第百八十號

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抄錄

環狀斷面の管に沿うて流れる渦亂流に於ける速度分布に就いて

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眞直ぐな圓管内を流れる渦亂流に於ける平均速度の分布は 混合距離の概念を基礎とする PRANDTL の運動量輸送の理論及び TAYLOR の渦動度輸送の變形理論を使つて, TAYLOR が理論的に計算してゐるが, それによると, 渦動度輸送の變形理論の與へる結果の方が運動量輸送の理論の與へる結果よりも, STANTON 及び NIKURADSE の實驗結果によく合ふようである.

圓管以外の管に沿うて流れる渦亂流に於ける平均速度の分布を, 此等二つの渦亂流の理論によつて理論的に研究することは極めて興味深いことであるが, 未だ如何なる場合の理論的研究も遂行されてゐない樣である. 實際, 矩形とか三角形とか又は梯形の斷面を有する管に沿うての渦亂流に於ける平均速度の分布に關するかなり詳しい實驗的研究は, 例へば NIKURADSE によつてなされてゐるが, しかし此等の場合を理論的に研究することは殆んど不可能なのである.

ところが, 最近 MIKRJUKOV は, 二つの同心圓によつて圍まれた環狀斷面を有する眞直ぐな管に沿うて流れる渦亂流に於ける平均速度の分布を, 種々の異なれる壓力勾配の場合に測定した結果を發表したが, この場合は平均速度の分布が管の軸に關し

て對稱であるために，これを理論的に吟味することも比較的容易である．斯様な環狀斷面の管に沿うて流れる渦亂流の場合に，果して渦動度輸送の變形理論及び運動量輸送の理論の孰れが實測結果に近い理論的結果を與へるであらうか．これを吟味することは極めて興味深く且つ重要であると思はれる．

本論文は，著者等が環狀斷面の管に沿うて流れる渦亂流に於ける平均速度の分布を渦動度輸送の變形理論及び運動量輸送の理論によつて計算した結果を示すこと及びそれ等を MIKRJUKOV の實測結果と比較することをその目的とする．圓管の場合の様に，環狀斷面の管の場合に於いても，渦動度輸送の變形理論の方が運動量輸送の理論よりも實驗結果によく合ふ結果を與へることが知れた．

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On the Velocity Distribution in Turbulent Flow through a Straight Pipe of Annular Cross-section.

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I. Introduction.

§ 1. The distribution of mean velocity in turbulent flow of a fluid flowing under pressure through a straight pipe of circular cross-section and between parallel planes has been discussed theoretically by TAYLOR(1), on the basis of both the modified vorticity transport and the momentum transport theories of turbulent motion. To do this he has assumed that the turbulence is isotropic and that the mixing length is proportional to the distance from the nearest point of the wall or walls. Thus, it has been found that the calculated results obtained by applying the modified vorticity transport theory are generally in better agreement

(1) G. I. TAYLOR, Flow in Pipes and between Parallel Planes. Proc. Roy. Soc., London, A 159 (1937), 496-506.

with the experimental results of STANTON and NIKURADSE than those obtained by the momentum transport theory.

The problem has also been discussed by GOLDSTEIN(1), applying the similarity theory of turbulence to the momentum transport and to the (unmodified) vorticity transport theories.

On the other hand, any theoretical discussion on the distribution of mean velocity in turbulent flow through straight pipes of non-circular cross-sections seems to have not yet been made, so far as the present writers are aware. Although some experimental results concerning the velocity distribution in turbulent flow through various pipes with rectangular, or triangular, or trapezoidal cross-sections are known(2), yet these cases do not seem to be capable of being subjected to mathematical treatment.

Recently, MIKRJUKOV(3) has observed the distribution of mean velocity in turbulent flow of water flowing under pressure through a straight pipe of annular cross-section bounded by two concentric circles, and using the results so obtained, he has also discussed the intensities of turbulent skin friction on the inner and outer cylindrical walls. The mean motion in this case being evidently symmetrical about the axis of the pipe, theoretical discussion of the distribution of mean velocity across any cross-section seems to be comparatively easy.

Since it is in general of great interest to investigate the applicability of the modified vorticity transport and the momentum transport theories of turbulent motion to the case of turbulent flow through straight pipes of non-circular cross-sections, the present writers have

(1) S. GOLDSTEIN, The Similarity Theory of Turbulence, and Flow between Parallel Planes and through Pipes. Proc. Roy. Soc., London, A **159** (1937), 473-496.

(2) For example, J. NIKURADSE, Untersuchungen über turbulente Strömungen in nicht kreisförmigen Rohren. Ing.-Arch., **1** (1930), 306-332. In this paper, the experimental results for the velocity distribution in turbulent flow through pipes of circular cross-section with one or two grooves are also given.

(3) V. MIKRJUKOV, A Study of the Turbulent Flow of a Fluid in a Straight Pipe of Annular Cross-section. Techn. Phys. USSR, **4** (1937), 961-977.

calculated the distribution of mean velocity in turbulent flow through a pipe of annular cross-section on the basis of these two theories and have compared the calculated results with the observed results of MIKRJUKOV. The object of the present paper is to describe the results of our mathematical analysis, together with the result of comparison of the calculated results with observations.

II. Mikrjukov's Experimental Results.

§ 2. Before entering into the mathematical analysis it will be convenient to describe briefly the principal results of MIKRJUKOV's experiments.

Taking two circular tubes made of seamless brass, MIKRJUKOV constructed a pipe of annular cross-section bounded by two concentric circles. The external diameter of the inner tube was 4·5 cm, while the internal diameter of the outer tube was 8·3 cm. Thus, the distance between the inner and outer walls of the annular space was 1·9 cm. Also, the length of the pipe was 4·3 metres.

Water was used as the fluid and it was made to flow through the pipe under pressure at various constant values of the pressure gradient along the pipe. The distribution of mean velocity was measured across such a cross-section of the pipe where the turbulence was conceived to be fully developed. Use was made of a specially prepared PITOT tube, which was made of platinum and consisted of two small tubes, the internal diameter of the inner tube and the external diameter of the outer one being 0·35 and 0·5 mm respectively. The movement of the PITOT tube across the annular space was attained by means of a micrometer screw.

