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抄　錄

球の境界層に於ける流れの層狀より渦亂狀への遷移に就いて

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流れの中に置かれた物體の表面に於ける境界層中の流れが，表面上の如何なる場所で層狀から渦亂狀に遷移するか，又その遷移點が流れの REYNOLDS 數と共に如何に移動するかといふ問題は物體の抵抗その他の問題に關聯して極めて重要であるが，圓柱及び球の如き比較的簡單な形狀の物體に於いても未だ判然たることは知れてゐない樣である．

球面上の境界層に就いての詳細な實驗的研究は最近英國 N. P. L. の FAGE 氏によつて遂行され，その結果は興味ある一論文として發表されてゐるが，FAGE 氏は球面上の垂直壓力の分布曲線に於ける一つの彎曲點と表面摩擦の强度の分布曲線に於ける極小とが互に對應することに着目し，境界層に於ける流れの層狀から渦亂狀への遷移が始まる點は，壓力分布曲線が一つの彎曲點を示し且つ表面摩擦の强度が極小値を採る樣な球面上の場所であるとの結論に到達した．果してこの樣な FAGE 氏の結論が妥當であるか否かを吟味檢討することは，問題の重要性に鑑み，極めて重要で且つ興味ある問題であると考へられる．

本論文に於いて，著者等は，FAGE 氏とは全然異なる立場から球の境界層の流れに於ける層狀より渦亂狀への遷移に關しての研究を試みてゐる．即ち，球及び飛行船の胴體の如き廻轉體の表面に於ける境界層に對する一般の運動量積分方程式を球の場合に適用し，境界層の流れは層狀であると假定して，POHLHAUSEN が二次元の流れの場合に行つた樣に，速度分布に對しては物體表面からの距離に就いての四次式を假定した．而して，境界層の外部に於ける流れの速度分布に對しては，FAGE 氏が實測した球面上の垂直壓力の分布から容易に計算して得られる實際の速度分布のうち代表的な三つを選んで使つた．即ち，流れの REYNOLDS 數が所謂臨界領域に丁度含まれる程度の大いさの場合，その中央にある場合，及びその領域を丁度超える位の大いさの場合の三つの場合に於ける FAGE 氏の實驗結果を採用して計算を遂行し，境界層の厚さ，境界層の REYNOLDS 數，表面摩擦の强度等重要な諸種の量を求め，それ等を詳細に圖示した．吾々の得た結果を要約すれば次の樣である．

流れの REYNOLDS 數が小さくて臨界領域に丁度含まれる場合には，層流と假定して計算した表面摩擦の强度の分布は，境界層の剝離點に至る迄實測の結果とよく一致する．且つ，剝離點に對する計算値と實測値との一致も亦極めて良好である．

このことは，吾々が本論文で採用した運動量積分方程式及び速度の四次式が層狀境界層に對しては極めて妥當に使はれること，從つてそれから得られる計算結果も亦實測結果によく合ふ信據すべきものであることを示すものと信ずる.

故に，流れの REYNOLDS 數が大きい場合に，若しも層狀境界層と假定しての計算結果と實測結果との間に何等かの相違 認められれば，その相違は境界層中の流れが假定の樣に層狀になつてゐないで實は多少渦亂狀であることを示すものと考へて差支へないと思はれる. 換言すれば，REYNOLDS 數が小さい場合に實測と良く合ふ結果を與へる運動量積分方程式及び速度の四次式の假定が惡いのではなくて，境界層中の流れが層狀であると假定したことが REYNOLDS 數が大きい場合には事實と合はない結果を與へると考へて差支へない.

實際，流れの REYNOLDS 數が臨界領域の中央にある場合には，垂直壓力の分布曲線が極小になる樣な場所から，表面摩擦の强度の實測値は層流と假定して計算した强度の分布曲線から外れ始め，その外れは後方に進むに從つて益々大きくなつてゐる樣に見える. 從つて，この場合には，極小壓力の點の近傍に於いて境界層中の流れに或種の擾亂が生じ，それが段々成長するために，この點から後では境界層の流れが純粹の層狀でなく幾分渦亂狀になるものと考へられる. 壓力の極小點の近傍に於いて起る擾亂は恐らく流れの中に含まれてゐる不規則に振動する小さい壓力勾配によつて起る境界層の局部的剝離に因るものと考へられるが，斯樣な擾亂は流れの REYNOLDS 數が小さい場合に於いても極小壓力の點の近傍に於いて起つてゐる樣に認められるけれども，REYNOLDS 數が小さい場合には擾亂はやがて老衰して流れは再び元の純粹な層狀に復歸して遂に層流としての剝離が起るのである.

流れの REYNOLDS 數が更に大きくなつて臨界領域を超えると，極小壓力の點のかなり前方の點で表面摩擦の强度の實測値はそれの理論的分布曲線から外れ始め，REYNOLDS 數が大きいために擾亂は益々成長して實測値と計算値との差違は段々大きくなり，境界層の流れは遂に完全な渦亂狀になることが認められる.

此等の結果を綜合して，吾々は，流れの REYNOLDS 數が大きくて臨界領域の中央にあるか更にそれを超える場合には，垂直壓力の極小點の近傍乃至それの多少前方に於いて，境界層に於ける流れの層狀から渦亂狀への遷移が始まるといふ結論に到達した. ところで，若しも吾々の考察が正しいものならば，斯樣な遷移の始まる點に於ける境界層の REYNOLDS 數に對する所謂臨界値は二つの場合に就いて殆んど等しい値を採る筈である. 吾々の計算によると，境界層の REYNOLDS 數に對する臨界値は流れの REYNOLDS 數が臨界領域の中央にある時に凡そ 1450 で，臨界領域を丁度超えてゐる時に凡そ 1440 であつて，二つの値は豫想通り良く一致する. これは層流より渦亂流への遷移に關する吾々の考へ方の妥當なことを示すものと考へられ，球面上の境界層に於いては層流より渦亂流への遷移は境界層の REYNOLDS 數の値が凡そ 1450 になる點から始まると結論される. 流れの REYNOLDS 數が丁度臨界領域に含まれる場合には境界層の REYNOLDS 數は 1400 を超えない. 從つてこの場合には境界層中の流れは剝離點に至る迄到る處層狀であることを知るが，この結果は實測結果と一致するものである.

若しも FAGE 氏の云ふ樣に，壓力分布曲線に於ける一つの彎曲點が層流より渦亂流への遷移の始まる點に對應するとすれば，吾々の取扱つたすべての場合に於いて境界層は層狀のままで剝離する筈であつて，且つその剝離點はその樣な彎曲點よりも前方にあるから，境界層に於ける流れの層狀から渦亂狀への遷移は考へられないことになる. このことは FAGE 氏の結論の妥當でないことを示すものであると思ふ.

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On the Transition from Laminar to Turbulent Flow in the Boundary Layer of a Sphere.

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I. Introduction.

§ 1. In a previous paper[1], one of the present writers made a preliminary study on the laminar boundary layer on the surface of a sphere placed in a uniform stream of an incompressible fluid. Employing the general momentum integral equation for the boundary layer on a body of revolution and assuming a quartic form for the velocity profile in the boundary layer, as done by POHLHAUSEN in the case of two-dimensional stream, the differential equation for determining the thickness of the boundary layer was solved and thus various characteristic quantities for the boundary layer were discussed. For the velocity distribution outside the boundary layer, use was made of the well-known theoretical distribution as well as a distribution found

(1) S. TOMOTIKA, The Laminar Boundary Layer on the Surface of a Sphere in a Uniform Stream. British Aeronautical Research Committee, R. and M., No. 1678 (1935). In what follows this paper will be referred to as paper I.

experimentally by O. FLACHSBART when the REYNOLDS number of the stream was below the critical REYNOLDS number of a sphere. However, no further theoretical investigations on the subject were tried in the previous paper, because detailed experimental evidence concerning the boundary layer of a sphere was not available at that time which could be compared with theoretical results.

Recently, A. FAGE[1] has made some detailed experimental researches on the boundary layer of a sphere, at REYNOLDS numbers below, within, and above the critical range over which the drag coefficient of a sphere experiences a large fall. The principal object of his experiments was to determine the influence of REYNOLDS number and turbulence in the free stream on the transition from laminar to turbulent flow in the boundary layer of a sphere, for REYNOLDS numbers within the critical range. In order to obtain the information on the matter, FAGE has measured distributions of normal pressure and surface friction over the surface of a sphere.

In view of the importance of the problem, the present writers have recently performed various calculations similar to those in the previous paper for the boundary layer of a sphere, by employing actual distributions of velocity over the surface of a sphere which have been obtained from the distributions of normal pressure determined experimentally by FAGE, and the results of the calculations have been compared with FAGE's observations.

A part of the results of our investigations has already been given briefly in a recent paper[2]. That paper has dealt with a case when the REYNOLDS number of the stream is just within the critical range so that, according to FAGE's observations, the flow in the boundary layer

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(1) A. FAGE, Experiments on a Sphere at Critical REYNOLDS Numbers. British Aeronautical Research Committee, R. and M., No. 1766 (1936).

(2) S. TOMOTIKA and I. IMAI, The Distribution of Laminar Skin Friction on a Sphere placed in a Uniform Stream. Proc. Phys.-Math. Soc. Japan, [3], 20 (1938), 288–303. This paper will be referred to as paper II in the present paper.

is everywhere laminar up to the point of separation of the layer from the surface of the sphere.

It has been found that the calculated point of separation is in good accordance with the observed one and that the agreement between the calculated distribution of the intensity of laminar skin friction and the observation is also quite satisfactory.

Thus, we have been led to the conclusion that the momentum integral equation for the laminar boundary layer on a body of revolution, together with the assumed quartic form for the velocity distribution in the layer, can be used, with sufficiently good approximation, to describe the laminar boundary layer on the surface of a sphere.

