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抄　　錄

運動量輸送の理論の環狀斷面の管に沿うて流れる渦亂流に對する應用に就いて

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吾々は，前論文（報告第 180 號）に於いて，渦動度輸送の變形理論及び運動量輸送の理論を應用して，環狀斷面の眞直ぐな管に沿うて流れる渦亂流に於ける平均速度の分布を計算し，その結果を MIKRJUKOV の實測結果と比較したが，その樣にして吾々の到達した結論は，環狀斷面の管に沿うて流れる渦亂流の場合に於いても，圓管に沿うて流れる渦亂流の場合と同樣に，渦動度輸送の變形理論の方が運動量輸送の理論よりも實驗結果によく合ふ結果を與へるといふのであつた．

ところが，最近に至つて，前論文に於ける計算のうち，運動量輸送の理論の應用に關する部分は，殘念ながら多少近似的に過ぎることが見出された．即ち，吾々は環狀斷面の管に沿うての流れが恰も二次元的であるかの如く取扱つてゐたことが見出されたのである．從つて，前論文に於いて到達した結果が果してそのままでよいか或は變更を必要とするのではないかといふ樣な疑問も起つて來た．尤も，斯樣な近似的取扱ひと雖も恐らく全然誤つてゐるものではなく，MIKRJUKOV の實驗に於ける樣に，環狀領域の幅がかなり狹い場合には，却つて許容されるものの樣にも思はれるけれども，運動量輸送の理論を使つて，問題をもつと妥當に且つもつと正確に取扱ふことは望ましいことである．

そこで，吾々は，運動量輸送の理論を再び應用して，環狀斷面の管に沿うて流れる渦亂流に於ける平均速度の分布を計算し直し，その計算の結果を，前論文に於けると同樣に，MIKRJUKOV の實測結果と比較したが，本論文はそれ等に就いて報告することをその主な目的とする．

計算の結果，本論文に於いて得られた結果と前論文に於いて近似的取扱ひによつて得られた結果との間には，辨別し得る樣な差異が殆んど存在しないことが知れた．從つて，渦動度輸送の變形理論の方が運動量輸送の理論よりも實測結果によく一致する樣な結果を與へるといふ前論文に於いて到達した結論には何等の變更をも必要としない樣に思はれる．

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Note on the Application of the Momentum Transport Theory to the Turbulent Flow through a Straight Pipe of Annular Cross-section.

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I. Introduction.

§ 1. An application of the momentum transport theory of turbulent motion to the calculation of the distribution of mean velocity in turbulent flow of an incompressible fluid flowing under pressure through a straight pipe of annular cross-section has been made in a previous paper(1), and the calculated velocity distribution has been compared with MIKRJUKOV's observation(2). Also, the results have been compared with those obtained on the basis of the modified vorticity transport theory of turbulence.

(1) S. TOMOTIKA and I. IMAI, On the Velocity Distribution in Turbulent Flow through a Straight Pipe of Annular Cross-section. Report Aeron. Res. Inst., Tokyo Imp. Univ., No. 180 (1939).

(2) The essential part of MIKRJUKOV's experimental results has been described in our previous paper above cited.

It has recently been found however that the treatment of the problem in the previous paper is unfortunately somewhat too approximate. In fact, we have treated the flow through an annular pipe as if it were two-dimensional. Although even such an approximate treatment may perhaps not be quite erroneous, but may be rather allowable, at least when the annular region of the pipe is narrow as in the case of MIKRJUKOV's experiment, where the radii of the inner and outer walls of the annular region were 2·25 cm and 4·15 cm respectively so that the breadth of the annular region was 1·9 cm, yet an appropriate exact treatment of the problem is desirable. In the present paper, therefore, an improved calculation of the distribution of mean velocity in turbulent flow of a fluid flowing under pressure through an annular pipe is carried out on the basis of the momentum transport theory.

As in the previous paper, the annular region is divided into two regions, i.e., into the so-called inner and outer regions, and it is assumed that in the inner region the mixing length is proportional to the distance from the inner wall, while in the outer region it is proportional to the distance from the outer wall.