MIKRJUKOV's observed results for the velocity distribution obtained in four cases where the pressure gradient $\partial P/\partial x$ was $-132{\cdot}7$, $-165{\cdot}6$, $-198{\cdot}4$ and $-277{\cdot}6$ dynes/cm^3 respectively are reproduced in Fig. 1, x being measured along the axis of the pipe in the direction of mean flow. In this figure, the ordinate gives the observed values of mean velocity U in cm/sec, and the abscissa shows the distance $(r-R_1)$ in centimeters measured from the inner wall, where r denotes the distance of any point in the annular space from the axis of the pipe and R_1 the radius of the inner cylindrical wall so that $R_1=\frac{1}{2}\times 4{\cdot}5=2{\cdot}25$ cm. The observed velocities are shown by ●, ○, ▲ and × respectively, corresponding, in this order, with the four values 132·7, 165·6, 198·4 and 277·6 dynes/cm³ of $-\partial P/\partial x$.

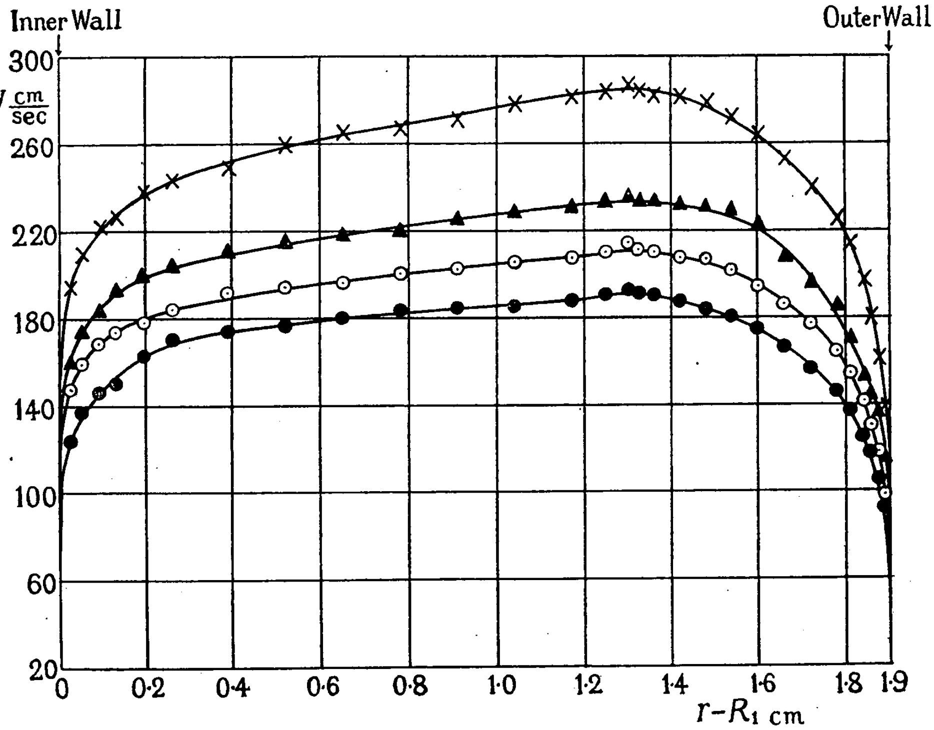

Fig. 1. Observed Velocity Distribution in the Case of Turbulent Flow (MIKRJUKOV).

Observed values: ●, when $-\partial P/\partial x$ = 132·7 dynes/cm³;
○, when $-\partial P/\partial x$ = 165·6 dynes/cm³;
▲, when $-\partial P/\partial x$ = 198·4 dynes/cm³;
×, when $-\partial P/\partial x$ = 277·6 dynes/cm³.

From this figure it will be seen that the distribution of mean velocity across any cross-section is remarkably asymmetric in the case of fully developed turbulent flow. The position of maximum velocity is not situated midway between the walls, but it is greatly shifted towards the outer wall. In MIKRJUKOV's experiment, the distance of the position of maximum velocity from the inner wall is always 1·3 cm, irrespective of the value of the pressure gradient, and this value, 1·3 cm, is about two-thirds of the distance, 1·9 cm, between the inner and outer walls. The fact that the position of maximum velocity is fixed independently of the value of the pressure gradient seems to be worthy of notice.

The observed asymmetry in the distribution of mean velocity in turbulent flow through an annular pipe indicates evidently that the intensities of turbulent skin friction on the inner and outer boundary walls of the annular space are different from each other. Analysing his observed results for the velocity distribution, MIKRJUKOV found that in his apparatus the intensity of turbulent skin friction on the inner wall was approximately three times greater than that on the outer wall.

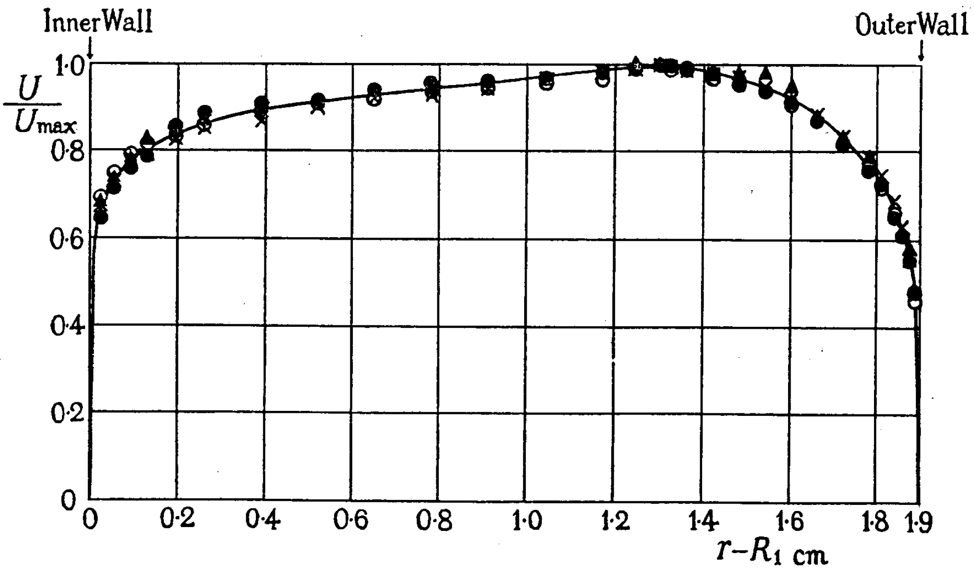

Fig. 2. Observed Velocity Distribution in the Case of Turbulent Flow (MIKRJUKOV).

Observed values: ●, when $-\partial P/\partial x = 132{\cdot}7$ dynes/cm³;
○, when $-\partial P/\partial x = 165{\cdot}6$ dynes/cm³;
▲, when $-\partial P/\partial x = 198{\cdot}4$ dynes/cm³;
×, when $-\partial P/\partial x = 277{\cdot}6$ dynes/cm³.