§ 2. On observing the calculated distribution curve and the corresponding observed points for the intensity of laminar skin friction on the surface of the sphere, it has been found that there are small systematic deviations of the experimental points from the theoretical curve near the point of minimum pressure and that beyond the point of minimum pressure the observed points fall again on the calculated curve. These deviations seem to indicate that some kinds of accidental disturbance had occurred near the pressure minimum and thus the flow in the boundary layer, becoming somewhat irregular, had departed from the purely laminar state. The fact that the observed points fall again on the calculated curve beyond the point of minimum pressure shows however that such disturbances had been unable to grow up so as to make the flow turbulent, but, on the contrary, they had soon decayed down and the motion had become again purely laminar, since the Reynolds number was not too large in the case discussed.

This reminds us of the well-known similar phenomenon in the case of flow through a pipe of circular section in the Reynolds experiments. It is well known that when the Reynolds number is below a certain critical value, initial disturbances, if any, do not grow up so as to make the flow turbulent, but, on the contrary, they are soon obliterated, and the flow becomes purely laminar.

On the other hand, it is also well established in the case of the REYNOLDS experiments that if the REYNOLDS number is greater than the critical value, the flow becomes sensitive to initial or accidental small disturbances, and the motion becomes ultimately fully turbulent.

Therefore, we may conjecture that a similar phenomenon may perhaps occur in the case of the flow in the boundary layer of a sphere. Thus, it may be expected that when the REYNOLDS number of the stream takes larger values, accidental disturbances, which are likely to be originated near the point of minimum pressure, would grow up more and more, and the flow in the boundary layer would ultimately become fully turbulent.

In order to investigate this point, the present writers have carried out various calculations similar to those in the former case for two more cases of larger REYNOLDS numbers, using, as before, the actual pressure distributions measured by FAGE, and it has been ascertained that the expectation above mentioned is not quite erroneous.

The object of the present paper is to describe all the results of our calculations. For reference the results of the previous paper II also are described in more detail.

II. The Momentum Integral Equation for the Boundary Layer on a Body of Revolution.

§ 3. We shall begin with the general momentum integral equation for the boundary layer on a body of revolution. We assume that a body of revolution is placed in a uniform stream of an incompressible fluid such that its axis of revolution is parallel to the direction of the undisturbed stream.

Let x be the length of the generator of the body of revolution measured from the forward stagnation point, which coincides, in the present case, with the point where the axis of revolution cuts the surface of the body, and let y be the distance of a point in the layer from the surface of the body measured along the normal to the surface. We denote the radius of the transverse cross-section of the body of revolution by r, which is a known function of x.

Further, let δ be the thickness of the boundary layer, and u be the velocity in the x-direction inside the layer; while the velocity and pressure just outside the boundary layer will be denoted by U and p respectively. Also, we denote by ρ the density of the incompressible fluid concerned, and by τ_0 the intensity of skin friction on the surface of the body.

Then, if we assume that y is very small in comparison with the longitudinal radius of curvature of the body and that the thickness of the boundary layer δ also is very small compared with r, we get, with the aid of the theorem of momentum, the momentum integral equation for the boundary layer on the body of revolution in the form:

$$\frac{1}{r}\frac{d}{dx}\left\{r\int_0^\delta u^2 dy\right\} - U\frac{1}{r}\frac{d}{dx}\left\{r\int_0^\delta u dy\right\} = -\delta\frac{1}{\rho}\frac{dp}{dx} - \frac{\tau_0}{\rho}. \quad (1)$$

If we use the relation:

$$\frac{1}{\rho}\frac{dp}{dx} = -U\frac{dU}{dx},$$

which follows immediately from BERNOULLI's theorem, this equation can be written as:

$$\frac{1}{r}\frac{d}{dx}\left\{r\int_0^\delta u^2 dy\right\} - U\frac{1}{r}\frac{d}{dx}\left\{r\int_0^\delta u dy\right\} = \delta\cdot U\frac{dU}{dx} - \frac{\tau_0}{\rho}. \quad (2)$$

This momentum integral equation is applicable to both the laminar and turbulent boundary layers on a body of revolution. In the case of turbulent boundary layer, however, we must take for u, U and dU/dx their respective mean values.

Further, it will be noticed that at least for a blunt-nosed body of revolution the above integral equation (2) may legitimately be used, as was proved by CLARK B. MILLIKAN[1], to describe the laminar or turbulent boundary layer, even in the neighbourhood of the forward stagnation point at the nose, where $r \to 0$ and therefore the condition that δ is very small compared with r is not necessarily satisfied.

§4. In the case of laminar boundary layer, τ_0 is the intensity of laminar skin friction and is given by

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \qquad (3)$$

where μ is the coefficient of viscosity of the fluid concerned.

Thus, the momentum integral equation for the laminar boundary layer on a body of revolution is

$$\frac{1}{r}\frac{d}{dx}\left\{r\int_0^\delta u^2 dy\right\} - U\frac{1}{r}\frac{d}{dx}\left\{r\int_0^\delta u dy\right\} = \delta \cdot U\frac{dU}{dx} - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \qquad (4)$$

where $\nu = \mu/\rho$ is the kinematic coefficient of viscosity of the fluid.

It may be remarked here that this momentum integral equation for the laminar boundary layer on a body of revolution can also be derived, as done by MILLIKAN in his paper cited above, from the well-known PRANDTL boundary layer equations for a body of revolution, by integrating them once.

III. The Differential Equation for Determining the Thickness of the Laminar Boundary Layer.

§5. In the present paper we perform various calculations under the assumptions that the flow in the boundary layer is everywhere laminar up to the point of separation, and therefore, the integral equation of the form (4) only is considered.

Now, equation (4) can be put in the form:

(1) C. B. MILLIKAN, The Boundary Layer and Skin Friction for a Figure of Revolution. Trans. Amer. Soc. Mech. Eng., Appl. Mech. Sec., 54 (1932), 29.

$$\frac{d}{dx}\int_0^\delta u^2dy - U\frac{d}{dx}\int_0^\delta udy + \frac{1}{r}\frac{dr}{dx}\left[\int_0^\delta u^2dy - U\int_0^\delta udy\right]$$
$$= \delta\cdot U\frac{dU}{dx} - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}. \qquad (5)$$

To solve this equation approximately we assume for u a quartic form in y, as assumed by POHLHAUSEN in the case of two-dimensional laminar boundary layer, namely:

$$u = a_1y + a_2y^2 + a_3y^3 + a_4y^4 , \qquad (6)$$

where a_1, a_2, a_3 and a_4 are functions of x.

The boundary conditions are

$$\left.\begin{aligned} &u = 0, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{1}{\nu}U\frac{dU}{dx}, \quad \text{at} \quad y = 0; \\ &u = U, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{at} \quad y = \delta. \end{aligned}\right\} \qquad (7)$$

The first condition that $u = 0$ at $y = 0$ is satisfied by the assumed expression for u and the remaining four conditions determine a_1, a_2, a_3, a_4. We thus have

$$\left.\begin{aligned} a_1 &= \frac{U}{6\delta}(12+\lambda), \\ a_2 &= -\frac{U\lambda}{2\delta^2}, \\ a_3 &= -\frac{U}{2\delta^3}(4-\lambda), \\ a_4 &= \frac{U}{6\delta^4}(6-\lambda), \end{aligned}\right\} \qquad (8)$$

where λ stands for the non-dimensional quantity $\frac{\delta^2}{\nu}\frac{dU}{dx} = \frac{\delta^2 U'}{\nu}$, namely:

$$\lambda = \frac{\delta^2 U'}{\nu}. \qquad (9)$$

In this as well as in subsequent equations, dashes denote differentiation with respect to x.

The separation of the boundary layer from the surface of the body occurs at a point where $\partial u/\partial y$ becomes zero. Therefore, it will be seen from (6) and (8) that the point of separation is determined by

$$\lambda = -12 , \tag{10}$$

as in the two-dimensional case.

Making use of (8) we easily find that

$$\left.\begin{aligned} \int_0^\delta u dy &= \frac{U\delta}{120}(84+\lambda) , \\ \int_0^\delta u^2 dy &= \frac{U^2\delta}{1260}\left(734+\frac{71}{6}\lambda+\frac{5}{36}\lambda^2\right) , \end{aligned}\right\} \tag{11}$$

and the substitution of these values in the momentum integral equation (5) gives a differential equation of the form:

$$\frac{dz_*}{dx} = \frac{1}{U}f(\lambda)-\left(\frac{1}{r}\frac{dr}{dx}\frac{U}{U'}\right)\frac{1}{U}f^*(\lambda)+z_*^2 U''g(\lambda) , \tag{12}$$

where

$$z_* = \frac{\delta^2}{\nu} , \tag{13}$$

and

$$f(\lambda) = \frac{4-\frac{232}{315}\lambda+\frac{79}{3780}\lambda^2+\frac{1}{2268}\lambda^3}{\frac{37}{315}-\frac{1}{315}\lambda-\frac{5}{9072}\lambda^2}$$

$$= \frac{7257\cdot6-1336\cdot32\,\lambda+37\cdot92\,\lambda^2+0\cdot8\,\lambda^3}{213\cdot12-5\cdot76\,\lambda-\lambda^2} ; \tag{14a}$$

$$f^*(\lambda) = \frac{\dfrac{74}{315}\lambda - \dfrac{2}{945}\lambda^2 - \dfrac{1}{4536}\lambda^3}{\dfrac{37}{315} - \dfrac{1}{315}\lambda - \dfrac{5}{9072}\lambda^2}$$

$$= \frac{426 \cdot 24\lambda - 3 \cdot 84\lambda^2 - 0 \cdot 4\lambda^3}{213 \cdot 12 - 5 \cdot 76\lambda - \lambda^2}\,; \qquad (14b)$$

$$g(\lambda) = \frac{\dfrac{2}{945} + \dfrac{1}{2268}\lambda}{\dfrac{37}{315} - \dfrac{1}{315}\lambda - \dfrac{5}{9072}\lambda^2}$$

$$= \frac{3 \cdot 84 + 0 \cdot 8\lambda}{213 \cdot 12 - 5 \cdot 76\lambda - \lambda^2}\,. \qquad (14c)$$

By solving the differential equation (12) the thickness δ of the laminar boundary layer on a body of revolution can be determined.

