第二百一號

（昭和十五年十一月發行）

抄録

楕圓柱の周りの壓縮性流體の流れに關する研究

囑託 理學博士 友近晋
理學士 玉田珖

物體が小さい速度で空氣中を動く場合には，空氣の壓縮性の影響は極めて微々たるものであつて，それを非壓縮性流體と看做しても差支へない位であることは，周知の通りである．實際，今まで，航空力學に於いては，空氣を非壓縮性流體の樣に取扱つて物體に働らく力等を計算し，かなりよく實驗的事實を說明することが出來てゐるのである．然しながら，物體の速度が大きくなると，この樣に空氣を非壓縮性流體と假定することは最早許されなくなる．

飛行機の速度の最近に於ける急激な增加は，空氣の壓縮性の影響に對する研究を益〻必要ならしめ，その結果，種々の形の物體の周りに於ける空氣の非廻轉運動に對する壓縮性の影響が，理論的及び實驗的に色々研究されてゐるが，數學的解析の困難さのために理論的研究に於ける進步は遲々たるものであり，又實驗的研究に於いても得られた結果は未だ多くない現狀である．

實驗的研究に於いて今までに得られた結果のうち，航空力學上最も重要で且つ興味深いことは，亂されない流れ即ち物體から充分遠く離れた場所の流れに對する所謂 MACH 數が或値よりも大きくなると，翼の揚力は急激に減少し，反對に抗力は

急激に增加するといふ結果であらう．斯様な揚力の急激な減少及び抗力の急激な增加は，MACH 數が或値よりも大きくなつたために翼の周りの流れが最早非廻轉的ではなくなつたことを示すものと考へられるが，かかる現象を壓縮性失速と稱する．一般に，流れの場に於ける最大速度が丁度局所音速に等しくなる様な MACH 數の臨界値を臨界 MACH 數と稱してゐるが，最近の實驗的研究の結果によれば，斯様な壓縮性失速は，流れの速度が大きくなつてそれに對する MACH 數が臨界 MACH 數を超える時に起る様である．從つて，翼その他の物體に對する臨界 MACH 數を實驗的に求めることが必要であると同時に，若し出來れば，これを理論的に計算することも興味深い問題である．

既に述べた様に，數學的解析の非常な困難さのために，圓柱，楕圓柱，JOUKOWSKI 對稱翼型，球の様な比較的に簡單な形の物體の周りの壓縮性流體の流れでさへも，それに對する理論的研究は近似的に遂行されてゐるに過ぎない．即ち，斯様な物體の周りに於ける非壓縮性流體の流れに對する既知の解を假りに壓縮性流體の流れに對する第零次の近似解と考へ，その近似度を高めて第一次近似解，第二次近似解等を順次求めるといふ所謂逐次近似法が多く採用されてゐるのである．この方法は，圓柱の場合に JANZEN 及び LORD RAYLEIGH が採用した方法で，通常 JANZEN-RAYLEIGH の方法と呼ばれてゐるが，所謂 POGGI の方法も逐次近似法であるといふ點では JANZEN-RAYLEIGH の方法と變りがない．

さて，圓柱の場合は今井によつて第三次近似解まで，又球の場合は玉田によつて第二次近似解まで求められ，此等の場合に就いては大體眞相に近いことが既に知られてゐるが，楕圓柱及び JOUKOWSKI 對稱翼型の場合は第一次近似解が得られてゐるに過ぎない．しかも，臨界 MACH 數の計算は極めて特別な迎角の場合に就いて遂行されてゐるに過ぎないのである．その他の實用的に興味深い物體の場合に就いては，理論的に第一次近似解を求めることさへ多くは極めて困難である．

楕圓柱の周りの壓縮性流體の流れの問題は，既に HOOKER 及び KAPLAN によつて研究され，第一次近似解が求められてゐるが，楕圓柱表面上の速度及び壓力の分布とか臨界 MACH 數の計算は，楕圓の長軸が無限遠の流れに平行な様な特別な場合に就いて遂行されてゐるに過ぎず，しかも HOOKER の解は所謂厚み比が相當大きい楕圓でないと適用されない様な近似解である．又，KAPLAN は楕圓柱に働らく壓力の合モーメントも計算してゐるが，詳しい數値的の研究は遂行してゐない．尙ほ，最近今井及び相原は新しい方法で楕圓柱の場合の第一次近似解を一般に求めてゐるが，短