The results of calculation are compared with MIKRJUKOV's observation, and also they are compared with those obtained by the approximate treatment in the previous paper. It is thus found that there is no distinguishable difference between the results obtained by the exact treatment in the present paper and those obtained by the approximate treatment in the previous paper. It seems therefore that the conclusion of the previous paper that the modified vorticity transport theory gives results in better agreement with observation than does the momentum transport theory suffers no alteration.

II. Application of the Momentum Transport Theory to the Turbulent Flow through an Annular Pipe.

§ 2. We consider the turbulent flow of an incompressible fluid flowing under pressure through a straight pipe of annular cross-secion.

Let the radii of the inner and outer walls of the annular space of the pipe be denoted by R_1 and R_2 respectively. Also, let the distance of the position of maximum mean velocity from the axis of the pipe be denoted by R_0. In MIKRJUKOV's experiment, the values of R_1 and R_2 were, as mentioned already, 2·25 cm and 4·15 cm respectively so that the breadth of the annular region was $R_2-R_1 = 1 \cdot 9$ cm. The value of R_0 was 3·55 cm.

Taking the axis of the pipe as the x-axis, we assume that the mean pressure gradient, $-\partial P/\partial x$, along the pipe is constant. Also, the mean motion, with velocity U, is assumed to be steady and symmetrical about the axis of the pipe so that U is a function of r only, where r denotes the distance of any point in the annular region from the axis.

Now, we assume, with PRANDTL, that the REYNOLDS shearing stress τ is, as a first approximation, given by

$$\tau = \rho l^2 \frac{dU}{dr}\left|\frac{dU}{dr}\right|, \tag{1}$$

where ρ is the density of the fluid and l is a mixing length.

Then, remembering that $dU/dr > 0$ in the so-called inner region where $R_1 \leqq r \leqq R_0$ and $dU/dr < 0$ in the so-called outer region where $R_0 \leqq r \leqq R_2$, the equation of mean motion on the momentum transport theory for the inner region is given by

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{1}{r}\frac{d}{dr}\left\{l^2 r\left(\frac{dU}{dr}\right)^2\right\}, \tag{2}$$

while the corresponding equation for the outer region is

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{1}{r}\frac{d}{dr}\left\{l^2 r\left(\frac{dU}{dr}\right)^2\right\}. \tag{3}$$

These two equations may conveniently be combined together, and we have[1]

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \mp\frac{1}{r}\frac{d}{dr}\left\{l^2 r\left(\frac{dU}{dr}\right)^2\right\}, \quad \left\{\begin{array}{l} -,\ R_1 \leqq r \leqq R_0\,; \\ +,\ R_0 \leqq r \leqq R_2\,, \end{array}\right\} \quad (4)$$

where, as indicated, the negative sign on the right-hand side is taken for the inner region $R_1 \leqq r \leqq R_0$, while the positive sign for the outer region $R_0 \leqq r \leqq R_2$.

Next, if we denote the intensities of turbulent skin friction on the inner and outer walls of the pipe by τ_1 and τ_2 respectively, we easily have, by considering the forces acting on the fluid contained in a thin portion with length dx of the annular pipe,

$$-\frac{\partial P}{\partial x} = \frac{2\pi R_1\tau_1 + 2\pi R_2\tau_2}{\pi(R_2^2 - R_1^2)}. \quad (5)$$

As mentioned in the previous paper[2], MIKRJUKOV has found experimentally that in turbulent flow through an annular pipe the maximum velocity occurs at nearly the same position, irrespective of the values of the pressure gradient; in other words, the value of R_0 is independent of the values of $-\partial P/\partial x$. We may put therefore

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{2(R_1\tau_1 + R_2\tau_2)}{\rho(R_2^2 - R_1^2)} = \frac{2U_\tau^2}{R_0}, \quad (6)$$

U_τ being a friction velocity.