§6. At the forward stagnation point, i.e., at the origin $x = 0$, the velocity U vanishes and therefore it will be seen from equation (12) that unless the following quantity:

$$f(\lambda) - \lim_{x \to 0}\left(\frac{1}{r}\frac{dr}{dx}\frac{U}{U'}\right)f^*(\lambda)$$

becomes also zero at that point, no integral of (12) exists having a finite value at the origin. Thus, when $x = 0$

$$f(\lambda_0) - \lim_{x \to 0}\left(\frac{1}{r}\frac{dr}{dx}\frac{U}{U'}\right)f^*(\lambda_0) = 0\,, \qquad (15)$$

where λ_0 is the value of λ at the origin and is determined by this equation.

Now, the practically important bodies of revolution such as spheres and airship-shaped bodies have a blunt nose and in those cases the neighbourhood of the forward stagnation point at the nose can be generally approximated by a portion of the surface of a sphere with the longitudinal radius of curvature at the nose (R_0, say) as its radius. Thus, in the vicinity of the forward stagnation point we may write approximately

$$x = R_0\varphi\ , \quad r = R_0\varphi\ , \quad U = c\varphi\ ,$$

c being a constant, so that

$$\lim_{x\to 0}\left(\frac{1}{r}\frac{dr}{dx}\frac{U}{U'}\right) = 1\ .$$

Thus, remembering that the fundamental momentum integral equation adopted in the present paper can legitimately be used, as mentioned already, to describe the boundary layer on a body of revolution even in the vicinity of the forward stagnation point at the nose of the body when it has a blunt nose, if we confine ourselves to such a practically important blunt-nosed body of revolution, the equation for determining λ_0, i.e., the value of $\lambda = U'z_*$ at the origin, becomes

$$f(\lambda_0) - f^*(\lambda_0) = 0\ ,$$

or

$$7257{\cdot}6 - 1762{\cdot}56\,\lambda_0 + 41{\cdot}76\,\lambda_0^2 + 1{\cdot}2\,\lambda_0^3 = 0\ . \qquad (16)$$

It has been proved in the previous paper I that this cubic equation has one negative root and two positive roots, 4·71601 and 21·14. Of these three roots, the negative one must be rejected owing to the fact that at the forward stagnation point $U' > 0$ so that λ_0 must necessarily be positive, while the greater one of the positive roots, i.e., 21·14, also cannot be adopted[(1)]. Hence, the appropriate root of (16) is $\lambda_0 = 4{\cdot}71601$

(1) As done by HOWARTH in the case of two-dimensional flow, this can be proved as follows. We assume that $\lambda_0 = 21{\cdot}14$ at the forward stagnation point. Then, the value of λ decreases continuously from 21·14 to −12, which corresponds with the point of separation. In this case, the common denominator: $213{\cdot}12 - 5{\cdot}76\lambda - \lambda^2 \equiv (12-\lambda)(\lambda + 17{\cdot}76)$ of the three functions $f(\lambda)$, $f^*(\lambda)$ and $g(\lambda)$ vanishes at some point between the forward stagnation point and the point of separation, and the corresponding value of dz_*/dx is infinite. The infinite value of dz_*/dx implies, however, an infinite value for δ, indicating the breakdown of the present approximate method. Thus, the root $\lambda_0 = 21{\cdot}14$ cannot be adopted.

Cf., L. HOWARTH, On the Calculation of Steady Flow in the Boundary Layer near the Surface of a Cylinder in a Stream. British Aeronautical Research Committee, R. and M., No. 1632 (1934).

and the required integral of the differential equation (12) is the one defined by

$$\lambda = U'z_* = 4{\cdot}71601\ , \quad \text{at} \quad x = 0\ . \tag{17}$$

Some of the values of the functions $f(\lambda)$, $f^*(\lambda)$ and $g(\lambda)$ for values of λ ranging from 5 to -12 have been tabulated in Table I in the previous paper I above referred to and use has been made of them in the present paper in performing the graphical integration of the differential equation (12).

Next, we determine the value of dz_*/dx at $x = 0$, i.e., $(dz_*/dx)_0$. From (12) we have

$$\left(\frac{dz_*}{dx}\right)_0 = \lim_{x\to 0}\left[\frac{f(\lambda)-\left(\frac{1}{r}\frac{dr}{dx}\frac{U}{U'}\right)f^*(\lambda)}{U}\right] + (z_*^2 U'')_0 g(\lambda_0)\ . \tag{18}$$

But, in the case of a blunt-nosed body of revolution,

$$\lim_{x\to 0}\left[\frac{f(\lambda)-\left(\frac{1}{r}\frac{dr}{dx}\frac{U}{U'}\right)f^*(\lambda)}{U}\right] = \left[\frac{\frac{d(f-f^*)}{d\lambda}\lambda' + \frac{1}{2}\frac{U''}{U'}f^*}{U'}\right]_{x=0}$$

$$= \left[\frac{d(f-f^*)}{d\lambda}\left\{\frac{U''}{U'}z_* + \frac{dz_*}{dx}\right\} + \frac{1}{2}\frac{U''}{U'^2}f^*\right]_{x=0}.$$

Inserting this in (18) and taking (17) into account, the value of $(dz_*/dx)_0$ can be obtained by simple algebraic calculations. We thus find

$$\left(\frac{dz_*}{dx}\right)_0 = -3{\cdot}420\left(\frac{U''}{U'^2}\right)_0. \tag{19}$$

With (17) and (19) as the initial conditions, the graphical solution of (12) can be conveniently carried out by the method described in the previous paper I.

§ 7. In the particular case of a sphere, with which we are specially concerned in the present paper, if we introduce the central angle θ, we have

$$x = a\theta\ , \qquad r = a \sin\theta\ ,$$

a being the radius of the sphere.

Then, if we write

$$z = \frac{U_0}{a} z_* = \frac{U_0 \delta^2}{a\nu}\ , \tag{20}$$

where U_0 is the velocity of the undisturbed stream, z is the non-dimensional function of θ only and we have, from (12), the differential equation for z in the form:

$$\frac{dz}{d\theta} = \frac{U_0}{U} f(\lambda) - \frac{\cos\theta}{\sin\theta} \frac{U_0}{\frac{dU}{d\theta}} f^*(\lambda) + z^2 \frac{1}{U_0} \frac{d^2U}{d\theta^2} g(\lambda)\ . \tag{21}$$

One of the initial conditions is, by (17) and (20),

$$\lambda = \frac{1}{U_0} \frac{dU}{d\theta} z = 4{\cdot}71601\ , \quad \text{at} \quad \theta = 0\ . \tag{22}$$

In the vicinity of the forward stagnation point of a sphere, the velocity distribution outside the boundary layer is represented, as proved experimentally(1), by the well-known theoretical formula $U = \frac{3}{2} U_0 \sin\theta$, irrespective of the values of the REYNOLDS number of the stream. Thus, we can always put

$$\left(\frac{dU}{d\theta}\right)_{\theta=0} = \frac{3}{2} U_0\ ,$$

and therefore the above condition can also be written in the form:

$$z = \frac{2}{3} \times 4{\cdot}71601 = 3{\cdot}14401\ , \quad \text{at} \quad \theta = 0\ . \tag{23}$$

The other condition is

$$\frac{dz}{d\theta} = 0\ , \quad \text{at} \quad \theta = 0\ , \tag{24}$$

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(1) See, e.g., FAGE's paper, loc. cit.

which follows immediately from (19), since in the case of a sphere the value of U'' is equal to zero at the forward stagnation point.

Our next problem is therefore to integrate the above differential equation (21) subject to these two initial conditions (23) and (24), and for this purpose we have to know the expression for U, namely the velocity distribution over the surface of a sphere.

In the previous paper I, use was made of the theoretical velocity distribution $U = \frac{3}{2} U_0 \sin\theta$ as well as an experimentally determined distribution, and some characteristic quantities for the laminar boundary layer on a sphere were calculated in both cases. However, no comparison of the calculated results with observations were made there, because at that time detailed experimental evidence concerning the boundary layer of the sphere was not available.

Recently A. FAGE[1] has carried out detailed measurements on the distributions of normal pressure and skin friction over the surface of a sphere, for a wide range of REYNOLDS number which included the so-called critical range, and thus he has discussed the influence of REYNOLDS number and turbulence in the free stream on the transition from laminar to turbulent flow in the boundary layer of the sphere.

In view of the importance of the problem, therefore, similar calculations to those in the previous paper I have been repeated, using this time actual velocity distributions on a sphere which have been obtained from the distributions of normal pressure determined experimentally by FAGE. The results of the calculations will be described in the following lines.

A brief account of the results for a case when the REYNOLDS number of the stream was just within the critical range so that the flow in the boundary layer was, according to FAGE's observation, everywhere laminar up to the point of separation has already been given in the recent paper

(1) A. FAGE, loc. cit.

II. For the sake of comparison with the results for other cases, however, the previous results also will be given in the first place.

IV. Calculations using Experimentally Determined Velocity Distributions.

§ 8. The diameter, D, of the sphere used in FAGE's experiments was 6 inches, and the observations were made in three wind tunnels, with or without honeycomb in front of the sphere, at several wind speeds, U_0. For the detailed description of these tunnels and of the position of the sphere used reference should be made to FAGE's original paper.

The values of the pressure coefficient $(p-p_0)/\frac{1}{2}\rho U_0^2$, where p is the pressure on the sphere and p_0 the pressure in the undisturbed free stream, were obtained from general explorations made at seven speeds, ranging from $U_0 = 35$ ft. per sec. to $U_0 = 135$ ft. per sec. in No. 2 open jet tunnel, of the National Physical Laboratory. Additional coefficients were also obtained from more detailed explorations made at four of these speeds ($U_0 = 50$, 80, 95 and 135 ft. per sec.). No honeycomb was placed in front of the sphere in No. 2 open jet tunnel, so that no artificial initial turbulences were introduced in the flow impinging upon the sphere. The range of the REYNOLDS number DU_0/ν covered, 110,000 to 424,500, included the critical range, 140,000 to 330,000.