軸が無限遠に於ける流れに平行な場合に就いて臨界 MACH 數を計算した程度で，その他の詳しい數値計算は遂行してゐない．

JOUKOWSKI 對稱翼型の場合に關する研究も POGGI や KAPLAN によつて遂行されてゐて，特に KAPLAN の研究は相當詳しいが，更に一般の場合に就いて詳しい數値計算を必要とする樣に考へられる．しかも KAPLAN の解析には多少疑點があり，從つて結果が多少間違つてゐることを著者等は見出してゐるが，これ等に關しての議論は近く發表豫定の次の論文に讓ることにする．

本論文では，著者の研究室で遂行した壓縮性流體の流れに關する種々の研究のうち，先づ楕圓柱の周りの流れに就いての研究結果を述べる．著者等の採用した方法は，KAPLAN の論文に於ける樣に，POGGI の方法であつて，別に新しくはないが，KAPLAN の論文に於けるよりもかなり事柄を一般的に取扱つてゐる．即ち，KAPLAN は主として楕圓柱の表面上の事柄に着目してゐるが，著者等は先づ任意點に於ける速度ポテンシャルを求め，然る後に楕圓柱の表面に於ける事柄を論じた．本論文の第一の目的は，詳細な數値計算を遂行して，迎角が 0° 及び 90° 以外の値を採る場合に於ける臨界 MACH 數を種々の厚み比の楕圓柱に就いて算出することであつて，著者等は迎角の値として實用的にも重要と思はれる 5° 及び 10° の二つの値を採用した．計算の結果は第 2 圖に示してあるが，これから知れるところの最も興味あることは，迎角が或與へられた値を採る時，或適當な厚み比に對して臨界 MACH 數が極大になるといふことであると思はれる．例へば，迎角 5° の場合には厚み比が大凡 0·2 の楕圓柱に對する臨界 MACH 數が他の如何なる厚み比の楕圓柱に對するものよりも大きく，又迎角 10° の場合には厚み比が大凡 0·3 の楕圓柱に對する臨界 MACH 數が極大である．斯樣な點はかなり興味深く且つ重要なのではないかと思ふ．著者等は此等に關する實驗的研究の遂行されんことを望むものである．

著者等は更に楕圓柱に働らく壓力の合モーメントを計算し，流體の壓縮性がモーメントに對して如何なる影響を及ぼすかをしらべた．モーメントに對する一般式は既に KAPLAN の得たものと一致するが，KAPLAN は別に數値的にこれを検討してゐないので，著者等は二つの特別な楕圓柱を例に採り，詳しい數値計算を遂行して，迎角が 5°, 10° 及び 15° である三つの場合に，モーメントが MACH 數と共に如何に變るかを調べた．その結果は第 5 及び第 6 兩圖に示してあるが，そこでは，GLAUERT-PRANDTL の近似理論によるモーメントの近似式の與へる結果との比較も行つてゐる．

本論文の緒論では，今までに遂行された理論的及び實驗的研究のうちの主なものに就いての歴史的概觀の樣なものを述べてゐる．多少冗長と思はれるかも知れないが，著者等が本論文で研究してゐることや，續いて發表豫定の二三の論文に於いて著者等が行つてゐる研究が，如何なる意味のものであるかを了解して貰ふために，敢へて書いた次第である．同じ樣な問題に對する著者等の數篇の論文の全體に對する緒論と解して頂き度いと思ふ．しかしながら，文獻の蒐集には完全を期さなかつたことを附言する．

No. 201.

(Published November, 1940.)

Studies on the Subsonic Flow of a Compressible Fluid past an Elliptic Cylinder.

By

Susumu TOMOTIKA, *Rigakuhakusi*,
Research Associate of the Institute,

and

Kō TAMADA, *Rigakusi*.