Further, denoting the maximum mean velocity by U_{max}, the values of the ratio U/U_{max} have been calculated using the observed values of U. Fig. 2 shows those values of U/U_{max} against the distance $(r-R_1)$ from the inner wall. It should be noticed that a single smooth curve can be drawn through the points.

§ 3. Now, it will be of interest to show here, as an addendum, the theoretical velocity distribution in laminar flow flowing under pressure through a straight pipe of annular cross-section, for the sake of comparison with the preceding observed distribution of mean velocity for the case of turbulent flow. As in the case of laminar flow through a circular pipe, such a theoretical velocity distribution can be realized in actual flow when the REYNOLDS number of the flow assumes reasonably small values so that the flow through the annular pipe is certainly laminar.

If we denote the radii of the inner and outer cylindrical walls of a straight pipe of annular cross-section by R_1 and R_2 respectively, the velocity distribution is given by[1]

$$U=\frac{1}{4\mu}\frac{\partial P}{\partial x}\left\{r^2-\frac{R_2^2\log(r/R_1)-R_1^2\log(r/R_2)}{\log(R_2/R_1)}\right\}, \qquad (1)$$

or, expressing in a somewhat different form,

$$U=-\frac{1}{4\mu}\frac{\partial P}{\partial x}\left\{R_1^2-r^2+\frac{R_2^2-R_1^2}{\log(R_2/R_1)}\log\frac{r}{R_1}\right\}, \qquad (2)$$

where r is the distance of any point in the annular space from the axis of the pipe, μ the coefficient of viscosity of the fluid, and $\partial P/\partial x$ the pressure gradient, x being taken along the axis of the pipe in the direction of flow.

The distance R_0 of the position of maximum velocity from the axis of the pipe is obtained from the equation $(dU/dr)_{r=R_0}=0$. Thus, we readily have

$$R_0=\sqrt{\frac{R_2^2-R_1^2}{2\log(R_2/R_1)}}. \qquad (3)$$

(1) H. LAMB, Hydrodynamics, 6th edit. (1932), 586.

Also, using this value of R_0, the maximum velocity U_{max} is given by

$$U_{max} = -\frac{1}{4\mu}\frac{\partial P}{\partial x}\left\{R_1^2 - R_0^2 + \frac{R_2^2 - R_1^2}{\log(R_2/R_1)}\log\frac{R_0}{R_1}\right\}. \tag{4}$$

Taking $R_1 = \frac{1}{2} \times 4{\cdot}5 = 2{\cdot}25$ cm and $R_2 = \frac{1}{2} \times 8{\cdot}3 = 4{\cdot}15$ cm, as in MIKRJUKOV's apparatus, the value of R_0 has been first calculated by (3) and we get approximately

$$R_0 = 3{\cdot}15 \text{ cm}. \tag{5}$$

Since $R_0 - R_1 = 0{\cdot}90$ cm and $R_2 - R_0 = 1{\cdot}00$ cm, it is seen that the position of maximum velocity lies almost midway between the inner and outer walls of the pipe.

Also, the values of U for various values of r have been calculated(1). The curve of U/U_{max} plotted against the distance, $r - R_1$, of any point in the annular space measured from the inner wall is shown in Fig. 3.

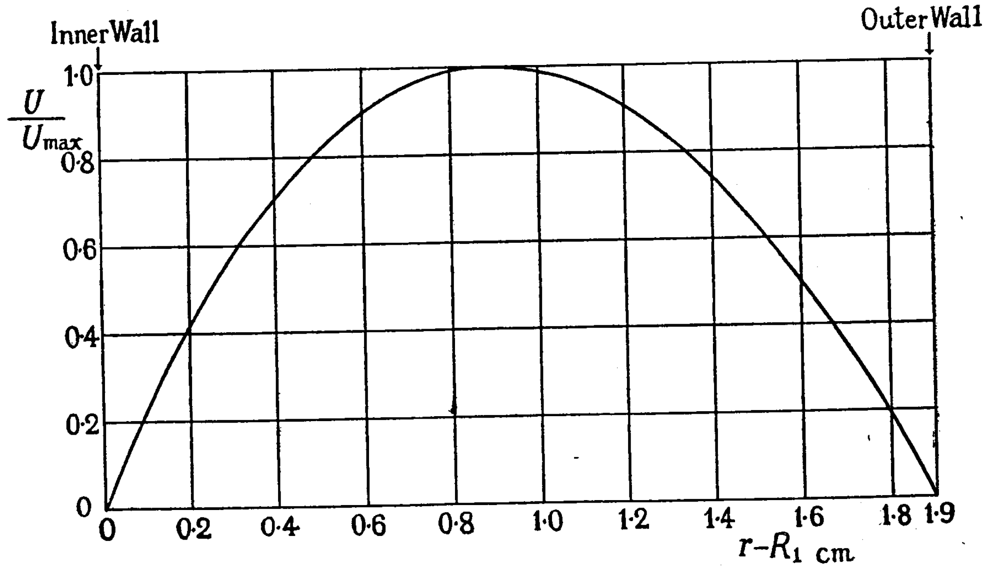

Fig. 3. Calculated Velocity Distribution in the Case of Laminar Flow.

(1) The numerical calculations in this paragraph have been kindly carried out by Mr. H. UMEMOTO.

From this figure it will be seen that the velocity distribution in laminar flow through an annular pipe is nearly symmetrical, and this result is of great interest when compared with the remarkable asymmetry of the distribution of mean velocity in turbulent flow. Further, it will be noticed that while the position of maximum velocity is greatly shifted towards the outer wall in the case of turbulent flow, it is rather shifted slightly towards the inner wall in the case of laminar flow.

The approximate symmetry of the velocity distribution in the case of laminar flow seems to suggest that the intensities of laminar skin friction on the walls are nearly equal to each other. In fact, if we denote the intensities of laminar skin friction on the inner and outer walls by τ_1 and τ_2 respectively, an easy calculation shows that

$$\frac{\tau_1}{\tau_2} = 1{\cdot}23 \qquad (6)$$

approximately.

III. Application of the Modified Vorticity Transport Theory.

§ 4. We shall now proceed to the calculation of the distribution of mean velocity U in turbulent flow of a fluid flowing under pressure through a straight pipe of annular cross-section. Taking the axis of the pipe as the x-axis, we denote the cylindrical coordinates by r, θ and x. Also, we denote, as before, the radii of the inner and outer cylindrical walls of the pipe by R_1 and R_2 respectively and the distance of the position of maximum velocity from the x-axis by R_0.