In the present paper, three typical cases in which $U_0 = 50$, 80 and 135 ft. per sec. respectively have been subjected to mathematical analysis. The values of the corresponding REYNOLDS number, $R = DU_0/\nu$, are 157,200, 251,300 and 424,500 respectively. The first one is just within the critical range and the second is at its middle, while the remaining largest one is just beyond this range.

The observed values of the pressure coefficient for these three cases are reproduced in Fig. 1, where the pressure distribution corresponding with the theoretical velocity distribution $U = \frac{3}{2} U_0 \sin\theta$ is also shown by a dotted-line curve for comparison. Using these observed values, the corresponding actual velocity distribution on the sphere has been found in each case, with the aid of BERNOULLI's theorem:

$$\left(\frac{U}{U_0}\right)^2 = 1 - \frac{p-p_0}{\frac{1}{2}\rho U_0^2}. \tag{25}$$

In what follows, the results of the analysis for each of these three cases will be described separately.

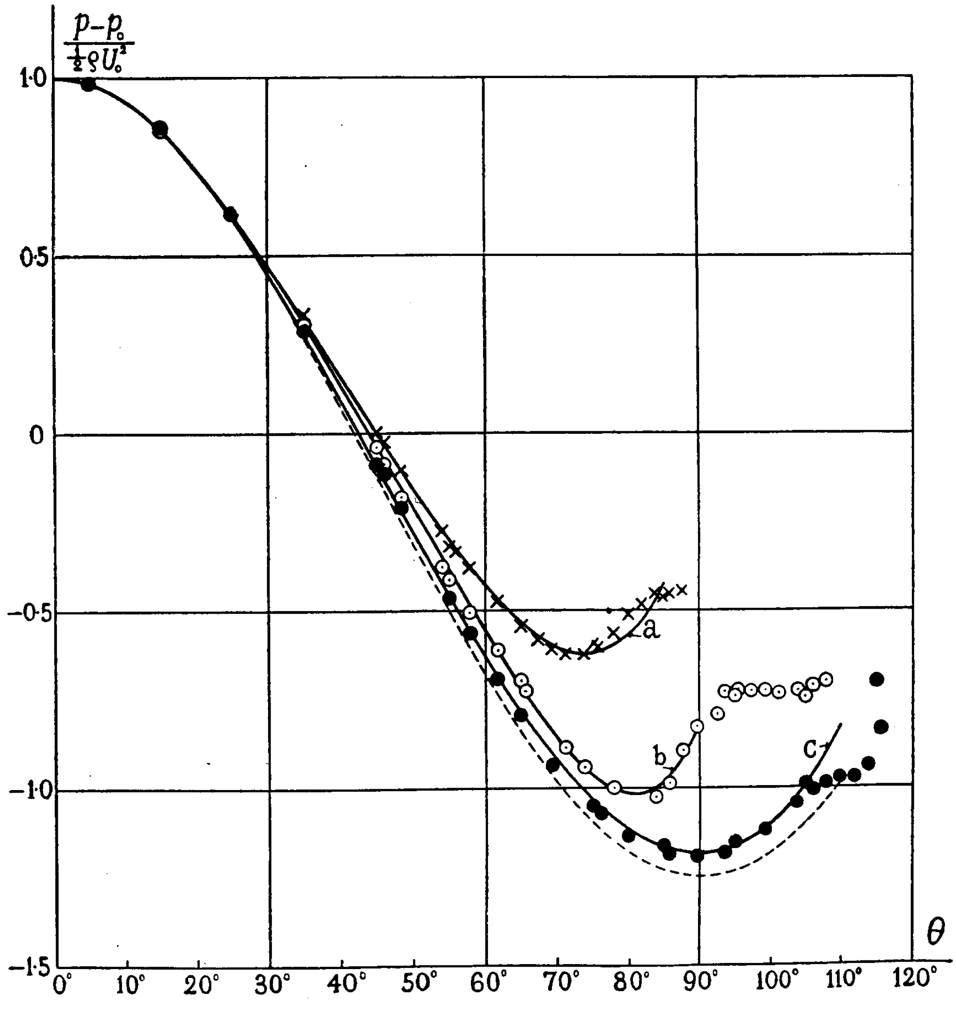

Fig. 1. × : observed points, a : calculated curve, for $U_0 = 50$ ft./sec.
⊙ : observed points, b : calculated curve, for $U_0 = 80$ ft./sec.
● : observed points, c : calculated curve, for $U_0 = 135$ ft./sec.

(a) Case when $U_0 = 50$ ft. per sec.

§9. In case when $U_0 = 50$ ft. per sec. and therefore the value of $R = DU_0/\nu$, being 157,200, was just within the critical range, FAGE observed that the flow in the boundary layer was everywhere laminar up to the point of separation, which was found to occur at $\theta = 83°$.

Using the observed values of the pressure coefficient $(p-p_0)/\frac{1}{2}\rho U_0^2$, the velocity distribution just outside the boundary layer has been calculated. In the range $0° \leqq \theta \leqq 85°$, this can be expressed approximately by the formula:

$$\frac{U}{U_0} = 1{\cdot}5\,\theta - 0{\cdot}43707\,\theta^3 + 0{\cdot}148097\,\theta^5 - 0{\cdot}042329\,\theta^7\,. \tag{26}$$

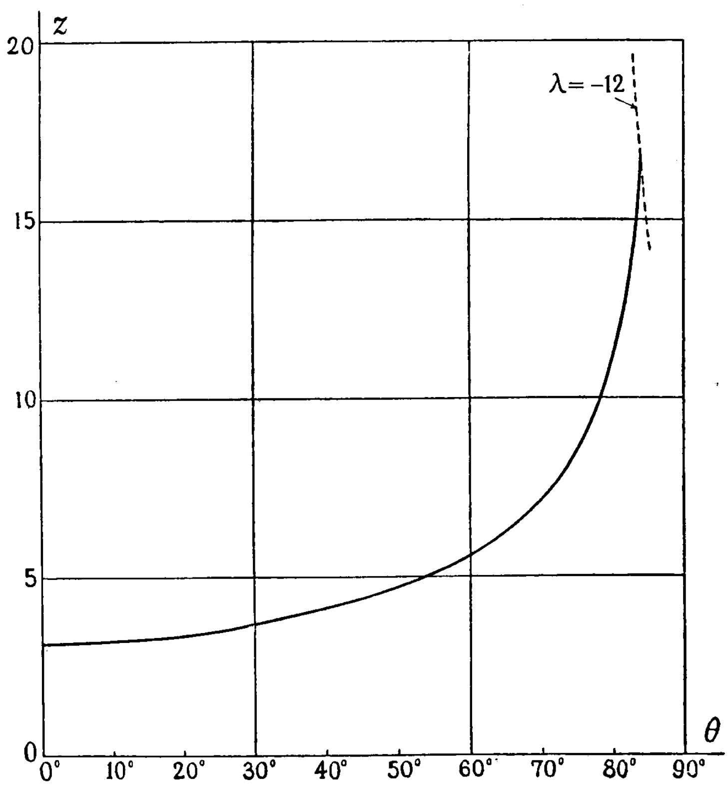

Fig. 2. ($U_0 = 50$ ft./sec.)

The pressure distribution corresponding with this velocity distribution is shown in Fig. 1 by a full-line curve (a). It will be seen that in the present case the above formula (26) represents, with sufficient approximation, the actual velocity distribution on the sphere for the range $0° \leqq \theta \leqq 85°$.

The values of the three quantities:

$$\frac{U}{U_0}, \quad \frac{1}{U_0}\frac{dU}{d\theta}, \quad \frac{1}{U_0}\frac{d^2U}{d\theta^2},$$

occurring in the differential equation (21), have been calculated with the aid of the formula (26), and making use of them as well as the values of the functions $f(\lambda)$, $f^*(\lambda)$ and $g(\lambda)$, the graphical integration of the differential equation (21) has been carried out, subject to the initial conditions (23) and (24). The result is shown in Fig. 2. It will be seen that the point of separation is approximately at $\theta = 84°$. This value should be compared with the observed one, 83°, and we find that the agreement between the calculation and the observation is quite satisfactory.

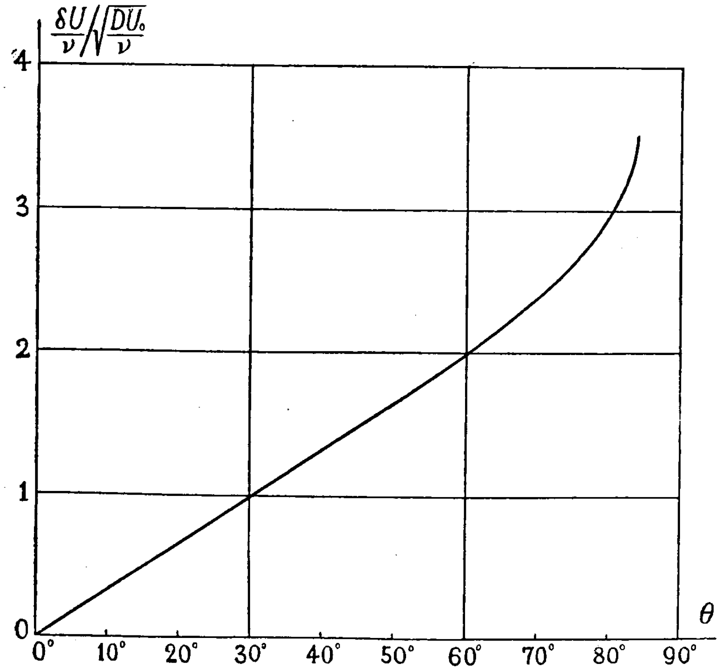

Fig. 3. ($U_0 = 50$ ft./sec.)