Taking this into account, equations (4) now become

$$\frac{1}{r}\frac{d}{dr}\left\{l^2 r\left(\frac{dU}{dr}\right)^2\right\} = \mp\frac{2U_\tau^2}{R_0}, \quad \left\{\begin{array}{l} -,\ R_1 \leqq r \leqq R_0\,; \\ +,\ R_0 \leqq r \leqq R_2\,. \end{array}\right\} \quad (7)$$

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(1) These equations may also be put in the forms:

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \mp\left[\frac{d}{dr}\left\{l^2\left(\frac{dU}{dr}\right)^2\right\} + \frac{1}{r}l^2\left(\frac{dU}{dr}\right)^2\right],$$

and if we neglect the last term $r^{-1}l^2(dU/dr)^2$ on the right-hand side, assuming that r is not small, these equations are reduced to those used in the previous paper. However, such neglection evidently implies, as mentioned already, that we treat the flow through the annular pipe as if it were two-dimensional.

(2) S. TOMOTIKA and I. IMAI, loc. cit.

These equations may be integrated immediately, and if the constant of integration thus introduced is determined by the condition that $dU/dr = 0$ at $r = R_0$, we have

$$l^2 r\left(\frac{dU}{dr}\right)^2 = \mp\frac{U_\tau^2}{R_0}(r^2 - R_0^2), \quad \begin{cases} -, & R_1 \leqq r \leqq R_0; \\ +, & R_0 \leqq r \leqq R_2. \end{cases} \tag{8}$$

Thus, we get

$$l^2\left(\frac{dU}{dr}\right)^2 = \mp\frac{U_\tau^2}{R_0}\frac{r^2 - R_0^2}{r}, \quad \begin{cases} -, & R_1 \leqq r \leqq R_0; \\ +, & R_0 \leqq r \leqq R_2. \end{cases} \tag{9}$$

In order to calculate the velocity U as a function of r on the basis of these equations, we have to make some suitable assumptions for the mixing length l. As in the previous paper, it will be assumed here that in the inner region the mixing length l is proportional to the distance from the inner wall, while in the outer region it is proportional to the distance from the outer wall. Thus, we put

$$l = \begin{cases} B_1(r - R_1), & (R_1 \leqq r \leqq R_0); \\ B_2(R_2 - r), & (R_0 \leqq r \leqq R_2), \end{cases} \tag{10}$$

where B_1 and B_2 are constants.

Since the equation of mean motion takes different forms for the inner and outer regions, these two regions will be discussed separately.

§ 3. It will be of some interest to calculate here, as an addendum, the value of the ratio τ_1/τ_2 of the intensities of turbulent skin friction on the inner and outer walls, by using the values of R_1, R_2 and R_0 in MIKRJUKOV's experiment. Analysing his experimental results, MIKRJUKOV himself found that the value of τ_1 was appoximately three times greater than that of τ_2.

Now, we take, from the annular pipe, a thin portion of length dx bounded by two adjacent planes perpendicular to the axis of the pipe, and we consider that this thin annular portion is divided, as before, into the inner and outer regions by the cylindrical surface of radius

R_0 passing through the position of maximum mean velocity. Then, considering all the forces acting on the fluid contained in the inner region, we get

$$-\frac{\partial P}{\partial x} = \frac{2\pi R_1 \tau_1}{\pi(R_0^2 - R_1^2)} . \qquad (11)$$

Similarly, by considering the forces acting on the fluid in the outer region, we obtain

$$-\frac{\partial P}{\partial x} = \frac{2\pi R_2 \tau_2}{\pi(R_2^2 - R_0^2)} . \qquad (12)$$

Combining these two, we easily get again the previous relation (5).

Elimination of $-\partial P/\partial x$ from the above two equations gives readily the expression for the ratio τ_1/τ_2 in the form:

$$\frac{\tau_1}{\tau_2} = \frac{R_2(R_0^2 - R_1^2)}{R_1(R_2^2 - R_0^2)} . \qquad (13)$$

Thus, knowing the values of R_1, R_2 and R_0, we can calculate the value of τ_1/τ_2 by this formula.