I. Introduction.

§ 1. When a solid body moves in the air at low speeds, the effect of compressibility of the air is inappreciable and there is complete justification of the usual assumption that the air may be regarded effectively as an incompressible medium. In effect, application of the hydrodynamic theory to aeronautics have hitherto been made, with great success, on the assumption that the air behaves like an incompressible fluid in regard to the forces which it exerts on bodies moving through it. This assumption, however, ceases to be valid in cases when a solid body moves through the air with speeds approaching that of sound.

In recent years the rapid increase in the speed of modern aeroplanes has very much increased the importance of the study of the effect of compressibility on air flow. Various attempts, both theoretical and experimental, have therefore been made to find the effect of compressibility on irrotational flows of the air past solid bodies of various shapes. Owing to the considerable difficulties of the mathematical analysis, however, only little progress has hitherto been made in theoretical investigations. Also, experimental results so far obtained are not at all abundant.

In the first place, a brief survey of the principal results of both the theoretical and the experimental investigations so far carried out will be made.

The first attempts to find the effect of compressibility on irrotational flow round a body moving in a fluid were those of JANZEN(1) and of the late LORD RAYLEIGH(2), who obtained, by employing the method of successive approximations, the first approximation to the velocity potential for the irrotational flow of a compressible fluid past a circular cylinder, starting with the well-known solution for the irrotational flow of an incompressible fluid past a circular cylinder as the zero approximation. RAYLEIGH considered that the true solution could be ultimately obtained if we would carry the method of successive approximations far enough and that the solution involving irrotational motion would hold till the speed of the undisturbed stream at a great distance from the obstacle reached that of sound. But, no further approximations have been obtained until recently, owing to the difficulty in carrying out the successive approximations.

In 1932, POGGI(3) has contrived an ingenious method to attack the problems in compressible fluid flow. The essential feature of the method,

---

(1) O. JANZEN, Beitrag zu einer Theorie der stationären Strömung kompressibler Flüssigkeiten. Phys. Zeits., **14** (1913), 639-643.

(2) LORD RAYLEIGH, On the Flow of Compressible Fluid past an Obstacle. Phil. Mag., **32** (1916), 1–6; or, Scientific Papers, VI, 402–406.

(3) L. POGGI, Campo di velocità in una corrente piana di fluido compressibile. L'Aerotecnica, **12** (1932), 1579–1593.

L. POGGI, Campo di velocità in una corrente piana di fluido compressibile. Parte II: Caso di profili ottenuti con rappresentazione conforme dal cerchio ed in particolare dei profili JOUKOWSKI. L'Aerotecnica, **14** (1934), 532–549.

which is now usually called POGGI's method for simplicity, is to replace the compressible fluid flow past a body by an incompressible fluid flow due to a suitable distribution of sources throughout the fluid region outside the body. With the aid of his method, POGGI has investigated the irrotational flow of a compressible fluid past a circular cylinder and the second approximate solution has been obtained. In the second part of his paper, POGGI has also discussed the irrotational subsonic flow of a compressible fluid past a JOUKOWSKI aerofoil.

Recently, IMAI[1] has re-investigated the compressible fluid flow past a circular cylinder, by employing the JANZEN-RAYLEIGH method, and the second approximate solution has been obtained. His result is not in agreement with that of POGGI, however, and by a careful examination, it has been found that POGGI's result is unfortunately erroneous.

In one of his recent papers[2], KAPLAN has also calculated, as an addendum to the discussion of the compressible fluid flow past an elliptic cylinder, the second approximate solution for the compressible fluid flow past a circular cylinder, obtaining a result which is not in accord with both the results of POGGI and of IMAI[3], the latter of which is believed, as mentioned just in the above, to be the correct second approximation. The junior author of the present paper has also re-calculated[4], in conjunction with SAITŌ, the second approximate

---

(1) I. IMAI, On the Flow of a Compressible Fluid past a Circular Cylinder. Proc. Phys.-Math. Soc., Japan, **20** (1938), 636–645. In the second part of his paper, which will be published shortly elsewhere, IMAI has obtained the third approximate solution for the problem. See, I. IMAI, On the Flow of a Compressible Fluid past a Circular Cylinder, II..

(2) C. KAPLAN, Two-dimensional Subsonic Compressible Flow past Elliptic Cylinders. N. A. C. A. Report No. **624** (1938). It may be noted here that in this paper of KAPLAN, the known solution for the incompressible fluid flow is called the first approximation to the compressible fluid flow.

