_t}$$

$$=\theta-tg^{-1}\frac{\dfrac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}}{\omega r-\dfrac{N\Gamma}{4\pi r}}$$

$$\doteqdot\theta-tg^{-1}\frac{\dfrac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}}{\omega r}$$

while we have

$$\Gamma=\frac{1}{2}c_a tV\doteqdot\frac{1}{2}c_a t\omega r$$

$$\therefore\quad \alpha=\theta-tg^{-1}\frac{\dfrac{\sqrt{N\omega\,.\,\dfrac{1}{2}c_a t\omega r}}{2\sqrt{\pi}}}{\omega r}$$

$$=\theta-tg^{-1}\frac{\sqrt{Nc_a t}}{2\sqrt{2\pi r}}\quad\text{radians} \tag{89}$$

Thus we have an important property of an airscrew at a fixed point :

Angle of incidence of each blade element of an airscrew at a fixed point is independent of the angular velocity.

4. Airscrew with Finite Number of Blades.

Airscrew with finite number of blades can be regarded as equivalent to an airscrew with infinite number of blades with its radius reduced to $R'=R/k$ as was said in the Section 7 of the first part where

$$k=\frac{1}{1-\dfrac{1.386}{N\sqrt{1+z^2}}}$$

$$z=\frac{\text{circumferential velocity}}{\text{mean translational velocity}}=\frac{\omega R}{\dfrac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}}$$

$$=\frac{6\lambda\,\omega R}{\pi} \quad \text{for optimum airscrew.} \tag{90}$$

Then the values of thrust and torque become

$$T=\frac{8\pi\,\rho R^2}{9\lambda^2 k^2}\left\{1-c_f\frac{3\,N\epsilon\omega\,R\lambda\,k^2}{16\,\pi}\right\} \tag{91}$$

$$Q=\frac{16\pi\,\rho R^2}{27\lambda^3\,\omega k^2}\left\{1+c_f\frac{27\,N\epsilon\omega^3 R^3\lambda^3 k^2}{128\,\pi}\right\} \tag{92}$$

$$\frac{T}{P}=\frac{T}{\omega Q}=\frac{3\lambda}{2}\cdot\frac{1-c_f\dfrac{3\,N\epsilon\omega\,R\lambda\,k^2}{16\pi}}{1+c_f\dfrac{27\,N\epsilon\omega^3 R^3\lambda^3 k^2}{128\,\pi}} \tag{93}$$

In the Table 6 I give the values of T/P, both experimental and theoreotical, of the airscrews tested by Durand and Lesley.

Table 6. Values of T/P.

Airscrew	T/P (experimental)	T/P ($c_f=0$)	T/P ($c_f=0.02$)
Durand 5	0.10	0.16	0.14
Durand 10	0.123	0.17	0.14
Durand 20	0.105	0.148	0.13
Durand 24	0.124	0.174	0.137
Durand 80	0.0725	0.152	0 136

5. Effect of Side Wind on the Performance of Stationary Airscrew.

In Section 1. I have obtained the following formulae for the expressions of thrust and torque

$$dT=\left[\rho N\Gamma\left(\omega r-\frac{N\Gamma}{4\pi r}\right)-\frac{1}{2}c_f\rho Nt\,\omega r\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\right]dr \qquad (94)$$

$$dQ=\left[\rho N\Gamma r\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}+\frac{1}{2}c_f\rho Nt\,\omega^2 r^3\right]dr \qquad (95)$$

In the expression of thrust the term due to the frictional resistance is exceedingly small. Therefore neglecting this and the term due to the rotation of the slip stream we have for the thrust :

$$dT=\rho N\Gamma\,\omega r\,dr \qquad (96)$$

The term due to the frictional resistance in the expression of the torque is retained, for it is comparatively large.

Now let us make the following assumptions :

a) The instantaneous tangential velocity of the blade element at radius r is $\omega r+u\cos\theta$; the existence of the velocity component $u\sin\theta$ along the radius of the blade is neglected.

b) The aerodynamical force acting on the blade element at each instant is equal to that on the corresponding element when the airscrew is supposed stationary and rotating with angular velocity $\omega r + u \cos\theta$.