In MIKRJUKOV's experiment, the values of R_1 and R_2 were, as mentioned already, 2·25 cm and 4·15 cm respectively, and the value of R_0 was nearly constant and equal to 3·55 cm, irrespective of the values of the pressure gradient $-\partial P/\partial x$.

Taking these values, the value of τ_1/τ_2 has been calculated by using formula (13). The result is

$$\frac{\tau_1}{\tau_2} = 3{\cdot}01 . \qquad (14)$$

This result should be compared with MIKRJUKOV's result previously mentioned that in his apparatus the intensity of turbulent skin friction on the inner wall was approximately three times greater than that on the outer wall. It is found that the agreement of the calculated result with MIKRJUKOV's result derived from his observation is satisfactory.

A. Inner Region.

§ 4. For the inner region in which $R_1 \leqq r \leqq R_0$, the equation of mean motion is, by (9),

$$l^2\left(\frac{dU}{dr}\right)^2 = -\frac{U_\tau^2}{R_0}\frac{r^2-R_0^2}{r}, \tag{15}$$

and the assumed form for the mixing length is

$$l = B_1(r-R_1). \tag{16}$$

Putting this expression for l in (15) and remembering that $r \leqq R_0$ and $dU/dr > 0$ in the inner region, we get

$$\frac{dU}{dr} = \frac{U_\tau}{B_1\sqrt{R_0}}\frac{\sqrt{R_0^2-r^2}}{\sqrt{r}(r-R_1)}. \tag{17}$$

Thus, integrating this and determining the constant of integration by the condition that $U = U_{\max}$ at $r = R_0$, we get the expression for $(U_{\max}-U)/U_\tau$ in the form:

$$\frac{U_{\max}-U}{U_\tau} = \frac{1}{B_1\sqrt{R_0}}\int_r^{R_0}\frac{\sqrt{R_0^2-r^2}}{\sqrt{r}(r-R_1)}dr, \tag{18}$$

or, putting $\xi = r/R_1$ as in the previous paper,

$$\frac{U_{\max}-U}{U_\tau} = \frac{1}{B_1}\sqrt{\frac{R_1}{R_0}}\int_\xi^{R_0/R_1}\frac{\sqrt{R_0^2/R_1^2-\xi^2}}{\sqrt{\xi}(\xi-1)}d\xi. \tag{19}$$

Further, if we put, for simplicity,

$$f_1(\xi) = \frac{\sqrt{R_0^2/R_1^2-\xi^2}}{\sqrt{\xi}(\xi-1)}, \tag{20}$$

and

$$F_1(\xi) = \sqrt{\frac{R_1}{R_0}}\int_\xi^{R_0/R_1} f_1(\xi)d\xi, \tag{21}$$

the expression for $(U_{max}-U)/U_{\tau}$ may be put in the form :

$$\frac{U_{max}-U}{U_{\tau}}=\frac{1}{B_1}F_1(\xi)\,. \tag{22}$$

Taking $R_1=2{\cdot}25$ cm and $R_0=3{\cdot}55$ cm as in MIKRJUKOV's experiment, the values of the integral $\int_{\xi}^{R_0/R_1}f_1(\xi)d\xi$ as well as of the function $F_1(\xi)$ have been calculated for various values of ξ ranging from 1·01 to $R_0/R_1=1{\cdot}578$ by the method of numerical integration. Some of the results are given in Table I.

TABLE I.