(3) It may be remarked here, however, that KAPLAN gives the correct result in his paper with the same title as above which is reproduced in the Twenty-Fourth Annual Report of the National Advisory Committee for Aeronautics, 1938, without referring to unfortunate mistakes in his paper issued separately.

(4) K. TAMADA and Y. SAITŌ, Note on the Flow of a Compressible Fluid past a Circular Cylinder. Proc. Phys.-Math. Soc., Japan, **21** (1939), 403–409.

solution for the problem under discussion, by using POGGI's method, and thus IMAI's result has been confirmed.

On the other hand, TAYLOR has suggested, in his well-known paper[1], that a mathematical analogy between the flow of a compressible fluid in two dimensions and the flow of electricity in a sheet of conductor of variable thickness might be used to obtain successive approximations to the true solution for the compressible fluid flow past an obstacle of any shape. The first application of this mechanical method was made by TAYLOR himself to JANZEN-RAYLEIGH's case of the circular cylinder. In this case the method produces, mechanically, successive approximations which are practically identical with those which would result from successive applications of JANZEN-RAYLEIGH's mathematical method. One of the most important results of TAYLOR's work is the discovery that the failure of convergence occurs, contrary to RAYLEIGH's conjecture, not when the speed of the undisturbed flow reaches that of sound associated with the undisturbed stream but at some lower speed. TAYLOR has found that in the case of a circular cylinder convergence fails when the maximum velocity in the field of flow reaches the speed of sound in the air at that point and this first occurs when the velocity of the undisturbed flow is between 0·4 and 0·5 of that of sound. In other words, TAYLOR has found that when the MACH number is formed with the velocity of the undisturbed flow of air and the speed of sound associated with the undisturbed stream, the value of the so-called critical MACH number at which the local speed of sound is first attained in the field of flow lies between 0·4 and 0·5 in the case of a circular cylinder placed in a stream of air.

Using the second approximate solution, IMAI has calculated the value of the critical MACH number, denoted by M_{crit} in the present paper, in the case of the flow of air past a circular cylinder, obtaining

(1) G. I. TAYLOR and C. F. SHARMAN, A Mechanical Method for solving Problems of Flow in Compressible Fluids. Proc. Roy. Soc., London, A 121 (1928), 194–217; or, British A. R. C. Reports and Memoranda No. 1195 (1928).

the result that $M_{crit} = 0{\cdot}409$. This result should be compared with TAYLOR's result of observation above mentioned, and it is found that the agreement between the theory and the observation is satisfactory.

§ 2. The irrotational subsonic flow of a compressible fluid past an elliptic cylinder was first discussed by HOOKER[1], who has obtained, by making use of JANZEN-RAYLEIGH's method, a first order correction to the velocity potential for the compressible fluid flow past an elliptic cylinder in case when the direction of the undisturbed flow is parallel to its major axis. But, owing to the necessity for expanding a certain function in the analysis, the thickness ratio of the ellipse to which his result applies is limited, the thickness ratio of an ellipse being here defined as the ratio of the semi-minor axis to the semi-major axis of the ellipse.

In a recent paper[2], KAPLAN has used POGGI's method to calculate the effect of compressibility upon the irrotational flow past an elliptic cylinder, confining himself to cases in which the angle of attack is zero and there is no circulation round it. He has obtained the first approximate solution in a closed form, contrary to HOOKER's result in a form of a series, and using his solution, he has calculated the distributions of velocity and of pressure over the surface of an elliptic cylinder with a definite thickness ratio. Also, the so-called critical MACH number has been found as a function of the thickness ratio of the ellipse.

In another recent paper[3], KAPLAN has further investigated the effect of compressibility upon the moment about the axis of an elliptic cylinder of fluid pressures acting on the surface of the cylinder, but no detailed numerical discussions have been made.

---

(1) S. G. HOOKER, The Two-dimensional Flow of Compressible Fluids at Subsonic Speeds past Elliptic Cylinders. British A.R.C. Reports and Memoranda No. 1684 (1936).

