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On the Lateral Vibration of a Rectangular Plate Clamped at four Edges.*

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Abstract.

The solution of the problem of the lateral vibration of a rectangular plate clamped at four edges is obtained in view of getting some clue regarding the stiffness of a machinery bed or of a hull. The solution is obtained to satisfy the differential equation

$$D_1\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=0,$$

by allowing very small residual slope at some portion of the boundary. In the above equation, w is the deflection of the plate, ρ the density, h the thickness of the plate and D_1 defined by

$$D_1=\frac{1}{12}\frac{Eh^3}{(1-\sigma^2)}.$$

For a square plate, a few modes of nodal vibrations are discussed; namely, the cases of no nodal line, of one nodal line parallel to an edge, of two nodal lines parallel to an edge and of two perpendicular nodal lines parallel to different edges. For a rectangular plate having different ratios of length to breadth, merely fundamental vibration is considered. The author treats further of the problem of the vibration of a rectangular plate, in which a pair of opposite edges is supported and the other is clamped. These problems may be solved without difficulty by Ritz's

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or other methods based upon the principles of energy. But, if one start from the differential equation of a plane plate, such cases are hardly be dealt with without the present method of analysis.

Although the problem of the lateral vibration of a rectangular plate has no much meaning in regard to musical instruments, they are rather important in connection with the stiffness of a machinary bed or of a hull. Moreover, the case of clamped edges is of some interest in the sense that such problem has not yet been solved from the differential equations of a plane plate. Ritz(1) treated of a plate having free periphery, yet his solution does not satisfy the differential equation of the vibration of a plate and is merely based upon the consideration of the kinetic and strain energies corresponding to an assumed deformation and also upon the application of Hamilton's principle. In my present analysis of a clamped plate, the solution has been obtained to satisfy the differential equation by allowing very small residual slope at some portion of the boundary. A similar treatment will enable us to solve the problem of a plate having free periphery.

Let x and y be the coordinate axes passing through the centre of the plate and parallel to the edges of the same plate, and w be the deflection, while D_1 is defined by

$$D_1=\frac{1}{12}\frac{Eh^3}{(1-\sigma^2)}, \quad \cdots\cdots\cdots \quad (1)$$

where E is Yonng's modulus, σ Poisson's ratio and h the thickness of the plate. The equation of the vibration is then denoted by

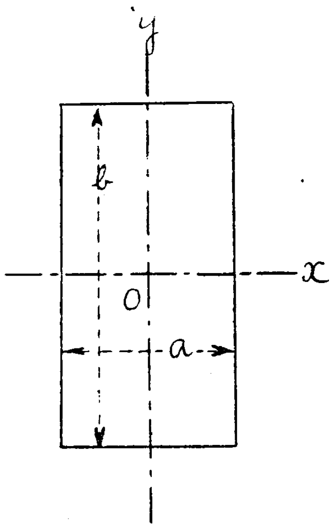

Fig. 1.

(1) W. Ritz, "Theorie der Transversalschwingungen einer quadratischen Platte mit freien Rändern," Ann. Phys., [4], 28 (1909), 737-786.

$$D_1\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=0\,, \quad\ldots\ldots\ldots \quad (2)$$

in which ρ is density of the plate.

When four edges of this plate are clamped, we have

$$x=\pm\frac{a}{2}\,; \qquad w=0\,, \qquad \frac{\partial w}{\partial x}=0\,, \quad\ldots\ldots\ldots\ldots \quad (3)$$

$$y=\pm\frac{b}{2}\,; \qquad w=0\,, \qquad \frac{\partial w}{\partial y}=0\,, \quad\ldots\ldots\ldots\ldots \quad (4)$$

where b and a are the length and breadth of the plate. Write in (2)

$$w=\left[W_1\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{m\pi x}{a}+W_2\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{n\pi y}{b}\right]\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}pt\,, \quad\ldots\ldots\ldots \quad (5)$$

in which $\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}pt$ is the normal coordinate, $2\pi/p$ being period of free vibrations, and the expression within the square bracket is the normal function. In the expression (5) W_1 and W_2 are functions of y and x respectively, m and $n=1, 3, 5, \ldots\ldots$ for $\cos\frac{m\pi x}{a}$ and $\cos\frac{n\pi y}{b}$, and m and $n=2, 4, 6, \ldots\ldots$ for $\sin\frac{m\pi x}{a}$ and $\sin\frac{y\pi y}{b}$.