Then, when the mean motion is symmetrical about the axis of the pipe, the equation of mean motion according to the modified vorticity transport theory is[1]

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \overline{l_r v}\frac{d^2U}{dr^2} + \overline{l_\theta w}\frac{1}{r}\frac{dU}{dr}, \qquad (7)$$

(1) G. I. Taylor, loc. cit., p. 502.

where v, w are the turbulent velocity components in the r- and θ-directions respectively and l_r, l_θ are the r- and θ-components of the mixing length. ρ is the density of the fluid.

If the mixing length l is small and turbulence isotropic, we have

$$\overline{l_r v} = \overline{l_\theta w}\,. \tag{8}$$

We assume, with PRANDTL and TAYLOR, that the components of eddy motion are proportional to $l dU/dr$. Then, remembering that $\overline{l_r v}$, $\overline{l_\theta w}$ are considered as essentially positive and that $dU/dr > 0$ in the region $R_1 \leqq r \leqq R_0$ and $dU/dr < 0$ in the region $R_0 \leqq r \leqq R_2$, we put

$$\overline{l_r v} = \overline{l_\theta w} = \pm l^2 \frac{dU}{dr}, \quad \begin{Bmatrix} +,\ R_1 \leqq r \leqq R_0\,; \\ -,\ R_0 \leqq r \leqq R_2\,. \end{Bmatrix} \tag{9}$$

Equation (7) then reduces to

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \pm l^2 \frac{dU}{dr}\left(\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr}\right), \quad \begin{Bmatrix} +,\ R_1 \leqq r \leqq R_0\,; \\ -,\ R_0 \leqq r \leqq R_2\,. \end{Bmatrix} \tag{10}$$

Also, we assume that in the region $R_1 \leqq r \leqq R_0$, the mixing length l is proportional to the distance from the inner wall, while in the region $R_0 \leqq r \leqq R_2$ it is proportional to the distance from the outer wall. Thus, we put

$$l = \begin{Bmatrix} B_1(r - R_1)\,, & (R_1 \leqq r \leqq R_0)\,; \\ B_2(R_2 - r)\,, & (R_0 \leqq r \leqq R_2)\,, \end{Bmatrix} \tag{11}$$

where B_1, B_2 are constants. In TAYLOR's paper, similar assumption has been made for the mixing length in turbulent flow through a circular pipe.

Next, if we denote the intensities of turbulent skin friction on the inner and outer cylindrical walls by τ_1 and τ_2 respectively, we easily have

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{2\pi R_1 \tau_1 + 2\pi R_2 \tau_2}{\pi\rho(R_2^2 - R_1^2)}\,. \tag{12}$$

As mentioned previously MIKRJUKOV has found experimentally that in turbulent flow through an annular pipe the maximum velocity occurs at nearly the same position, irrespective of the values of the pressure gradient; in other words, the value of R_0 is independent of the values of $\partial P/\partial x$. Remembering this fact, we may put therefore

$$-\frac{1}{\rho}\frac{\partial P}{\partial x}=\frac{2(R_1\tau_1+R_2\tau_2)}{\rho(R_2^2-R_1^2)}\equiv\frac{2U_\tau^2}{R_0}, \tag{13}$$

U_τ being a friction velocity.

Combining (10) with (13), we have finally

$$\frac{2U_\tau^2}{R_0}=\mp l^2\frac{dU}{dr}\left(\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}\right),\quad \begin{Bmatrix}-,\ R_1\leqq r\leqq R_0;\\ +,\ R_0\leqq r\leqq R_2.\end{Bmatrix} \tag{14}$$

In what follows, the region in which $R_1\leqq r\leqq R_0$ is called "the inner region", whilst the region $R_0\leqq r\leqq R_2$ is called "the outer region", for the sake of convenience. Since the equation of mean motion takes different forms for the inner and outer regions, it is necessary to discuss these two regions separately.

A. Inner Region.

§ 5. For the inner region in which $R_1\leqq r\leqq R_0$, the equation of mean motion is

$$\frac{2U_\tau^2}{R_0}=-l^2\frac{dU}{dr}\left(\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}\right), \tag{15}$$

and according to our assumption the mixing length is given by

$$l=B_1(r-R_1). \tag{16}$$

Inserting (16) into (15) and writing

$$\xi=\frac{r}{R_1},\qquad u=\frac{U}{U_\tau}, \tag{17}$$

we have

$$2=-B_1^2\frac{R_0}{R_1}(\xi-1)^2\frac{du}{d\xi}\left(\frac{d^2u}{d\xi^2}+\frac{1}{\xi}\frac{du}{d\xi}\right), \tag{18}$$

which can also be written in the form:

$$\frac{d}{d\xi}\left\{\left(\xi\frac{du}{d\xi}\right)^2\right\} = -\frac{4R_1}{B_1^2R_0}\frac{\xi^2}{(\xi-1)^2}. \qquad (19)$$

This equation can be integrated once and if we determine the constant of integration thus introduced by the condition that $du/d\xi = 0$ when $\xi = R_0/R_1$, which corresponds with the condition that $U = U_{max}$ at $r = R_0$, we have

$$\left(\xi\frac{du}{d\xi}\right)^2 = \frac{4R_1}{B_1^2R_0}\left\{\frac{R_0}{R_1}-\xi+2\log\frac{R_0/R_1-1}{\xi-1}+\frac{1}{\xi-1}-\frac{1}{R_0/R_1-1}\right\},$$

and remembering that $du/d\xi > 0$ in the inner region in which $R_1 \leqq r \leqq R_0$ and therefore $1 \leqq \xi \leqq R_0/R_1$, we get

$$\frac{du}{d\xi} = \frac{2}{B_1}\sqrt{\frac{R_1}{R_0}}\left\{\frac{R_0}{R_1}-\xi+2\log\frac{R_0/R_1-1}{\xi-1}+\frac{1}{\xi-1}-\frac{1}{R_0/R_1-1}\right\}^{\frac{1}{2}}\frac{1}{\xi}. \qquad (20)$$

Integrating this again and determining the integration constant by the condition that $u = U_{max}/U_\tau$ when $\xi = R_0/R_1$, i.e., $U = U_{max}$ at $r = R_0$, we have, since $u = U/U_\tau$,

$$\frac{U_{max}-U}{U_\tau} = \frac{2}{B_1}\sqrt{\frac{R_1}{R_0}}\int_\xi^{R_0/R_1}\left\{\frac{R_0}{R_1}-\xi+2\log\frac{R_0/R_1-1}{\xi-1}+\frac{1}{\xi-1}-\frac{1}{R_0/R_1-1}\right\}^{\frac{1}{2}}\frac{d\xi}{\xi}. \qquad (21)$$

If, for simplicity, we put

$$f_1(\xi) = \left\{\frac{R_0}{R_1}-\xi+2\log\frac{R_0/R_1-1}{\xi-1}+\frac{1}{\xi-1}-\frac{1}{R_0/R_1-1}\right\}^{\frac{1}{2}}\frac{1}{\xi}, \qquad (22)$$

and

$$F_1(\xi) = 2\sqrt{\frac{R_1}{R_0}}\int_\xi^{R_0/R_1} f_1(\xi)d\xi, \qquad (23)$$

this may also be put in the form:

$$\frac{U_{max}-U}{U_\tau} = \frac{1}{B_1}F_1(\xi). \qquad (24)$$

Taking $R_1=2{\cdot}25$ cm and $R_0=3{\cdot}55$ cm as in MIKRJUKOV's experiment, the values of the integral $\int_{\xi}^{R_0/R_1} f_1(\xi)d\xi$ for various values of ξ ranging from 1 to $R_0/R_1=1{\cdot}578$ have been calculated by the method of numerical integration. Also, the values of the function $F_1(\xi)$ have been obtained. The results are shown in Table I.