(2) C. KAPLAN, Two-dimensional Subsonic Compressible Flow past Elliptic Cylinders. N. A. C. A. Report No. 624 (1938).

(3) C. KAPLAN, A Theoretical Study of the Moment on a Body in a Compressible Fluid. N. A. C. A. Report No. 671 (1939).

Quite recently, TSIEN[1] has discussed the two-dimensional irrotational subsonic flow of a compressible fluid past a solid body, by applying the so-called hodograph method, due originally to MOLENBROEK and TSCHAPLIGIN, in which the magnitude of velocity and the inclination of velocity to a chosen axis are used as independent variables. To do this, he has used a tangent line to the adiabatic pressure-volume curve as an approximation to the curve itself. A similar approximation was already used by DEMTCHENKO[2] and BUSEMANN[3], but they limited themselves to the use of the tangent at the state of the gas corresponding with the stagnation point of flow, thus making their theory to be applicable only to a flow with velocities smaller than about one-half of the speed of sound.

Following KÁRMÁN's suggestion, TSIEN has generalised the theory by using the tangent at the state of the gas corresponding with undisturbed parallel flow. The theory has been put into a form by which, knowing the incompressible fluid flow past a body, the compressible fluid flow past an approximately similar body can be calculated.

Applying the theory, TSIEN has calculated the flow of a compressible fluid obeying the above-mentioned approximate adiabatic law past a nearly elliptic cylinder at zero angle of attack. The results obtained have been compared with the first approximate solutions already referred to of HOOKER and of KAPLAN for the compressible fluid flow past a true elliptic cylinder. Also, the results have been compared with those calculated by GLAUERT-PRANDTL's linear theory[4], in which the disturbance

(1) H. S. TSIEN, Two-dimensional Subsonic Flow of Compressible Fluids. Journ. Aeron. Soc., 6 (1939), 399-407.

(2) B. DEMTCHENKO, Sur les mouvements lents des fluides compressible. C. R., 194 (1932), 1218-1220; Variation de la résistance aux faibles vitesses sous l'influence de la compressibilité. C R., 194 (1932), 1720-1723.

(3) A. BUSEMANN, Die Expansionsberichtigung der Kontraktionsziffer von Blenken. Forschung, 4 (1933), 186-187; Hodographenmethode der Gasdynamik. Z. A. M. M., 12 (1937), 73-79:

(4) H. GLAUERT, The Effect of Compressibility on the Lift of an Aerofoil. Proc. Roy. Soc., London, A 118 (1928), 113-119; also, British A. R. C. Reports and Memoranda No. 1135 (1928).

of parallel rectilinear flow, due to the presence of a solid body, is assumed to be small.

The hodograph method has also been applied quite recently by the junior author(1) of the present paper to the compressible fluid flow past a circular cylinder.

Further, IMAI and AIHARA(2) have lately developed a new method of solving problems in the compressible fluid flow past a body, and applying the method they have investigated the irrotational subsonic flow of a compressible fluid past an elliptic cylinder, obtaining a first approximation to the velocity potential. No detailed numerical discussions have been made by them however; they have only calculated the value of the critical MACH number in a special case in which the minor axis of the ellipse is parallel to the undisturbed flow, the corresponding case when the major axis is parallel to the direction of the undisturbed flow having already been discussed by KAPLAN in the paper(3) to which reference has been made previously.

From the point of view of aeronautics perhaps the most important problem so far solved is that of the thin aerofoil. The solution of this problem, due to PRANDTL and GLAUERT(4), follows exactly the same lines as that of a thin aerofoil in an incompressible fluid. Thus, making the same assumption that is made in the case of incompressible fluid, that the changes in velocity due to the aerofoil are small compared with the velocity, U, of the undisturbed flow, they have found that the lift of the aerofoil in a compressible fluid is increased by the factor

---

(1) K. TAMADA, Application of the Hodograph Method to the Flow of a Compressible Fluid past a Circular Cylinder. Proc. Phys.-Math. Soc., Japan, **22** (1940), 208–219.

(2) I. IMAI and T. AIHARA, On the Subsonic Flow of a Compressible Fluid past an Elliptic Cylinder. Report Aeron. Res. Inst., Tokyo Imp. Univ., No. **194** (1940).