$\xi=\frac{r}{R_1}$	$\int_{\xi}^{R_0/R_1}f_1(\xi)d\xi$	$F_1(\xi)$
1·01	4·094	3·260
1·02	3·263	2·597
1·03	2·782	2·215
1·06	1·979	1·575
1·09	1·526	1·215
1·12	1·218	0·970
1·15	0·988	0·787
1·18	0·808	0·644
1·21	0·663	0·528
1·24	0·543	0·432
1·27	0·442	0·352
1·30	0·357	0·284
1·33	0·285	0·227
1·36	0·223	0·177
1·39	0·170	0·135
1·42	0·125	0·100
1·45	0·087	0·070
1·48	0·056	0·045
1·51	0·031	0·025
1·54	0·012	0·010
1·57	0·0010	0·0008
1·578	0	0

The calculated results are compared with MIKRJUKOV's observation in Fig. 1. The theoretical curve for $(U_{max}-U)/U_{\tau}$, shown by a full-line curve in the figure, is made to fit the observation at $\xi = r/R_1 = 1{\cdot}15$, by taking $B_1 = 0{\cdot}560$.

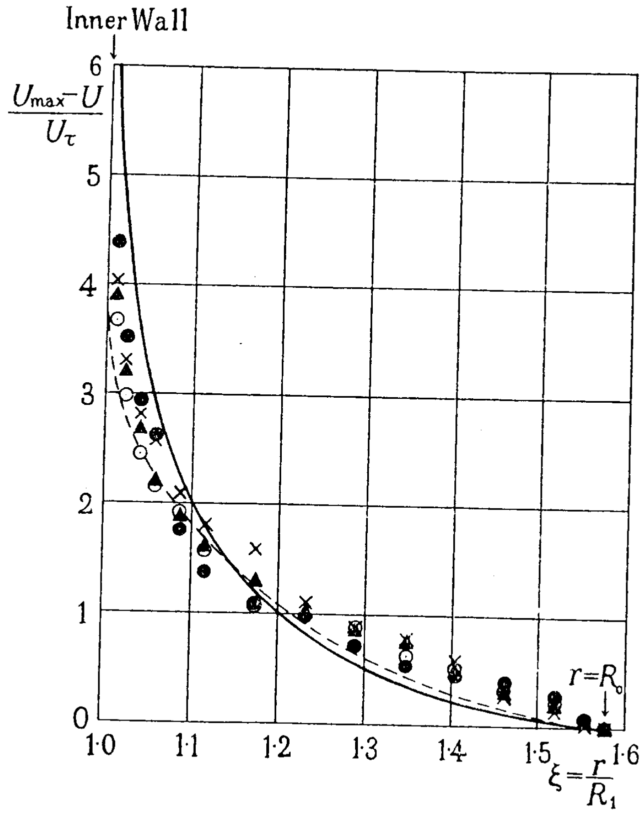

Fig. 1. Inner Region.

———, Momentum Transport Theory, $B_1 = 0{\cdot}560$;
- - - - -, Modified Vorticity Transport Theory, $B_1 = 0{\cdot}557$.

For the sake of comparison, the theoretical curve of $(U_{max}-U)/U_{\tau}$ calculated in the previous paper on the basis of the modified vorticity transport theory is also shown in Fig. 1 by a broken-line curve, the curve being made, as before, to fit the observation at $\xi = 1{\cdot}15$.

It will be seen that the modified vorticity transport theory gives results in better agreement with observation than does the momentum transport theory.

Also, it is of great interest to notice that there is no distinguishable difference between the theoretical curve of $(U_{max}-U)/U_\tau$ in Fig. 1 above and the corresponding curve in Fig. 6 of the previous paper, the latter having been obtained on the momentum transport theory by treating, as mentioned already, the problem in a somewhat approximate manner.

B. Outer Region.

§ 5. In the case of the outer region in which $R_0 \leqq r \leqq R_2$, the equation of mean motion is, by (9),

$$l^2\left(\frac{dU}{dr}\right)^2 = \frac{U_\tau^2}{R_0}\frac{r^2-R_0^2}{r}, \tag{23}$$

and the assumed form for the mixing length is

$$l = B_2(R_2-r) . \tag{24}$$

Inserting this expression for l in (23) and taking account of the fact that $r \geqq R_0$ and $dU/dr < 0$ in the outer region, we get

$$\frac{dU}{dr} = -\frac{U_\tau}{B_2\sqrt{R_0}}\frac{\sqrt{r^2-R_0^2}}{\sqrt{r}(R_2-r)} . \tag{25}$$