The equation (2) then becomes

$$\left\{\left[\frac{d^4W_2}{dx^4}-2\left(\frac{n\pi}{b}\right)^2\frac{d^2W_2}{dx^2}+\left\{\left(\frac{n\pi}{b}\right)^4-\frac{\rho hp^2}{D_1}\right\}W_2\right]\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{n\pi y}{b}\right.$$

$$+\left[\frac{d^4W_1}{dy^4}-2\left(\frac{m\pi}{a}\right)^2\frac{d^2W_1}{dy^2}+\left\{\left(\frac{m\pi}{a}\right)^4\right.\right.$$

$$\left.\left.-\frac{\rho hp^2}{D_1}\right\}W_1\right]\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\left.\frac{m\pi x}{a}\right\}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}pt=0 \quad\ldots\ldots \quad (6)$$

The equation (6) may be satisfied by putting each of the expressions within two square brackets to zero. Solving two differential equations thus formed, we find the general solution of the type:

$$w=\left\{\left[A\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{\sqrt{\sqrt{\gamma^2+\beta^2}-\gamma}}{b}x+B\begin{matrix}\cosh\\ \sinh\end{matrix}\Big\}\frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{b}x\right]\begin{matrix}\cos\\ \sin\end{matrix}\Big\}\frac{n\pi y}{b}$$

$$+\left[C\begin{matrix}\cos\\ \sin\end{matrix}\Big\}\frac{\sqrt{\sqrt{\gamma^2+\delta^2}-\delta}}{a}y+D\begin{matrix}\cosh\\ \sinh\end{matrix}\Big\}\frac{\sqrt{\sqrt{\gamma^2+\delta^2}+\gamma}}{a}y\right]\begin{matrix}\cos\\ \sin\end{matrix}\Big\}\frac{m\pi x}{a}\Big\}\begin{matrix}\cos\\ \sin\end{matrix}\Big\}pt\,,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots \quad (7)$$

where

$$\alpha=(n\pi)^2,\ \gamma=(m\pi)^2,\ \beta^2=\frac{\rho hp^2}{D_1}b^4-(n\pi)^4,\ \delta^2=\frac{\rho hp^2}{D_1}a^4-(m\pi)^4. \quad (8)$$

As it is very difficult to make the solution satisfy all the boundary conditions, I have made the following approximation as in the case of buckling problem such that

$$x=\pm a/2\,,\qquad y=\pm b/2\,;\qquad w=0\,,\ \dots\dots\dots\dots \quad (9)$$

$$\left.\begin{array}{lll} x=\pm a/2\,, & y=0\,; & \dfrac{\partial w}{\partial x}=0\,, \\ y=\pm b/2\,, & x=0\,; & \dfrac{\partial w}{\partial y}=0\,. \end{array}\right\}\ \text{[or similar conditions.]}\quad (10)$$

From this assumption the solution of the problem becomes rigorous except very small residual slope at some portion of the boundary. Thus we have, when n is odd and m is even or odd,

$$A\begin{matrix}\cos\\ \sin\end{matrix}\Big\}\frac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{2b}a+B\begin{matrix}\cosh\\ \sinh\end{matrix}\Big\}\frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{2b}a=0\,,\ \dots \quad (11)$$

$$\mp\left\{ A\frac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{b}\begin{matrix}\sin\\ \cos\end{matrix}\right\}\frac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{2b}a$$

$$+B\frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{b}\begin{matrix}\sinh\\ \cosh\end{matrix}\Big\}\frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{2b}a\mp(C+D)\frac{m\pi}{a}\begin{matrix}\sin\\ \cos\end{matrix}\Big\}\frac{m\pi}{2}=0\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

$$C\cos\frac{\sqrt{\sqrt{\gamma^2+\delta^2}-\gamma}}{2a}b+D\cosh\frac{\sqrt{\sqrt{\gamma^2+\delta^2}+\gamma}}{2a}b=0\,,\ \ldots\ldots \quad (13)$$

$$\left.\begin{matrix}-(A+B)\\ -\left(A\sin\dfrac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{2mb}a+B\sinh\dfrac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{2mb}a\right)\end{matrix}\right\}\frac{n\pi}{b}\sin\frac{n\pi}{2}$$

$$=C\frac{\sqrt{\sqrt{\gamma^2+\delta^2}-\gamma}}{a}\sin\frac{\sqrt{\sqrt{\gamma^2+\delta^2}-\gamma}}{a}b$$

$$-D\frac{\sqrt{\sqrt{\gamma^2+\delta^2}+\gamma}}{a}\sinh\frac{\sqrt{\sqrt{\gamma^2+\delta^2}+\gamma}}{2a}b\,,\ \ldots \quad (14)$$

the upper and lower terms of the left-hand member of (14) and of the first and the third members of (12) being taken according as m is odd or even.