TABLE

$\xi=\frac{r}{R_1}$	$\int^{R_0/R_1} f_1(\xi)d\xi$	$F_1(\xi)$
1	1·279	2·036
1·01	1·077	1·715
1·02	0·991	1·578
1·03	0·925	1·473
1·04	0·870	1·385
1·05	0·821	1·307
1·10	0·631	1·005
1·15	0·491	0·782
1·20	0·380	0·605
1·25	0·289	0·460
1·30	0·214	0·341
1·35	0·151	0·240
1·40	0·099	0·158
1·45	0·058	0·092
1·50	0·027	0·043
1·55	0·005	0·008
1·578	0	0

In order to compare the calculated results with MIKRJUKOV's observations, experimental values of $(U_{max}-U)/U_\tau$ have been obtained from MIKRJUKOV's observed values of U, the values of U_τ having been calculated by (13) using the values 132·7, 165·6, 198·4 and 277·6 dynes/cm^3 for $-\partial P/\partial x$[1]. The observed values of $(U_{max}-U)/U_\tau$ thus obtained are shown in Fig. 4.

(1) The value of ρ has been taken to be unity, since the fluid used in MIKRJUKOV's experiments was water.

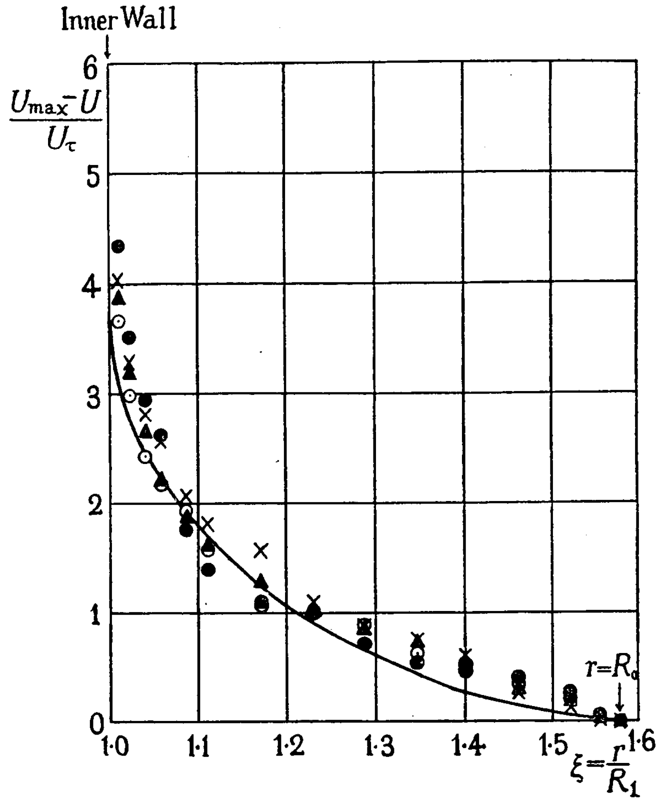

Fig. 4. Inner Region. Modified Vorticity Transport Theory. $B_1 = 0{\cdot}557$.

The theoretical curve for $(U_{max}-U)/U_\tau$ is made to fit MIKRJUKOV's observations at $\xi = 1{\cdot}15$, i.e., at the point $r = 1{\cdot}15\,R_1$, by taking $B_1 = 0{\cdot}557$, the distance of this point, $r = 1{\cdot}15\,R_1$, from the position of maximum velocity, $r = R_0$, being approximately seven-tenths of the distance between the inner wall and the position of maximum velocity. The theoretical curve when $B_1 = 0{\cdot}557$ is shown in Fig. 4. It will be seen that there is fairly good agreement with observation over the whole range except very close to the inner wall as well as in the neighbourhood of the position of maximum velocity.

B. Outer Region.

§ 6. In the case of the outer region in which $R_0 \leqq r \leqq R_2$, the equation of mean motion is

$$\frac{2U_\tau^2}{R_0} = l^2\frac{dU}{dr}\left(\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}\right), \tag{25}$$

and the assumed form for the mixing length is

$$l = B_2(R_2-r). \tag{26}$$

Inserting (26) into (25) and writing

$$\xi = \frac{r}{R_2}, \qquad u = \frac{U}{U_\tau}, \tag{27}$$

we have

$$2 = B_2^2\frac{R_0}{R_2}(1-\xi)^2\frac{du}{d\xi}\left(\frac{d^2u}{d\xi^2}+\frac{1}{\xi}\frac{du}{d\xi}\right), \tag{28}$$

which can also be put in the form:

$$\frac{d}{d\xi}\left\{\left(\xi\frac{du}{d\xi}\right)^2\right\} = \frac{4R_2}{B_2^2R_0}\frac{\xi^2}{(1-\xi)^2}. \tag{29}$$

This may be integrated once, and if the constant of integration is determined by the condition that $du/d\xi=0$ when $\xi=R_0/R_2$, which is equivalent to the condition that $U=U_{\max}$ at $r=R_0$, the resulting equation is

$$\left(\xi\frac{du}{d\xi}\right)^2 = \frac{4R_2}{B_2^2R_0}\left\{\xi-\frac{R_0}{R_2}+2\log\frac{1-\xi}{1-R_0/R_2}+\frac{1}{1-\xi}-\frac{1}{1-R_0/R_2}\right\},$$

and remembering that $du/d\xi<0$ in the outer region, we have

$$\frac{du}{d\xi} = -\frac{2}{B_2}\sqrt{\frac{R_2}{R_0}}\left\{\xi-\frac{R_0}{R_2}+2\log\frac{1-\xi}{1-R_0/R_2}+\frac{1}{1-\xi}-\frac{1}{1-R_0/R_2}\right\}^{\frac{1}{2}}\frac{1}{\xi}. \tag{30}$$