(3) C. KAPLAN, loc. cit. (Report No. **624**).

(4) H. GLAUERT, The Effect of Compressibility on the Lift of an Aerofoil. Proc. Roy. Soc., London, A **118** (1928), 113–119; also, British A. R. C. Reports and Memoranda No. **1135** (1928). The same result as GLAUERT's is quoted without proof by ACKERET in the Handbuch der Physik, VII (1927), 340, as given by PRANDTL in his lecture at Göttingen in 1922.

$(1-M^2)^{-\frac{1}{2}}$, where M is, as before, the MACH number formed with the velocity U of the undisturbed flow and the speed of sound, c_0, associated with the undisturbed flow so that $M=U/c_0$. It has been found that this result is in good agreement with the experimental results, provided that the value of the MACH number M of the undisturbed flow does not exceed a certain critical value.

Also, according to the PRANDTL-GLAUERT linear theory just referred to, it should be expected that the moment of fluid pressures acting on a body in a compressible fluid is increased by the effect of compressibility of the fluid in the ratio $(1-M^2)^{\frac{1}{2}}:1$.

A more exact discussion of the effect of the compressibility upon the aerodynamical characteristics of an aerofoil was first made in 1934 by POGGI[1], and later, in more detail, by KAPLAN[2]. In the earlier paper (Report No. 621) KAPLAN has obtained, confining himself to the case of a symmetrical JOUKOWSKI aerofoil, the approximate expressions for the lift and moment of the aerofoil which are valid only for very thin aerofoils placed at sufficiently small angles of attack, and also the pressure and velocity distributions on the surface of the aerofoil with a certain definite thickness ratio have been calculated in a special case when the angle of attack is zero, the thickness ratio of the aerofoil being here defined as the ratio of the maximum thickness to the chord of the aerofoil. Further, the critical MACH number has been calculated for two symmetrical JOUKOWSKI aerofoils with different thickness ratios.

In the subsequent paper (Report No. 671) KAPLAN has re-investigated the effect of compressibility on the moment of fluid pressures acting on both the symmetrical JOUKOWSKI aerofoil and the elliptic cylinder, by employing the extension to a compressible fluid of LAGALLY's theorem

(1) L. POGGI, Campo di velocità in una corrente piana di fluido compressibile. Parte II: Caso di profili ottenuti con rappresentazione conforme dal cerchio ed in particolare dei profili JOUKOWSKI. L'Aerotecnica, **14** (1934), 532–549.

(2) C. KAPLAN, Compressible Flow about Symmetrical JOUKOWSKI Profiles. N. A. C. A. Report No. **621** (1938).

on the moment on a body in an incompressible fluid and POGGI's method fo treating the flow of compressible fluids. The effect of compressibility on the position of the centre of pressure has also been discussed.

It seems however that KAPLAN's papers contain unfortunately various questionable points and mistakes, and consequently his results are not at all quite correct. Quite recently, UMEMOTO and the senior author[1] of the present paper have therefore re-investigated the problem, by using POGGI's method, and by performing detailed tedious numerical calculations, the effect of compressibility of air upon the aerodynamical characteristics of the symmetrical JOUKOWSKI aerofoil has been discussed in detail. It is hoped that the results of our investigations will be published in the near future.

So far we have concerned only with the case of two-dimensional flows of a compressible fluid past solid bodies of various shapes. Mention will now be made briefly about the case of three-dimensional subsonic flows of a compressible fluid. Up to the present time only the case of the subsonic irrotational flow of a compressible fluid past a sphere has been discussed. Starting with the well-known solution for the incompressible fluid flow past the sphere as the zero approximation for the compressible fluid flow, the late LORD RAYLEIGH[2] first obtained the first approximation to the velocity potential for the irrotational compressible fluid flow past the sphere. Quite recently the junior author[3] of the present paper has found the second approximate solution of the problem and it has been found that the theoretical result obtained is in agreement with experiment[4].

---

(1) S. TOMOTIKA and H. UMEMOTO, On the Subsonic Flow of a Compressible Fluid past a Symmetrical JOUKOWSKI Aerofoil.