When both of m and n are even, we have

$$A\sin\frac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{2b}a+B\sinh\frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{2b}a=0\,,\ \ldots\ldots \quad (15)$$

$$A\frac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{b}\cos\frac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{2b}a+B\frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{b}\cosh\frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{2b}a$$

$$+\left(C\sin\frac{\sqrt{\sqrt{\gamma^2+\delta^2}-r}}{2na}b+D\sinh\frac{\sqrt{\sqrt{\gamma^2+\delta^2}+\gamma}}{2na}b\right)\frac{m\pi}{a}\cos\frac{m\pi}{2}=0\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

$$C \sin \frac{\sqrt{\sqrt{\gamma^2+\delta^2}-\gamma}}{2a} b + D \sinh \frac{\sqrt{\sqrt{\gamma^2+\delta^2}+\gamma}}{2a} b = 0 , \quad \ldots\ldots \quad (17)$$

$$\left(A \sin \frac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{2mb} a + B \sinh \frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{2mb} a\right) \frac{n\pi}{b} \cos \frac{n\pi}{2}$$

$$+ C \frac{\sqrt{\sqrt{\alpha^2+\delta^2}-\gamma}}{a} \cos \frac{\sqrt{\sqrt{\gamma^2+\delta^2}+\gamma}}{2a} b$$

$$+ D \frac{\sqrt{\sqrt{\alpha^2+\delta^2}+\gamma}}{a} \cosh \frac{\sqrt{\sqrt{\gamma^2+\delta^2}+\alpha}}{2a} b = 0 \quad \ldots\ldots \quad (18)$$

Solving (11), (12), (13), (14) or (15), (16), (17), (18) by a tentative method, we may find the period of vibration $2\pi/p$ and the relative values of A, B, C, D. A few examples for a square plate will be given below:

	m, n	$\frac{\rho h p^2 a^4}{D_1 \pi^4}$	B/A	D/C	C/A	D/B	Nodal lines
I	$m=1$ $n=1$	14·4	0·0554	0·0554	1	1	No nodal line.
II	$m=2$ $n=1$	57·5	0·0156	0·00932	0·320	0·191	One nodal line parallel to an edge.
III	$m=1$ $n=2$	57·5	0·00932	0·0156	0·191	0·320	,,
IV (II+III)		57·5					One diagonal nodal line.
V	$m=3$ $n=1$	185·0	−0·00379	0·00112	0·201	−0·0595	Two nodal lines parallel to an edge.
VI	$m=2$ $n=2$	120·0	0·00386	0·00386	1	1	Two perpendicular nodal lines parallel to different edges.

The general tendency of the modes of vibrations is shown in the appended figure.

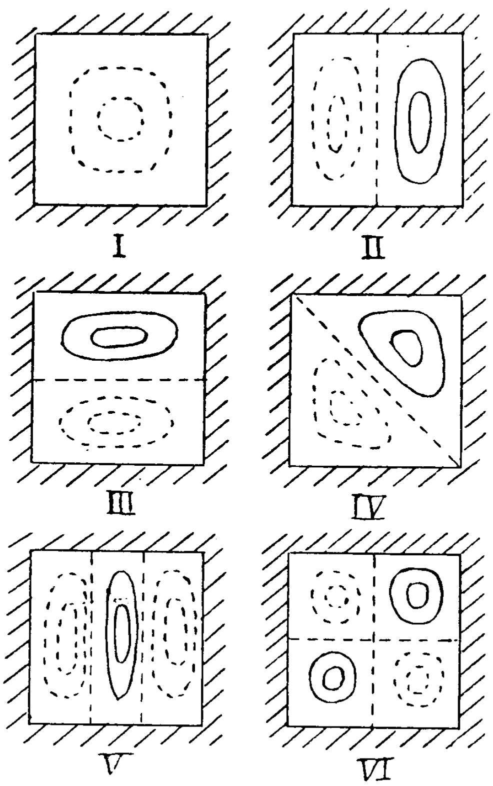

Fig. 2.

I have also calculated some cases of the fundamental vibration of a rectangular plate whose edges are not of equal length. Some of them are tabulated below:

b/a	$\dfrac{\rho h p^2 a^4}{D_1 \pi^4}$	B/A	D/C	C/A	D/B
1	14·4	0·0354	0·0554	1	1
3/2	8·13	0·0886	0·0194	0·350	0·0766
2	7·09	0·103	0·00302	0·253	0·00742
∞	5·07	0·133	—	0	0

The validity of these results may easily be checked by any experiment which is not so difficult. Several years ago Mr. K. Sudo and I[1] had the occasion of carrying out an experimental study of the vibration of a square plate clamped at four edges. Chief data of that experiment are as follows:

Vibration medium: Mixture of gelatine and glycerine in the ratio of 80 : 400.

Clamping edges: A certain wooden frame.

Effective length of each edge: 11 cm.

Thickness of vibration medium: 1·28 cm.

Density: 1·484÷1·28=1·16.

Young's modulus: 88·6 gr. / cm.^{2}

Poisson's ratio: 0·38.