Integrating this again and determining the integration constant thus introduced by the condition that $u=U_{\max}/U_\tau$ when $\xi=R_0/R_2$, i.e., $U=U_{\max}$ at $r=R_0$, we have, since $u=U/U_\tau$,

$$\frac{U_{\max}-U}{U_\tau} = \frac{2}{B_2}\sqrt{\frac{R_2}{R_0}}\int_{R_0/R_2}^{\xi}\left\{\xi-\frac{R_0}{R_2}+2\log\frac{1-\xi}{1-R_0/R_2}+\frac{1}{1-\xi}-\frac{1}{1-R_0/R_2}\right\}^{\frac{1}{2}}\frac{d\xi}{\xi}. \tag{31}$$

If we put

$$f_2(\xi)=\left\{\xi-\frac{R_0}{R_2}+2\log\frac{1-\xi}{1-R_0/R_2}+\frac{1}{1-\xi}-\frac{1}{1-R_0/R_2}\right\}^{\frac{1}{2}}\frac{1}{\xi},\quad(32)$$

and

$$F_2(\xi)=2\sqrt{\frac{R_2}{R_0}}\int_{R_0/R_2}^{\xi}f_2(\xi)d\xi,\quad(33)$$

the expression for $(U_{\max}-U)/U_\tau$ may also be put in the form:

$$\frac{U_{\max}-U}{U_\tau}=\frac{1}{B_2}F_2(\xi).\quad(34)$$

Taking $R_0=3{\cdot}55$ cm and $R_2=4{\cdot}15$ cm as in MIKRJUKOV's experiment, the numerical values of the integral $\int_{R_0/R_2}^{\xi}f_2(\xi)d\xi$ for various values of ξ ranging from $R_0/R_2=0{\cdot}855$ to 1 have been calculated by the method of numerical integration. Also, the values of the function $F_2(\xi)$ have been obtained. The results are shown in Table II.

TABLE II.

$\xi=\frac{r}{R_2}$	$\int_{R_0/R_2}^{\xi}f_2(\xi)d\xi$	$F_2(\xi)$
0·997	0·470	1·016
0·99	0·383	0·829
0·98	0·309	0·668
0·97	0·256	0·554
0·96	0·211	0·456
0·95	0·173	0·374
0·94	0·140	0·304
0·93	0·112	0·243
0·92	0·088	0·189
0·91	0·066	0·143
0·90	0·048	0·103
0·89	0·032	0·069
0·88	0·019	0·040
0·87	0·008	0·018
0·86	0·0015	0·0031
0·855	0	0

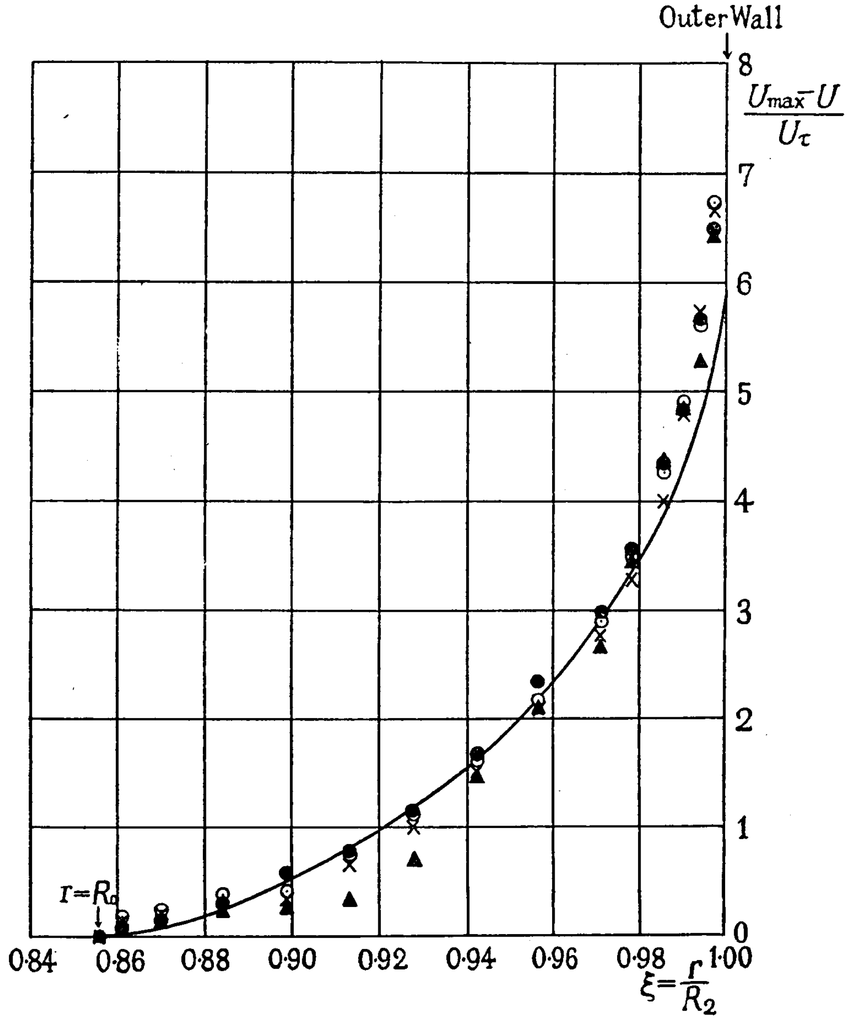

Fig. 5. Outer Region. Modified Vorticity Transport Theory. $B_2 = 0{\cdot}194$.

The calculated results are compared with MIKRJUKOV's observed results in Fig. 5. The experimental values of $(U_{max}-U)/U_\tau$ have been found using the observed values of U as in the case of the inner region. The theoretical curve for $(U_{max}-U)/U_\tau$ is made to fit the observation at $\xi=0{\cdot}96$, i.e., at the point $r=0{\cdot}96R_2$, by taking $B_2=0{\cdot}194$, the distance of this point, $r=0{\cdot}96R_2$, from the position of maximum velocity being nearly seven-tenths of the distance between the position of maximum velocity and the outer wall of the pipe. It will be seen that there is very good agreement with observation over the whole range except very close to the outer wall.

IV. Application of the Momentum Transport Theory.

§ 7. Next, we shall calculate the distribution of mean velocity in turbulent flow through an annular pipe on the basis of the momentum transport theory.