Integrating this and determining the constant of integration by the condition that $U = U_{max}$ at $r = R_0$, we get

$$\frac{U_{max}-U}{U_\tau} = \frac{1}{B_2\sqrt{R_0}}\int_{R_0}^{r}\frac{\sqrt{r^2-R_0^2}}{\sqrt{r}(R_2-r)}dr , \tag{26}$$

or, writing $\xi = r/R_2$,

$$\frac{U_{max}-U}{U_\tau} = \frac{1}{B_2}\sqrt{\frac{R_2}{R_0}}\int_{R_0/R_2}^{\xi}\frac{\sqrt{\xi^2-R_0^2/R_2^2}}{\sqrt{\xi}(1-\xi)}d\xi . \tag{27}$$

Further, if we put, for the sake of simplicity,

$$f_2(\xi) = \frac{\sqrt{\xi^2-R_0^2/R_2^2}}{\sqrt{\xi}(1-\xi)} , \tag{28}$$

and

$$F_2(\xi) = \sqrt{\frac{R_2}{R_0}} \int_{R_0/R_2}^{\xi} f_2(\xi) d\xi \,, \tag{29}$$

the expression for $(U_{max} - U) / U_\tau$ may be put in the form:

$$\frac{U_{max} - U}{U_\tau} = \frac{1}{B_2} F_2(\xi) \,. \tag{30}$$

Taking $R_0 = 3{\cdot}55$ cm and $R_2 = 4{\cdot}15$ cm as in MIKRJUKOV's experiment, the numerical values of the integral $\int_{R_0/R_2}^{\xi} f_2(\xi) d\xi$ and of the function $F_2(\xi)$ have been calculated for various values of ξ ranging from $R_0 / R_2 = 0{\cdot}855$ to 0·999 by the method of numerical integration. The results are shown in Table II.

TABLE II.

$\xi = \frac{r}{R_2}$	$\int_{R_0/R_2}^{\xi} f_2(\xi) d\xi$	$F_2(\xi)$
0·999	2·273	2·457
0·997	1·706	1·845
0·995	1·445	1·562
0·99	1·095	1·183
0·98	0·753	0·814
0·97	0·560	0·606
0·96	0·429	0·464
0·95	0·332	0·359
0·94	0·257	0·278
0·93	0·197	0·213
0·92	0·148	0·160
0·91	0·108	0·117
0·90	0·075	0·081
0·89	0·049	0·053
0·88	0·028	0·030
0·87	0·012	0·013
0·86	0·0019	0·0020
0·855	0	0

The calculated results are compared with MIKRJUKOV's observation in Fig. 2. The theoretical curve of $(U_{max}-U)/U_{\tau}$, shown by a full-line curve in the figure, is made to fit the observation at $\xi = r/R_2 = 0{\cdot}96$, by taking $B_2 = 0{\cdot}198$.