Next, the REYNOLDS number of the boundary layer, defined by $\delta U/\nu$, is expressed, in general, in the form:

$$\frac{\delta U}{\nu} = \frac{U}{U_0}\sqrt{\frac{z}{2}}\sqrt{\frac{DU_0}{\nu}}\,, \tag{27}$$

so that

$$\frac{\delta U}{\nu}\bigg/\sqrt{\frac{DU_0}{\nu}} = \frac{U}{U_0}\sqrt{\frac{z}{2}}\,. \tag{28}$$

The curve of $(\delta U/\nu)/\sqrt{DU_0/\nu}$ is shown in Fig. 3. With the aid of this curve we can calculate the values of $\delta U/\nu$ for any value of θ, since in the present case the value of DU_0/ν is known to be 157,200. The values of $\delta U/\nu$ thus calculated are shown in the second column in Table I.

TABLE I.

Values of the REYNOLDS number of the boundary layer, $\delta U/\nu$.

θ	$U_0 = 50$ ft./sec.	$U_0 = 80$ ft./sec.	$U_0 = 135$ ft./sec.
	$R = 157{,}200$	$R = 251{,}300$	$R = 424{,}500$
0°	0	0	0
10°	130	160	210
20°	260	330	430
30°	390	500	650
40°	520	660	870
50°	650	830	1090
60°	790	1000	1320
65°	870	1090	1440
70°	950	1180	1570
80°	1170	1390	1850
82°	1250	1450	1910
84°	1400	1520	1970
86°		1600	2040
88°		1750	2100
89°		1910	2140
90°			2180
92°			2250
94°			2340
96°			2420
98°			2530
100°			2660
102°			2830
104°			3100

Further, in order to see the way in which the non-dimensional quantity λ, i.e.,

$$\lambda = \frac{1}{U_0}\frac{dU}{d\theta}z\ , \tag{29}$$

changes with θ, the curve of λ is constructed in Fig. 4.

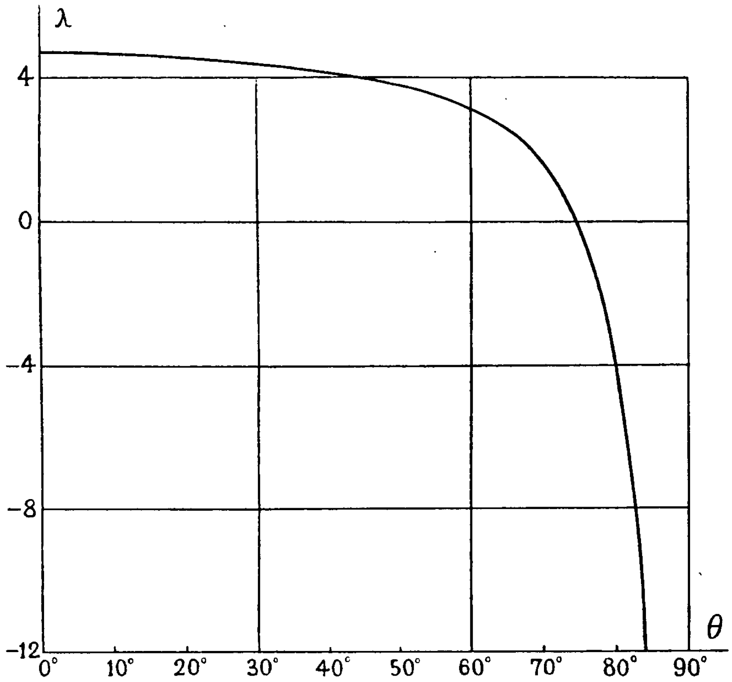

Fig. 4. ($U_0 = 50$ ft./sec.)

Lastly, the intensity of laminar skin friction τ_0 over the surface of the sphere is given by

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}\ . \tag{30}$$

Therefore, we have, by (6) and (8),

$$\tau_0 = \frac{\mu U}{6\delta}(12+\lambda)\ , \tag{31}$$

from which it follows, with the aid of (20),

$$\frac{\tau_0}{\frac{1}{2}\rho U_0^2} = \frac{\sqrt{2}}{3} \frac{1}{\sqrt{DU_0/\nu}} \frac{U}{U_0} \frac{1}{\sqrt{z}}(12+\lambda) . \qquad (32)$$

Making use of the known values of z, λ, U/U_0 and DU_0/ν, the values of $\tau_0/\frac{1}{2}\rho U_0^2$ have been calculated for various values of θ ranging from $0°$ to $84°$, the latter corresponding with the theoretical point of separation. The distribution of the intensity of laminar skin friction τ_0 thus calculated is shown in Fig. 11 by a full-line curve (a).

(b) Case when $U_0 = 80$ ft. per sec.

§ 10. In case when $U_0 = 80$ ft. per sec. and the corresponding REYNOLDS number $R = DU_0/\nu$ is 251,300, the velocity distribution on the sphere can be expressed, for the range $0° \leqq \theta \leqq 90°$, by the approximate formula:

$$\frac{U}{U_0} = 1{\cdot}5\,\theta - 0{\cdot}41803\,\theta^3 + 0{\cdot}173958\,\theta^5 - 0{\cdot}044398\,\theta^7 . \qquad (33)$$

The corresponding pressure distribution is shown in Fig. 1 by a full-line curve (b). It will be seen that this formula (33) represents, with sufficient accuracy, the actual velocity distribution on the sphere for the range $0° \leqq \theta \leqq 90°$.

Using the above formula (33), the graphical integration of the differential equation (21) has been carried out as in the preceding case, the result of which is shown in Fig. 5. From this figure it will be seen that if the flow in the boundary layer were assumed to be laminar up to the point of separation, separation would occur at $\theta = 89°$ approximately.

Next, the values of various characteristic quantities for the boundary layer have been calculated as before. Fig. 6 shows the curve of $(\delta U/\nu)/\sqrt{DU_0/\nu}$. Using this curve, the value of $\delta U/\nu$, the REYNOLDS number of the boundary layer, has been calculated. The results are given in the third column of the preceding Table I.

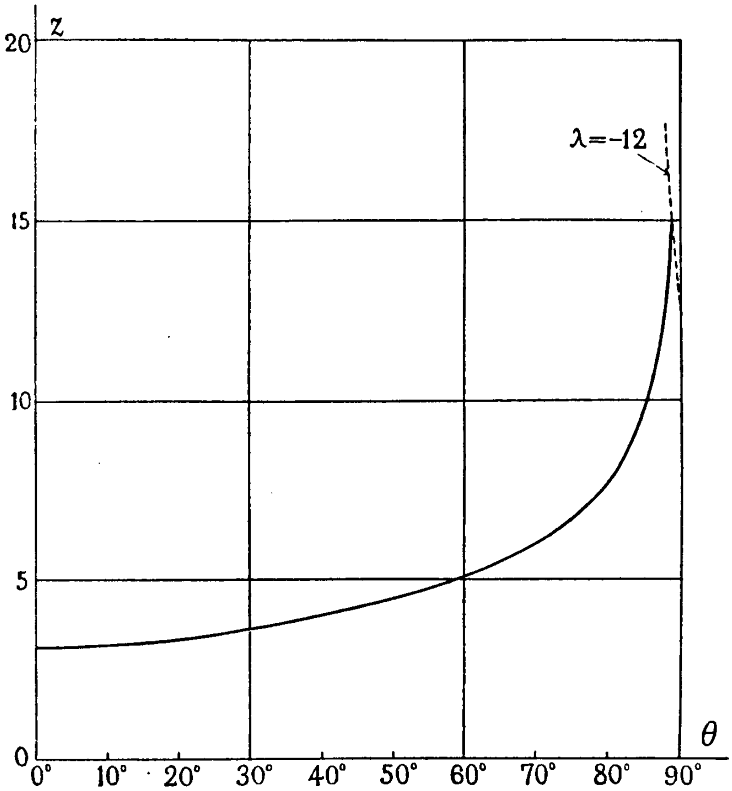

Fig. 5. ($U_0 = 80$ ft./sec.)

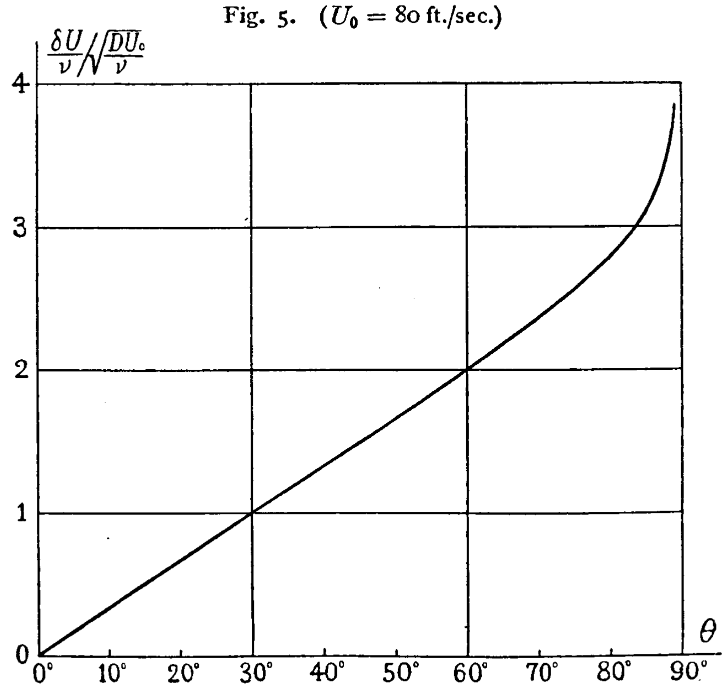

Fig. 6. ($U_0 = 80$ ft /sec.)