According to this theory, the equation of mean motion is, with the same notation as before,

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \mp\frac{d}{dr}\left\{l^2\left(\frac{dU}{dr}\right)^2\right\}, \quad \begin{Bmatrix} -, R_1 \leqq r \leqq R_0; \\ +, R_0 \leqq r \leqq R_2, \end{Bmatrix} \tag{35}$$

where the negative sign on the right-hand side is taken for the inner region $R_1 \leqq r \leqq R_0$, while the positive sign for the outer region $R_0 \leqq r \leqq R_2$.

We put, as in (13),

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{2U_\tau^2}{R_0}. \tag{36}$$

Then, combining this with (35) we have

$$\frac{d}{dr}\left\{l^2\left(\frac{dU}{dr}\right)^2\right\} = \mp\frac{2U_\tau^2}{R_0}, \quad \begin{Bmatrix} -, R_1 \leqq r \leqq R_0; \\ +, R_0 \leqq r \leqq R_2. \end{Bmatrix} \tag{37}$$

Equations (37) may be integrated to give $l^2(dU/dr)^2$, and if the constant of integration thus introduced is determined by the condition that $dU/dr=0$ at $r=R_0$, we have

$$l^2\left(\frac{dU}{dr}\right)^2 = \mp\frac{2U_\tau^2}{R_0}(r-R_0), \quad \begin{Bmatrix} -, R_1 \leqq r \leqq R_0; \\ +, R_0 \leqq r \leqq R_2. \end{Bmatrix} \tag{38}$$

As in the preceding application of the modified vorticity transport theory, the mixing length is assumed in the following form:

$$l=\begin{cases} B_1(r-R_1), & (R_1 \leqq r \leqq R_0); \\ B_2(R_2-r), & (R_0 \leqq r \leqq R_2). \end{cases} \tag{39}$$

Since the equation of mean motion takes different forms for the inner and outer regions, it is necessary to discuss these two regions separately.

A. Inner Region.

§ 8. For the inner region $R_1 \leqq r \leqq R_0$ the equation of mean motion is, by (38),

$$l^2\left(\frac{dU}{dr}\right)^2 = -\frac{2U_\tau^2}{R_0}(r-R_0), \tag{40}$$

and the assumed form for the mixing length is

$$l = B_1(r-R_1). \tag{41}$$

Inserting (41) into (40) and remembering that $r \leqq R_0$ and $dU/dr > 0$ in the inner region, we get

$$\frac{dU}{dr} = \frac{U_\tau}{B_1}\sqrt{\frac{2}{R_0}}\frac{\sqrt{R_0-r}}{r-R_1}. \tag{42}$$

This equation can be integrated immediately, and if we determine the constant of integration by the condition that $U=U_{\max}$ at $r=R_0$, we have

$$\frac{U_{\max}-U}{U_\tau} = \frac{1}{B_1}\sqrt{2\left(1-\frac{R_1}{R_0}\right)}\left\{\log\frac{1+\sqrt{\dfrac{R_0-r}{R_0-R_1}}}{1-\sqrt{\dfrac{R_0-r}{R_0-R_1}}} - 2\sqrt{\frac{R_0-r}{R_0-R_1}}\right\}, \tag{43}$$

or, writing $\xi=r/R_1$ and

$$F_3(\xi) = \sqrt{2\left(1-\frac{R_1}{R_0}\right)}\left\{\log\frac{1+\sqrt{\dfrac{R_0/R_1-\xi}{R_0/R_1-1}}}{1-\sqrt{\dfrac{R_0/R_1-\xi}{R_0/R_1-1}}} - 2\sqrt{\frac{R_0/R_1-\xi}{R_0/R_1-1}}\right\}, \tag{44}$$

this may be written in the form:

$$\frac{U_{max}-U}{U_\tau} = \frac{1}{B_1} F_3(\xi) . \qquad (45)$$

Taking $R_1=2{\cdot}25$ cm and $R_0=3{\cdot}55$ cm as in MIKRJUKOV's experiment, the values of the function $F_3(\xi)$ have been calculated for various values of ξ ranging from $1{\cdot}02$ to $R_0/R_1=1{\cdot}578$. The results are given in Table III.

TABLE III.

$\xi=\frac{r}{R_1}$	$F_3(\xi)$
1·02	2·368
1·03	2·029
1·04	1·790
1·05	1·607
1·10	1·052
1·15	0·744
1·20	0·538
1·25	0·389
1·30	0·276
1·35	0·188
1·40	0·121
1·45	0·069
1·50	0·031
1·55	0·006
1·578	0

The calculated results are compared with MIKRJUKOV's observations in Fig. 6. The theoretical curve is made to fit the observation at $\xi=1{\cdot}15$, as before, by taking $B_1=0{\cdot}530$.

Comparing Fig. 4 with Fig. 6, it will be seen that the modified vorticity transport theory gives results in better agreement with observation than does the momentum transport theory.

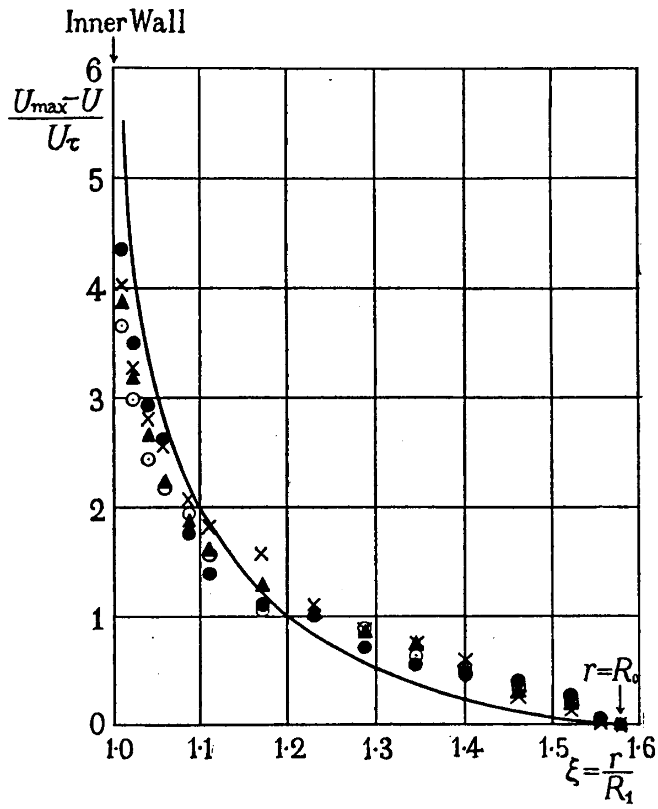

Fig. 6. Inner Region. Momentum Transport Theory. $B_1 = 0{\cdot}530$.