In the expressions (95) and (96) ωr and circulation Γ are function of θ (see fig. 8)

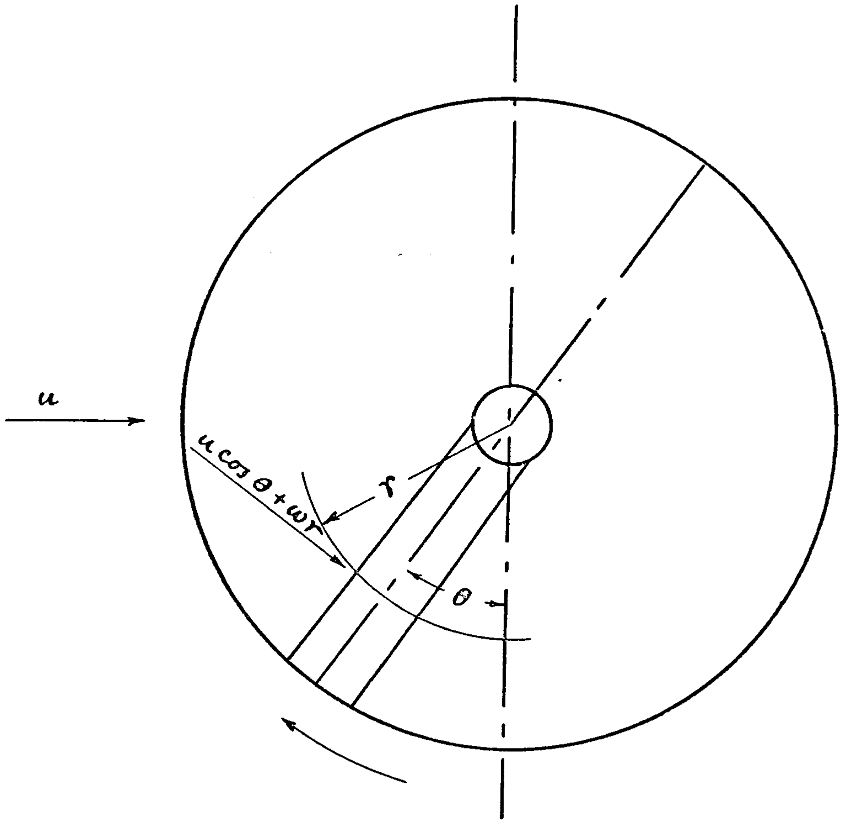


Fig. 8. Airscrew in side wind.

Therefore

$$dT = \frac{1}{2\pi}\int_0^{2\pi} \rho N \Gamma (\omega r + u \cos\theta)\, d\theta\, dr$$

while we have

$$\Gamma = \frac{1}{2} c_a t V \doteqdot \frac{1}{2} c_a t (\omega r + u \cos\theta)$$

$$\therefore \quad dT=\frac{1}{2\pi}\int_0^{2\pi}\rho N\frac{1}{2}c_a t(\omega r+u\cos\theta)^2 d\theta\, dr \tag{97}$$

The effective angle of incidence α and hence c_a of an airscrew at a fixed point is independent of the angular velocity. (Section 3. Part 2)

Therefore c_a is independent of θ and we have

$$dT=\rho N\frac{1}{2}c_a t\left(\omega^2 r^2+\frac{u^2}{2}\right)dr$$

$$=\rho N\Gamma\left(\omega r+\frac{1}{\omega r}\frac{u^2}{2}\right)dr \tag{98}$$

In the same way we can obtain the torque

$$dQ=\frac{1}{2\pi}\int_0^{2\pi}\rho N\frac{1}{2}c_a tr\frac{\sqrt{N\frac{1}{2}c_a t}}{2\sqrt{\pi r}}(\omega r+u\cos\theta)^2 d\theta\, dr$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}c_f\rho N tr(\omega r+u\cos\theta)^2 d\theta\, dr$$

$$=\left\{\rho N\Gamma r\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}+\frac{1}{2}c_f\rho Nt\omega^2 r^3\right\}\left\{1+\frac{u^2}{2\omega^2 r^2}\right\}dr \tag{99}$$

where Γ=circulation when there is no side wind $\doteqdot\frac{1}{2}c_a t\omega r$

In the above calculation we have supposed that $u<\omega r$ and therefore the expression of dQ is only true for outer region of the airscrew disc area.

To obtain the total thrust and torque let us limit the integration to this region only.

Let us integrate dT and dQ from $r=u/\omega$ to $r=R$, the radius of the airscrew.

Let us suppose Γ=constant.