The observed period of the fundamental vibration was 0·20 sec. From my present theory the period is

$$2\pi / p=\frac{2a^2}{\pi}\sqrt{\frac{\rho h}{14{\cdot}4 D_1}}=\frac{2a^2}{\pi}\sqrt{\frac{12\rho(1-\sigma^2)}{14{\cdot}4Eh^2}}\doteqdot 0.19 \text{ sec.}$$

Although the above experiment was not so accurate, yet we find that there is a certain agreement between theory and experiment.

Lastly, we shall add a theory of vibration of a rectangular plate clamped at a pair of opposite edges and supported at the other. This problem can be solved without any approximation.

In the equation of motion of a plate such that

$$D_1\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=0, \quad \ldots\ldots\ldots \quad (2')$$

(1) K. Sezawa, "Free Vibration of a Clamped Square Plate," (in Japanese) Jour. Aeron. Res. Inst., No. 6 (1924), 39-42.

we put

$$w = W \begin{matrix}\cos \\ \sin\end{matrix}\Big\} \frac{n\pi y}{b} \begin{matrix}\cos \\ \sin\end{matrix}\Big\} pt, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

in which b is the length of the plate between supported edges. Then, from (2′), we have

$$\frac{d^4W}{dx^4} - 2\left(\frac{n\pi}{b}\right)^2 \frac{d^2W}{dx^2} + \left\{\left(\frac{n\pi}{b}\right)^4 - \frac{\rho h p^2}{D_1}\right\} W = 0 \quad \ldots\ldots \quad (20)$$

Solving this differential equation, we arrive at finally

$$w = \left[A \begin{matrix}\cos \\ \sin\end{matrix}\Big\} \frac{\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}}{b} x + B \begin{matrix}\cosh \\ \sinh\end{matrix}\Big\} \frac{\sqrt{\sqrt{\alpha^2+\beta^2}+\alpha}}{b} x\right] \begin{matrix}\cos \\ \sin\end{matrix}\Big\} \frac{n\pi y}{b} \begin{matrix}\cos \\ \sin\end{matrix}\Big\} pt.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Some calculation of the fundamental vibration of a rectangular plate clamped at a pair of opposite edges and supported at the other has been made by means of the expression (21), the result being shown below:

b/a	1	3/2	2	∞
$\frac{\rho h p^2 a^4}{D_1 \pi^4}$	8·66	6·40	5·81	5·07

The fundamental vibration of a rectangular plate supported at four edges is well known and may be tabulated as follows:

b/a	1	3/2	2	∞
$\frac{\rho h p^2 a^4}{D_1 \pi^4}$	4·00	2·10	1·56	1·00

The fundamental vibrations of three cases of different edge conditions are plotted in Fig. 3.

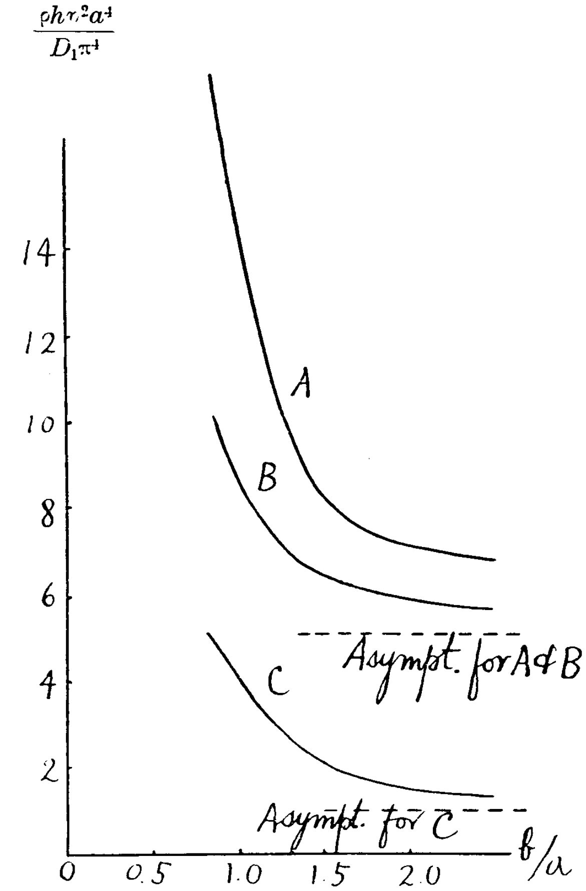

Fig. 3. A: four edges clamped. C: four edges supported. B: $\begin{cases} \text{two edges } x = \pm \frac{a}{2} \text{ clamped.} \\ \text{two edges } y = \pm \frac{b}{2} \text{ supported.} \end{cases}$

Many interesting features will be seen from this figure; yet I have not paid any attention on those features nor given any interpretation of them for the sake of simplicity.

In conclusion, my thanks are due to Mr. K. Kubo who has assisted me in preparing this paper.

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