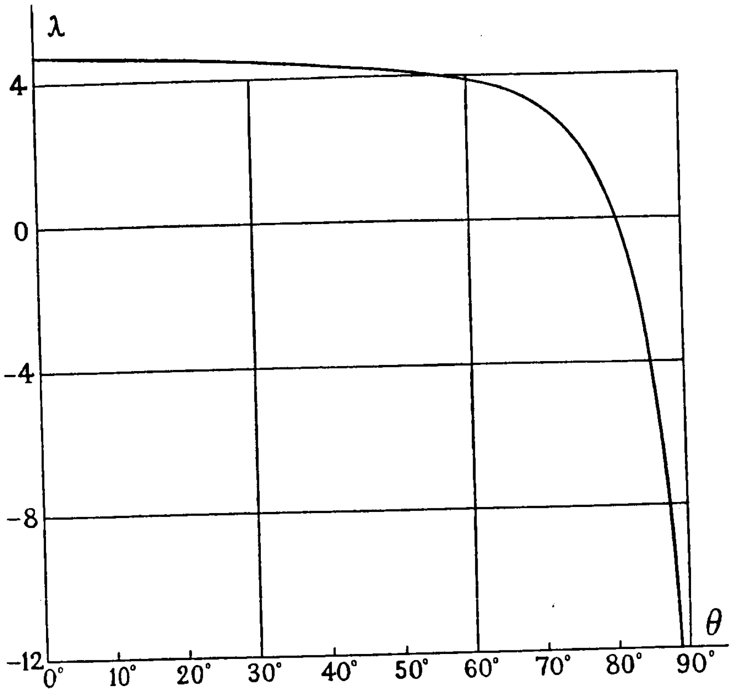

Fig. 7. ($U_0 = 80$ ft./sec.)

Also, Fig. 7 shows the curve of the non-dimensional quantity λ plotted against θ.

Further, the distribution of the intensity of laminar skin friction τ_0 calculated by the formula (32) is shown in Fig. 11 by a full-line curve (b).

(c) Case when $U_0 = 135$ ft. per sec.

§ 11. When $U_0 \doteq 135$ ft. per sec. and $DU_0/\nu = 424{,}500$, the approximate expression for U/U_0 is, for the range $0^\circ \leqq \theta \leqq 105^\circ$,

$$\frac{U}{U_0} = 1{\cdot}5\,\theta - 0{\cdot}29242\,\theta^3 + 0{\cdot}039031\,\theta^5 - 0{\cdot}0049565\,\theta^7 . \qquad (34)$$

The corresponding pressure distribution calculated by using this formula for U/U_0 is shown in Fig. 1 by a full-line curve (c). It will be seen that the above formula (34) can represent, with sufficient approximation, the actual velocity distribution on the sphere for the range $0^\circ \leqq \theta \leqq 105^\circ$.

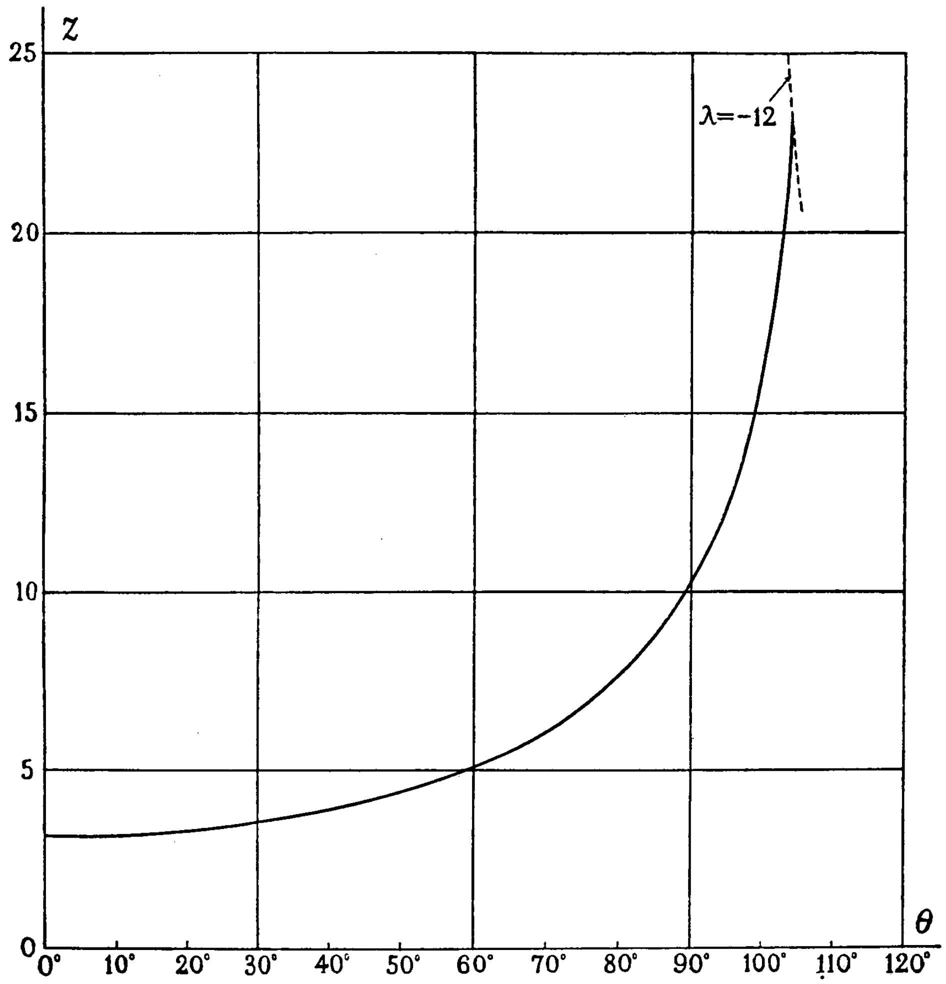

Fig. 8. ($U_0 = 135$ ft./sec.)

The results of the calculations similar to those in the preceding cases are shown in Figs. 8, 9 and 10. Fig. 8 shows the result of graphical integration of the differential equation (21). It will be seen that if the flow in the boundary layer were assumed to be laminar up to the point of separation, separation would occur at $\theta = 104^\circ$ approximately.

Fig. 9 gives the curve of $(\delta U/\nu)/\sqrt{DU_0/\nu}$ plotted against θ, while Fig. 10 shows how the non-dimensional quantity λ varies with θ.

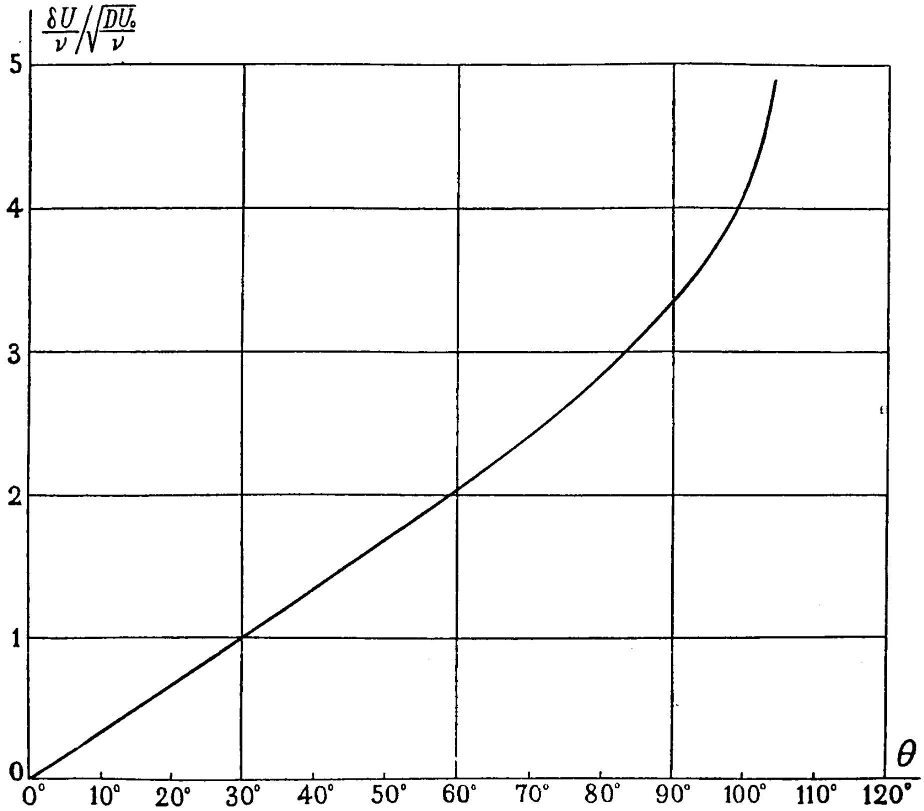

Fig. 9. ($U_0 = 135$ ft./sec.)

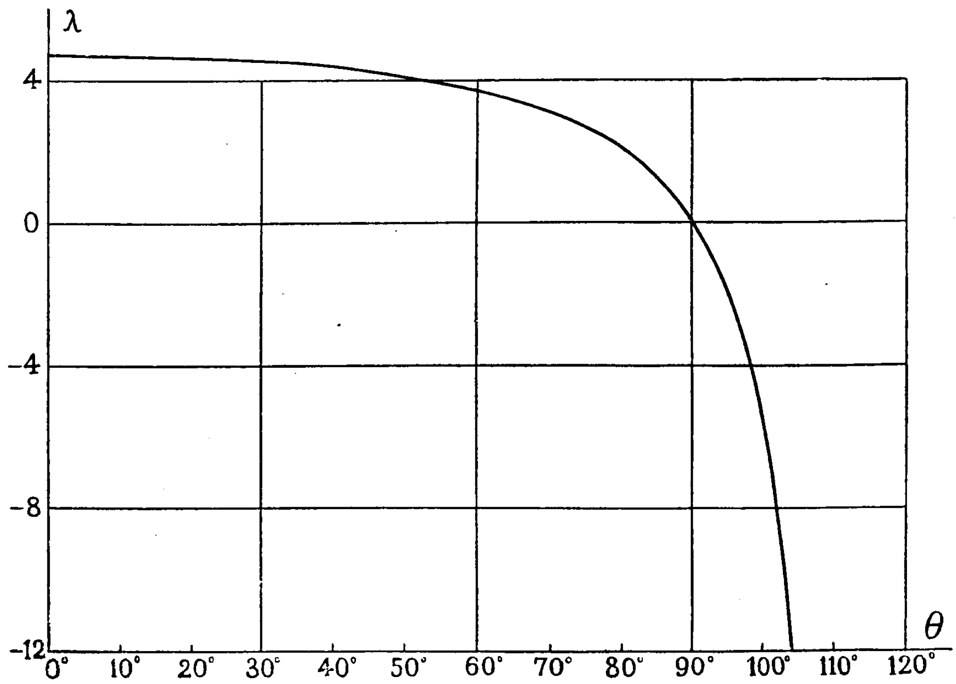

Fig. 10. ($U_0 = 135$ ft./sec.)