Then we have

$$T=\int_{\frac{u}{\omega}}^{R}\rho N\Gamma\left(\omega r+\frac{1}{\omega r}\frac{u^2}{2}\right)dr$$

$$=\rho N\Gamma\left\{\omega\frac{R^2-\frac{u^2}{\omega^2}}{2}+\frac{u^2}{2\omega}\,log\frac{\omega R}{u}\right\}$$

$$=\frac{\rho N\Gamma\,\omega R^2}{2}\left\{1-\frac{1}{\mu^2}(1-log\,\mu)\right\} \tag{100}$$

where $\mu=\frac{\omega R}{u}$

If $u=0$, integrating from $r=u/\omega$ to $r=R$ as before we have

$$T_0=\frac{\rho N\Gamma\omega\,R^2}{2}\left(1-\frac{1}{\mu^2}\right) \tag{101}$$

$$\therefore\quad T/T_0=\frac{1-\frac{1}{\mu^2}(1-log\,\mu)}{1-\frac{1}{\mu^2}} \tag{102}$$

Similarly

$$Q=\rho N\Gamma\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\,\frac{R^2}{2}\left\{1-\frac{1}{\mu^2}(1-log\,\mu)\right\}$$

$$+\frac{1}{8}c_f\,\rho Nt\,\omega^2 R^4\left(1+\frac{1}{\mu^2}-\frac{2}{\mu^4}\right) \tag{103}$$

while we have seen that for optimum airscrew at a fixed point

$$\Gamma=\frac{16\,\pi}{9\,\lambda^2\,\omega N}$$

Then

$$Q=\rho N\,I R^2\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\left\{1-\frac{1}{\mu^2}(1-log\,\mu)\right.$$

$$\left.+c_f\frac{27\,N\epsilon\lambda^3\omega^3R^3}{128\,\pi}\left(1+\frac{1}{\mu^2}-\frac{2}{\mu^4}\right)\right\} \tag{104}$$

and the torque when $u=0$ is, integrating from u/ω to R:

$$Q_0=\rho N I R^2 \frac{\sqrt{N\omega l}}{4\sqrt{\pi}}\left\{1-\frac{1}{\mu^4}\right.$$

$$\left. + c\ \frac{27\ N\epsilon\lambda^3\omega^3R^3}{128\ \pi}\left(1-\frac{1}{\mu^4}\right)\right\} \qquad (105)$$

$$\therefore\ Q/Q_0=\frac{1-\frac{1}{\mu^2}(1-\log\mu)+c_f\frac{27\ N\epsilon\lambda^3\omega^3R^3}{128\ \pi}\left(1+\frac{1}{\mu^2}-\frac{1}{\mu^4}\right)}{1-\frac{1}{\mu^2}+c_f\frac{27\ N\epsilon\lambda^3\omega^3R^3}{128\ \pi}\left(1-\frac{1}{\mu^4}\right)} \qquad (106)$$

From (102) and (106) we see that the side wind increases both thrust and torque. In table 7. I have calculated the values of T/T_0 and Q/Q_0

Table 7. Values of T/T_0 and Q/Q_0.

μ	T/T_0	Q/Q_0
15	1.01	1.01
10	1.02	1.017
8	1.035	1.025
6	1.05	1.04
4	1.09	1.07

6. Resistance of Airscrew in Horizontal Movement.

In this section let us examine the resistance in the direction of the side wind.

This resistance must arise from the inequality of the compoment force in the direction of side wind on the opposite blade.

The difference must be the resistance.

Call F this resistance. Then

$$dF=\frac{dQ}{r}\times\cos\theta$$

$$=\left\{\rho N\Gamma\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}+\frac{1}{2}c_f\rho Nt\omega^2r^2\right\}\frac{u}{\omega r}dr \qquad (107)$$

$$F=\int_{\frac{u}{\omega}}^{R}dF=\left\{\rho N\Gamma\frac{\sqrt{N\omega\Gamma}}{2\sqrt{\pi}}\,log\,\mu\right.$$

$$\left.+\frac{1}{2}c_f\rho\,Nt\,\omega^2R^3\left(1-\frac{1}{\mu^2}\right)\right\}\frac{u}{\omega} \qquad (108)$$

If we suppose as before

$$\Gamma=\frac{16\pi}{9\lambda^2\omega N}$$

$$F=\frac{32\rho\pi u}{27\lambda^3\omega^3}\left\{log\,\mu+c_f\frac{27\,N\epsilon\lambda^3\omega^3R^3}{128\pi}\left(1-\frac{1}{\mu^2}\right)\right\} \qquad (109)$$

From the last formula the power necessary to the propulsion of a helicopter will be easily calculated.

For example take a helicopter with following characteristics :

Total weight of helicopter............1500 kgs.
Radius of screw ....3.00 meters
Number of revolution................600 per min.
Power ....360 H. P.
Number of blades of screw 2
Mean blade width ....0.2 Radius

The powers necessary to overcome the resistance of the airscrew proper are calculated in Table 8.

Table 8.

Speed	H. P.
30 m/s	8.2
50	22.8

Next let us find the angle of inclination of the airscrew shaft to the vertical of the same helicopter necessary to overcome the total resistance by the component of the thrust.