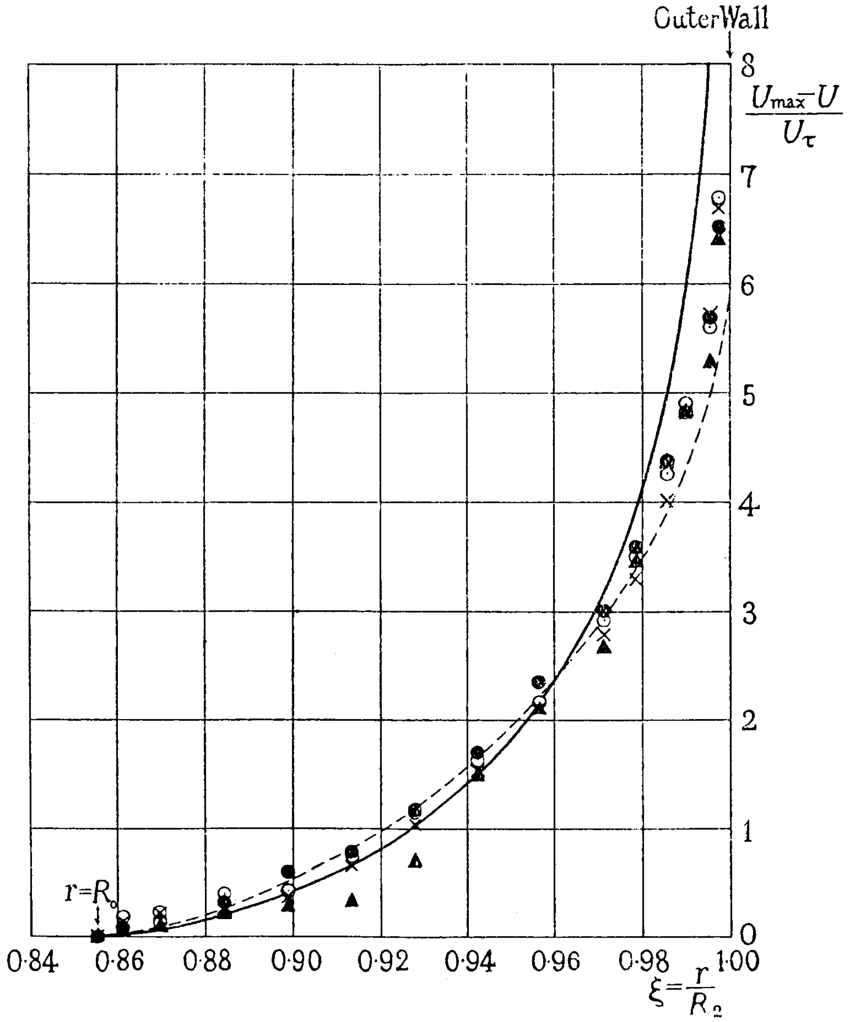

Fig. 2. Outer Region.

———, Momentum Transport Theory, $B_2 = 0{\cdot}198$;
- - - - -, Modified Vorticity Transport Theory, $B_2 = 0{\cdot}194$.

For comparison, the theoretical curve for $(U_{max}-U)/U_{\tau}$ calculated in the previous paper on the basis of the modified vorticity transport theory is also shown in Fig. 2 by a broken-line curve, the curve being made, as before, to fit the observation at $\xi = 0{\cdot}96$.

It will readily be seen that as in the case of the inner region, the modified vorticity transport theory gives results in better agreement with observation than does the momentum transport theory.

Further, it will be noticed that, as in the case of the inner region, there is no distinguishable difference between the theoretical curve of $(U_{max}-U)/U_\tau$ in Fig. 2 above and the corresponding curve in Fig. 7 in the previous paper, the latter having been obtained on the basis of the momentum transport theory by treating, as mentioned already, the problem in a somewhat approximate manner.

III. Summary.

§ 6. In the present paper, the distribution of mean velocity in turbulent flow of a fluid flowing under pressure through a straight pipe of annular cross-section is re-calculated on the basis of the momentum transport theory of turbulent motion. The treatment of the problem in the present paper is more appropriate and more exact than that given in our previous paper, where the flow through an annular pipe was treated as if it were two-dimensional.

As in the previous paper, the annular region is divided into the inner and outer regions, and it is assumed that in the inner region the mixing length is proportional to the distance from the inner wall of the annular pipe, while in the outer region it is proportional to the distance from the outer wall.

The calculated results are compared with MIKRJUKOV's experimental results and also with the theoretical results obtained in the previous paper on the basis of the modified vorticity transport theory.

It is thus found that the results obtained on the modified vorticity transport theory are in better agreement with observation than those calculated on the momentum transport theory. It seems therefore that the conclusion of the previous paper that the modified vorticity transport theory gives results in better agreement with observation than does the momentum transport theory does not suffer any alteration.

In conclusion, the writers wish to express their cordial thanks to Mr. K. TAMADA for his assistance in preparing the figures in the present paper.

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