The calculated values for the REYNOLDS number of the boundary layer, $\delta U/\nu$, are given in the fourth column of the preceding Table I.

Also, the intensity of laminar skin friction τ_0 has been calculated and its distribution curve is shown in Fig. 11 by a full-line curve (c).

V. Comparison of the Calculated Distributions of Laminar Skin Friction with FAGE's Observations.

§ 12. The calculated distributions of the intensity of laminar skin friction over the surface of the sphere have been compared with FAGE's observed values, which have been determined using a STANTON surface tube. The result of comparison is shown in Fig. 11, where the observed points for the three cases in which $U_0 = 50$, 80 and 135 ft. per sec. respectively are shown by ×, ⊙ and ● respectively, while the full-line curves a, b, c show the corresponding calculated distributions. In this figure, the calculated distribution of the intensity of laminar skin friction corresponding with the theoretical velocity distribution $U = \frac{3}{2}U_0 \sin\theta$ is also shown by a dotted-line curve, for the sake of comparison.

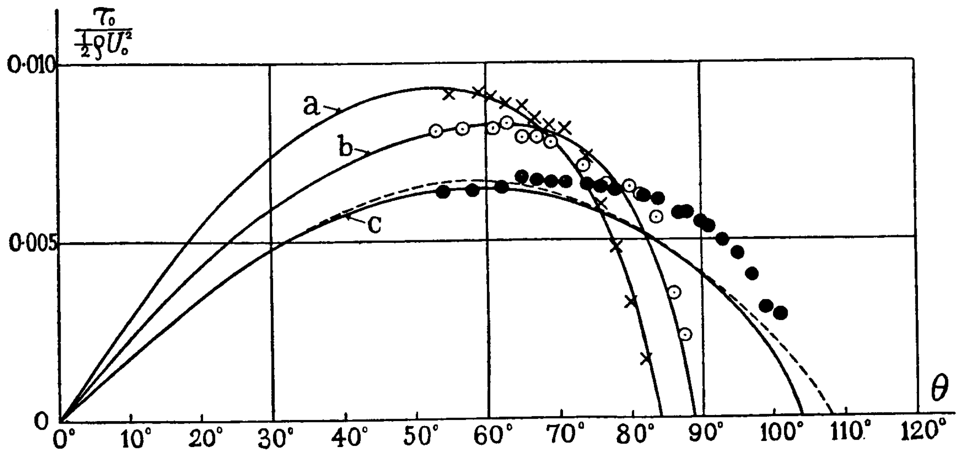

Fig. 11. × : observed points, a : calculated curve, for $U_0 = 50$ ft./sec.
⊙ : observed points, b : calculated curve, for $U_0 = 80$ ft./sec.
● : observed points, c : calculated curve, for $U_0 = 135$ ft./sec.

In the calculation, use was made of the results obtained in the previous paper I, and the REYNOLDS number of the stream is taken arbitrarily to be 424,500, which corresponds with the case $U_0 = 135$ ft. per sec. in FAGE's experiments.

In the case when $U_0 = 50$ ft. per sec. these observed points have been adjusted so as to fit the calculated curve near the point $\theta = 60°$, because the observed points are relatively dense in the neighbourhood of this point(1). In other words, we have used the re-estimated value of the distance from the surface, $\bar{y}$, to which the speed, u, deduced from the velocity head at the mouth of the surface tube, must be related to obtain the intensity of skin friction τ_0 by the relation $\tau_0 = \mu(u/\bar{y})$.

It will be seen that the agreement between the calculated distribution of the intensity of laminar skin friction and the observations is quite satisfactory. In the present case, the flow in the boundary layer is everywhere laminar and the layer separates from the surface of the sphere before the transition from laminar to turbulent flow can occur.

Thus, we see that when $U_0 = 50$ ft. per sec. and the corresponding REYNOLDS number R, being 157,200, is just within the critical range, the flow in the boundary layer is laminar up to the point of separation, in accordance with the result of FAGE's observations, and quite satisfactory agreement between the calculated results and the observations is found not only for the point of separation but also for the distribution of the intensity of laminar skin friction on the surface of the sphere.

This fact seems to indicate that the momentum integral equation (4) for the laminar boundary layer on a body of revolution, together with the assumed quartic form for the velocity distribution in the layer, can be used, with sufficient approximation, to describe the laminar boundary layer on the surface of a sphere.

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(1) It may be remarked that even if the observed points were adjusted so as to fit the calculated curve at $\theta = 55°$, where a single observation only has been taken, only slight unappreciable modifications would be necessary for Fig. 11.

§ 13. On a careful observation of the calculated distribution curve (a) of the intensity of laminar skin friction for the case when $U_0 = 50$ ft. per sec. and of the corresponding observed points denoted by $\times$, it will be found that there are small systematic deviations of the observed points from the calculated curve in the range from $\theta = 65^\circ$ to 75°, which includes the observed point of minimum pressure, at $\theta = 74^\circ$, in the present case.

In this connection it should be noticed, however, that these deviations cannot be considered to be due to the inaccuracy of the formula (26) for U/U_0, because it can represent, with sufficiently good approximation, the observed pressure distribution, especially for the range $\theta = 0^\circ$ to 75°, as shown in Fig. 1.

Thus, these deviations may be rather considered to indicate that some kinds of accidental disturbances had occurred near the point of minimum pressure and thus the flow in the boundary layer, becoming somewhat irregular, had departed from the purely laminar state. The fact that when $\theta > 75^\circ$ the observed points fall again on the calculated curve shows however that such disturbances had been unable to grow up so as to make the flow turbulent, but, on the contrary, they had soon decayed down and the motion had become again purely laminar, since the REYNOLDS number was not too large in the case under discussion.

This reminds us of the well-known similar phenomenon in the case of flow through a pipe of circular section in the REYNOLDS experiments. It is well known that when the REYNOLDS number of the stream, formed from the radius of the pipe and the velocity at the centre of the pipe, does not exceed a certain critical value, initial disturbances, if any, do not grow up so as to make the flow turbulent, but, on the contrary, they are soon obliterated and the flow becomes purely laminar.

On the other hand, it is also well established in the case of the REYNOLDS experiments that if the REYNOLDS number is greater than the critical value, the flow becomes sensitive to initial or accidental small disturbances, and the motion becomes ultimately turbulent.

We may therefore conjecture that a similar phenomenon may perhaps occur in the case of the flow in the boundary layer of a sphere. Thus, it may be expected that when the REYNOLDS number of the stream takes larger values than a certain critical value, accidental disturbances, which are likely to be originated near the point of minimum pressure, due perhaps to local separation of the boundary layer caused by small fluctuating pressure gradients(1), would grow up more and more, and the flow in the boundary layer would become turbulent.

The results for the cases of larger REYNOLDS numbers discussed in the present paper seem to show that such an expectation is not quite erroneous. In effect, it is seen from Fig. 11 that the deviations of the observed points from the calculated distribution of the intensity of laminar skin friction become greater as the REYNOLDS number of the stream increases, indicating that the flow in the boundary layer becomes more irregular and the degree of turbulence increases as the REYNOLDS number of the stream increases.

Thus, we may infer that the transition from laminar to turbulent flow in the boundary layer of a sphere begins near, or rather in front of, the point of minimum pressure. Before entering into the discussion on the transition from laminar to turbulent flow, however, some mention will be made about the results of comparison of the calculated distributions of the intensity of laminar skin friction with FAGE's observed points, in the two cases of larger REYNOLDS numbers.

§ 14. In the case when $U_0 = 80$ ft. per sec. and therefore the corresponding REYNOLDS number of the stream is $DU_0/\nu = 251,300$, the observed points for the intensity of skin friction, shown by ⊙ in Fig. 11, have been adjusted, as in the previous case for which $U_0 = 50$ ft.

(1) Cf. EASTMAN N. JACOBS, Laminar and Turbulent Boundary Layers as affecting Practical Aerodynamics. Journal Soc. Automotive Engineers, **41** (1937), 468–472.

per sec., so as to fit the calculated distribution curve of the intensity of laminar skin friction near the point $\theta = 55^\circ$, remembering that the flow in the layer near this point may be assumed to be laminar, in accordance with the results of observation. This adjustment is, as mentioned before, equivalent to having used the re-estimated value of the distance from the surface, $\bar{y}$, which is related to the speed u and the intensity of surface friction τ_0 by the relation $\tau_0 = \mu(u/\bar{y})$.

It will be seen from Fig. 11 that the agreement between the calculated curve (b) of the intensity of laminar skin friction and the observed points is quite satisfactory till the point where $\theta = 70^\circ$ approximately. This seems to indicate that the flow in the boundary layer is purely laminar up to the point $\theta = 70^\circ$. The observed points, however, begin to deviate from the calculated curve (b) at $\theta = 70^\circ$, and the deviation becomes greater more and more as θ increases, reaching a maximum at $\theta = 82^\circ$ approximately, which corresponds with the observed point of minimum pressure in the present case.

As mentioned before, such deviations seem to indicate that some kinds of accidental disturbances had occurred near the point of minimum pressure, due perhaps to local separation of the layer caused by small fluctuating pressure gradients(1), and thus the flow in the boundary layer, becoming somewhat irregular, had departed from the purely laminar state. The degree of irregularity, i.e., the degree of turbulence increases with increasing θ. But, Fig. 11 shows that the deviations somewhat decrease(2) beyond the point $\theta = 85^\circ$, and this seems to indicate that the degree of turbulence decreases to some extent. Since, however, the REYNOLDS number of the stream, being 251,300 in the present case, is at the middle of the so-called critical range and is greater than that for the previous case, the flow in the layer does not become again

(1) Cf. EASTMAN N. JACOBS's paper, loc. cit.