B. Outer Region.

§ 9. Finally, the equation of mean motion for the outer region is

$$l^2\left(\frac{dU}{dr}\right)^2 = \frac{2U_\tau^2}{R_0}(r-R_0)\,, \tag{46}$$

and according to our assumption the mixing length takes the form:

$$l = B_2(R_2-r)\,. \tag{47}$$

Putting (47) in (46) and remembering that $r \geqq R_0$ and $dU/dr < 0$ in the outer region, we get

$$\frac{dU}{dr} = -\frac{U_\tau}{B_2}\sqrt{\frac{2}{R_0}}\frac{\sqrt{r-R_0}}{R_2-r}\,. \tag{48}$$

This may be integrated to give U, and if the constant of integration is determined by the condition that $U=U_{max}$ at $r=R_0$, we have

$$\frac{U_{max}-U}{U_\tau}=\frac{1}{B_2}\sqrt{2\left(\frac{R_2}{R_0}-1\right)}\left\{\log\frac{1+\sqrt{\dfrac{r-R_0}{R_2-R_0}}}{1-\sqrt{\dfrac{r-R_0}{R_2-R_0}}}-2\sqrt{\frac{r-R_0}{R_2-R_0}}\right\}, \quad (49)$$

or, writing $\xi=r/R_2$ and

$$F_4(\xi)=\sqrt{2\left(\frac{R_2}{R_0}-1\right)}\left\{\log\frac{1+\sqrt{\dfrac{\xi-R_0/R_2}{1-R_0/R_2}}}{1-\sqrt{\dfrac{\xi-R_0/R_2}{1-R_0/R_2}}}-2\sqrt{\frac{\xi-R_0/R_2}{1-R_0/R_2}}\right\}, \quad (50)$$

this may also be put in the form:

$$\frac{U_{max}-U}{U_\tau}=\frac{1}{B_2}F_4(\xi). \quad (51)$$

Taking $R_0=3{\cdot}55$ cm and $R_2=4{\cdot}15$ cm as in MIKRJUKOV's experiment, we have calculated the values of $F_4(\xi)$ for various values of ξ ranging from $R_0/R_2=0{\cdot}855$ to 0·997. The results are shown in Table IV.

TABLE IV.

$\xi=\dfrac{r}{R_2}$	$F_4(\xi)$
0·997	1·902
0 99	1·217
0·98	0·834
0·97	0·620
0·96	0·474
0·95	0·366
0·94	0·283
0·93	0·216
0·92	0·162
0·91	0·118
0·90	0·082
0·89	0·053
0·88	0·030
0·87	0·013
0·86	0·002
0·855	0

The calculated results are compared with MIKRJUKOV's experimental results in Fig. 7. The theoretical curve is made to fit the observation at $\xi=0{\cdot}96$, as in § 6, by taking $B_2=0{\cdot}202$.

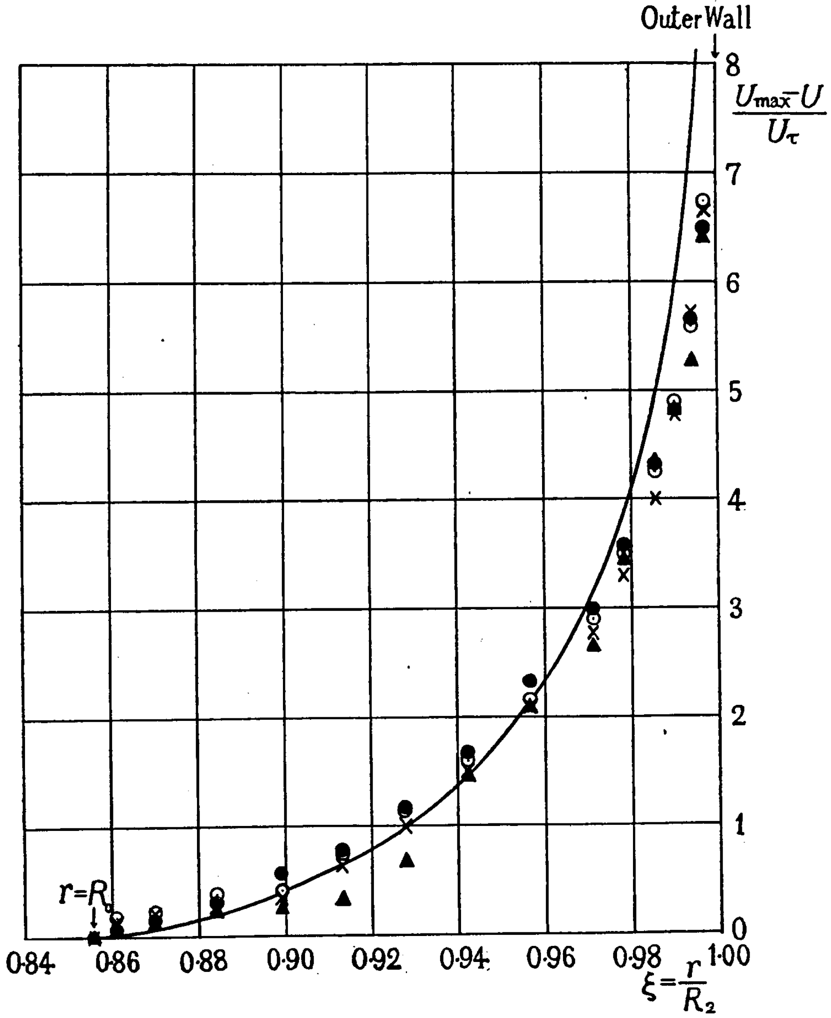

Fig. 7. Outer Region. Momentum Transport Theory. $B_2=0{\cdot}202$.

Comparing Fig. 5 with Fig. 7, it will be seen that as in the case of the inner region, the modified vorticity transport theory gives results in better agreement with observation than does the momentum transport theory.

V. Summary.

§ 10. In the present paper, the distribution of mean velocity in turbulent flow of a fluid flowing under pressure through a straight pipe of annular cross-section is discussed theoretically on the basis of both the modified vorticity transport and the momentum transport theories of turbulent motion. To do this, it is assumed that the turbulence is isotropic and that the mixing length is proportional to the distance from the inner wall in the case of the inner region, whilst in the outer region it is proportional to the distance from the outer wall.

The calculated results are compared with the results of MIKRJUKOV's recent experiments. It is found that the modified vorticity transport theory gives results in better agreement with observation than does the momentum transport theory for turbulent flow through a straight pipe of annular cross-section. In this connection it may be mentioned that a similar conclusion has been obtained by TAYLOR for the case of turbulent flow through a straight circular pipe.

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