The result is given in Table 9. We see from this that a helicopter can attain sufficient velocity of advance by an adequate inclination of the airscrew shaft.

Table 9. Angle of inclination necessary to the propulsion.

Velocity of advance	Parasite resistance	Angle of inclination
30 m/s.	0 kg.	45′
	50	2°—20′
	100	4°—40′
	200	8°—25′
50 m/s.	0	2°—10′
	50	4°
	100	6°
	200	9°—40′

List of Symbols.

R = radius of airscrew.

t = breadth of blade.

θ = blade angle.

i = apparent angle of incidence.

α = effective angle of incidence.

Γ = circulation round the blade.

N = number of blades.

c_a = lift coefficient for 1 : ∞ aspect ratio.

c_f = resistance coefficient for 1 : ∞ aspect ratio.

v = velocity of advance of airscrew.

ω = angular velocity of airscrew.

V = resultant velocity of air at the blade.

W_a = axial inflow velocity at the blade.

W_t = tangential (rotational) inflow velocity at the blade.

W_a' = axial added velocity in the slip stream.

W_t' = tangential added velocity in the slip stream.

ρ = density of air.

T_c = thrust coefficient.

第十四號

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抄錄

「プロペラ」の理論

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プラントルの翼の理論を基として「プロペラ」の空氣に對する作用を考へて見た.

その結果は第一部航空機用「プロペラ」に於ては

（イ）「インフロー」速度は「スリップストリーム」の速度の増しの半分ではないが之に極めて近いものである.

（ロ）「インフロー」のうち軸に向ふものは「プロペラ」の働きに大きい影響があるが圓周に沿ふものは大した影響がない.

（ハ）「ブレード」の表面の空氣の摩擦等によつて「トルク」増し推力へるが「トルク」の增加は推力の減少に比してその割合が大きい.

（ニ）併しその量は多くも全體の數「パーセント」にすぎぬから「ブレード」の表面を骨折つて滑かにしても效果は少ない.

（ホ）それよりも效率は「ブレード」に沿ふての「サーキユレーション」の分配に左右されることが大きい.

（ヘ）現在の最もよい「プロペラ」の效率は達し得べき極限に達したものと思はれる.

（ト）「プロペラ」の最大效率は主として推力係數.

$$\left(T_C=\frac{T}{\frac{1}{2}\rho\pi R^2 v^2}\right) \text{及} \ z=\frac{\omega R}{v} \ \text{の函數である.}$$

（チ）「ブレード」の各半徑に於ける入射角には外見的と實際的の二通りある．之は「インフロー」速度の爲起ることで從つて揚げ係數にも二通りあることゝなる．その關係は本文第四節に掲げた通りである.

（リ）二つの「プロペラ」をタンデムに置いたときは後方の「プロペラ」は單獨の場合に比して最大效率が落ちる．そのおちる量は「プロペラ」の推力係數及 $z=\frac{\omega R}{v}$ の値による。

前後兩「プロペラ」が同じ方向に回轉しておるときは反對

方向に回轉しておるときより効率の落ちが大きい.

(ヌ) 反對方向に回轉しておる同軸「プロペラ」及「コントラプロペラ」を備へた「プロペラ」の最大効率は何れもよくなる.「コントラプロペラ」は船舶用としては可なり有効であるが航空機用としては効果はない.

(ル) 以上は「ブレード」の數が多い場合の話であるがその數の少ない「プロペラ」は外徑が幾分小さくなつた「ブレード」の數の無限に多いものと同じと見られ上に書いた結果には變りを來さぬ.

第二部「ヘリコプタ」用「プロペラ」に於ては

(イ) から (ニ) までは第一部のものと同じ結論に達する.

(ホ) 結果を簡單にする爲圓周に向ふ「インフロー」を省略すれば同じ推力に對して最小の「パワー」は「サーキユレーション」が半徑に沿ふて一定の「プロペラ」に依つて得られる.

(ヘ) 此の場合空氣の摩擦を省略すれば推力 T とパワー P. 半徑 R の間には次の關係式が成立する.

$$T=(2\pi\rho)^{\frac{1}{3}}\ (PR)^{\frac{2}{3}}$$

ρ は空氣の密度である.

(ト) 各「ブレード」の空氣に對する入射角は回轉數にかゝわらず一定の値を有する.

(チ) 横風の存在は(ヘリコプタが水平飛行における場合)「プロペラ」の同じ回轉數に對して推力及「トルク」を增加せしめるが前者の增加する率は後者のそれより小し大きい.

(リ) 「プロペラ」それ自身の横風に對する抵抗は極めて小さい. 故に「ヘリコプタ」を進行せしめるには軸を垂直から少し傾ければ充分であるといふ結論に達する. (終り)