(2) Whether these deviations do actually decrease or not is not certain. Since, as mentioned by FAGE himself in his paper referred to before, the experimental determination by using a STANTON surface tube of the intensity of surface skin friction τ_0 is rather difficult in the region where $\frac{1}{2}\rho u^2$ is small, some uncertainty is unavoidable for the two observed values of τ_0 near $\theta = 87^\circ$.

purely laminar, but, on the contrary, it might rather become fully turbulent, because additional disturbances might be expected to occur near the point $\theta = 89^\circ$, at which, according to our calculations, separation of the boundary layer from the surface of the body would occur if the flow in the layer were everywhere laminar up to the point of separation. Thus, observing Fig. 11, we may infer that the transition from laminar to turbulent flow in the boundary layer *begins*, in the case under discussion, near the point $\theta = 82^\circ$ which corresponds with the observed point of minimum pressure.

§ 15. When $U_0 = 135$ ft. per sec. and the corresponding REYNOLDS number of the stream, being 424,500, is above the critical range, the deviations of the observed points from the calculated distribution of the intensity of laminar skin friction become quite conspicuous beyond a certain angular point. In this case, no adjustment has been necessary for making the observed points fit the calculated distribution curve in the range where the flow in the layer can be considered to be laminar.

Comparing the observed points denoted by ● with the calculated curve (c) in Fig. 11, it will be found that the agreement between the calculated result and the observation is quite satisfactory till the point $\theta = 62^\circ$, indicating that the flow in the boundary layer is laminar up to this point. However, the experimental points do not lie on the calculated curve when $\theta > 65^\circ$, and the departure of the observed points from the calculated curve increases with increasing θ. This shows evidently that the flow in the boundary layer becomes somewhat irregular and departs from the purely laminar state near the point $\theta = 65^\circ$, and that the degree of irregularity, i.e., the degree of turbulence increases as θ increases. Thus, we may infer that the transition from laminar to turbulent flow *begins* already at the point $\theta = 65^\circ$ approximately in the case under discussion, which lies about 25 degrees in front of the observed point of minimum pressure at $\theta = 90^\circ$. It seems that since the REYNOLDS number of the stream is sufficiently large in the present case, the flow becomes fully turbulent at a rather early stage.

VI. Discussions on the Transition from Laminar to Turbulent Flow in the Boundary Layer of a Sphere.

§ 16. The results of our mathematical analysis described in detail in the preceding paragraphs show that when the REYNOLDS number of the stream R, formed from the undisturbed velocity, U_0, of the stream and the diameter, D, of a sphere, assumes large values, the transition from laminar to turbulent flow in the boundary layer of the sphere begins to occur at a certain point near, or rather in front of, the point of minimum pressure. In effect, when $U_0 = 80$ ft. per sec. and $R = DU_0/\nu = 251,300$, the transition from laminar to turbulent flow seems to begin at $\theta = 82°$, which corresponds with the observed point of minimum pressure, and also, when $U_0 = 135$ ft. per sec. and $R = 424,500$, we may consider that the transition begins at $\theta = 65°$, which lies about 25 degrees in front of the observed point of minimum pressure, $\theta = 90°$.

The transition may perhaps be due to small accidental disturbances caused by local separation of the layer from the surface of the body(1) and the local separation of the layer itself may be considered to be due to small fluctuating pressure gradients existing in the stream, especially when the speed of the stream is large.

Now, it is well known in the case of the REYNOLDS experiments on the flow of liquid through a pipe of circular section that when the REYNOLDS number, formed from the diameter of the pipe and the speed along its axis, becomes larger than a certain critical value, the flow becomes sensitive to small accidental disturbances and the transition from laminar to turbulent flow occurs. In this connection it must be remarked however that the value of such a critical REYNOLDS number varies considerably with the degree of turbulence contained in the stream.

In the case of the boundary layer of the sphere, it may be assumed

(1) Cf. EASTMAN N. JACOBS's paper, loc. cit.

by analogy that when the REYNOLDS number of the boundary layer, $\delta U/\nu$, becomes greater than a certain critical value, the transition from laminar to turbulent flow begins to occur. If this assumption is correct, it might be expected that the REYNOLDS number of the boundary layer at a point where the transition begins assumes a certain definite value in the two cases where $U_0 = 80$ and 135 ft. per sec. respectively, since FAGE's experiments have been conducted under the same conditions of initial turbulence, if any[1], in these two cases.

Thus, it might be expected that the REYNOLDS number of the boundary layer at the point of transition $\theta = 82°$ in the case $U_0 = 80$ ft. per sec. is nearly equal to the corresponding quantity at the point $\theta = 65°$ in the case $U_0 = 135$ ft. per sec., where the transition from laminar to turbulent flow is considered to begin in this case.

The preceding Table I shows however that the REYNOLDS number of the boundary layer, $\delta U/\nu$, takes a value 1450 at $\theta = 82°$ when $U_0 = 80$ ft. per sec. and $R = 251,300$ and also it assumes a value 1440 at $\theta = 65°$ in the case when $U_0 = 135$ ft. per sec. and $R = 424,500$. It will be seen that, in conformity with our expectation above mentioned, the REYNOLDS number of the boundary layer at the point of beginning of the transition from laminar to turbulent flow assumes a nearly definite value 1450 in the two cases discussed in the present paper.

Thus, we may infer that the transition from laminar to turbulent flow in the boundary layer of a sphere begins to occur at a point at which the REYNOLDS number of the boundary layer assumes the value 1450. Table I shows also that when $U_0 = 50$ ft. per sec. and $R = 157,200$, the value of the REYNOLDS number of the boundary layer does not exceed 1400. This indicates that the flow in the layer is laminar and the layer separates from the surface of the sphere before the transition from laminar to turbulent flow can occur, agreeing with the result of observation.

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(1) As mentioned already, no artificial turbulence was introduced when No. 2 open jet tunnel was used in FAGE's experiments.

By measuring distributions of normal pressure and of the intensity of surface skin friction over the surface of the 6-inches sphere in the N. P. L. No. 2 open jet tunnel, FAGE has concluded, in his paper referred to before, that the transition from laminar to turbulent flow in the boundary layer of the sphere begins at a point of inflexion in the curve of normal pressure distribution, which also corresponds with a minimum in the distribution curve of the intensity of skin friction. Thus, FAGE has shown that the point of transition is $95°$ when $U_0 = 80$ ft. per sec. and $110°$ when $U_0 = 135$ ft. per sec.

The results of our analysis show, however, that if the flow in the boundary layer were laminar, the layer would separate from the surface of the sphere at $\theta = 89°$ when $U_0 = 80$ ft. per sec. and also separation would occur at $\theta = 104°$ when $U_0 = 135$ ft. per sec. Thus, if, as mentioned by FAGE, the flow in the layer be laminar up to the points $\theta = 95°$ and $\theta = 110°$ in the cases where $U_0 = 80$ and 135 ft. per sec. respectively, the layer would separate from the surface of the sphere before the transition from laminar to turbulent flow can occur. In this connection it should be again remarked that the momentum integral equation for the laminar boundary layer on a body of revolution, adopted in the present paper together with the assumed quartic form for the velocity distribution in the layer, can be used, with sufficient approximation, to describe the laminar boundary layer on the surface of a sphere, and therefore the theoretical results obtained in the present paper are reliable.

Thus, we are led to the conclusion that when the REYNOLDS number of the stream is sufficiently large the transition from laminar to turbulent flow in the boundary layer of a sphere begins to occur at a point near, or rather somewhat in front of, the point of minimum normal pressure, but not at a point of inflexion in the curve of normal pressure distribution, as mentioned by FAGE, and that the critical REYNOLDS number of the boundary layer at the point of transition is of the order of magnitude 1450.

VII. Summary.

§ 17. The momentum integral equation for the boundary layer on the surface of a body of revolution is applied to the case of the boundary layer of a sphere placed in a uniform stream. The flow in the boundary layer is assumed to be laminar and also the quartic form is assumed for the velocity distribution in the layer. For the velocity distribution just outside the boundary layer, use is made of each of three actual distributions which have been obtained from distributions of normal pressure measured by FAGE, at REYNOLDS numbers just within, near the middle of, and above, the critical range respectively, over which the drag coefficient of the sphere experiences a large fall.

The differential equation for determining the thickness of the laminar boundary layer is solved, in each case, by the method of graphical integration and thus various characteristic quantities for the layer are calculated.

In case when the REYNOLDS number of the stream is just within the critical range so that, according to FAGE's observation, the flow in the boundary layer is everywhere laminar up to the point of separation, it is found that the calculated point of separation is in good agreement with FAGE's observation and the agreement between the calculated distribution of the intensity of laminar skin friction and the observation is also quite satisfactory. These results indicate that the flow in the boundary layer is everywhere laminar up to the point of separation, in accordance with the result of FAGE's observation.

When the REYNOLDS number of the stream lies near the middle of the critical range, the departure of the observed values of the intensity of skin friction from the calculated distribution of the intensity of laminar skin friction occurs at a point near the observed point of minimum normal pressure. This seems to show that some kinds of accidental disturbances had occurred near the point of minimum pressure, due perhaps to local psearation of the boundary layer from the surface of the body caused by small fluctuating pressure gradients and the flow in the layer had departed from the purely laminar state. The departure of the observed points from the calculated curve increases with increasing θ and they never fall again on the theoretical curve. Thus, we may infer that the transition from laminar to turbulent flow in the boundary layer of the sphere begins at a point lying near the point of minimum pressure in the case discussed.

The deviations of the experimental points from the calculated distribution curve of the intensity of laminar skin friction become more conspicuous when the REYNOLDS number of the stream is above the critical range. It is then found that the transition from laminar to turbulent flow begins at an earlier stage than in the case of smaller REYNOLDS numbers, namely, it begins at a certain point in front of the point of minimum pressure.

The values of the critical REYNOLDS number of the boundary layer at the point of transition are nearly equal in the two cases discussed in the present paper, as we should have expected so if our assumption on the point of transition from laminar to turbulent flow in the boundary layer of the sphere is correct, and it is found that the said critical REYNOLDS number is of the order of magnitude 1450.

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