(2) LORD RAYLEIGH, On the Flow of Compressible Fluid past an Obstacle. Phil. Mag., 32 (1916), 1–6; or, Scientific Papers, VI, 402–406.

(3) K. TAMADA, On the Flow of a Compressible Fluid past a Sphere. Proc. Phys.-Math. Soc., Japan, 21 (1939), 743–752.

(4) K. TAMADA, Further Studies on the Flow of a Compressible Fluid past a Sphere. Proc. Phys.-Math. Soc., Japan, 22 (1940), 519–525.

§3. Next, an account will be given of the results which are obtained when aerofoils are placed in a stream of air moving at speeds comparable with that of sound. Various important experimental results have been obtained so far[1], and the following two are particularly of fundamental importance.

The first main result is that up to speeds of roughly 0·6 times the speed of sound the lift of an aerofoil increases with increasing speed till the speed of the stream rises to some critical value. Up to this critical speed the resistance of the aerofoil is low, and all the evidence points to the fact that the irrotational flow with JOUKOWSKI circulation would afford a good representation of the flow. In fact, the manner in which the lift increases with increasing speed can be explained fairly satisfactorily by the result referred to previously of PRANDTL-GLAUERT's linear theory[2] which, as mentioned already, is based on the assumptions that the flow is irrotational and the disturbances due to the presence of the aerofoil are small compared with the velocity of the undisturbed flow.

---

(1) L. J. BRIGGS, G. F. HULL and H. L. DRYDEN, Aerodynamic Characteristics of Airfoils at High Speeds. N. A. C. A. Report No. 207 (1925).

L. J. BRIGGS and H. L. DRYDEN, Pressure Distribution over Airfoils at High Speeds N. A. C. A. Report No. 255 (1927).

L. J. BRIGGS and H. L. DRYDEN, Aerodynamic Characteristics of Twenty-four Airfoils at High Speeds. N. A. C. A. Report No. 319 (1929).

T. E. STANTON, On the Distribution of Pressure over a Symmetrical JOUKOWSKI Section at High Speeds. British A. R. C. Reports and Memoranda No. 1280 (1929).

L. J. BRIGGS and H. L. DRYDEN, The Effect of Compressibility on the Characteristics of Airfoils. Proc. Third Intern. Congr. Appl. Mech., Stockholm. 1 (1930), 417-422.

L. J. BRIGGS and H. L. DRYDEN, Aerodynamic Characteristics of Circular Arc Airfoils at High Speeds. N. A. C. A. Report No. 365 (1930).

J. STACK, The N. A. C. A. High-speed Wind Tunnel and Tests of Six Propeller Sections. N. A. C. A. Report No. 463 (1933).

J. STACK and A. E. VON DOENHOFF, Tests of 16 Related Airfoils at High Speeds. N. A. C. A. Report No. 492 (1934).

E. N. JACOBS, Methods employed in America for the Experimental Investigation of Aerodynamic Phenomena at High Speeds. Proc. Fifth VOLTA Congress, Rome (1935), 369-393.

Other papers will be referred to on later pages at suitable opportunities.

(2) H. GLAUERT, loc. cit.

The second main result is that, at a speed which varies with the shape of the aerofoil employed and with its angle of attack, a marked increase occurs in the resistance and at the same time the lift rapidly decreases. This seems to indicate that the flow ceases to be even approximately irrotational.

This phenomenon of the breakdown of irrotational flow is now usually called the compressibility burble, and one of the important problems in the theory of the subsonic flow of a compressible fluid is to find a speed, if any, at which the compressibility burble occurs when a body of any given shape moves in a compressible fluid at speeds comparable with that of sound. It is also very important to investigate the physical nature of the compressibility burble.

The results of various recent experiments[1] seem to indicate that the breakdown of irrotational flow occurs when the velocity of the undisturbed flow exceeds a certain critical value at which the local speed of sound is first attained in the field of flow; in other words, the compressibility burble occurs when the MACH number, M, formed with the undisturbed velocity, U, and the speed of sound, c_0, in the undisturbed flow such that $M = U / c_0$, exceeds a certain critical value at which the local speed of sound is first attained in the field of flow. Such a critical value for the MACH number is, as mentioned before, usually called the critical MACH number, and it is of great importance to calculate, if possible, the value of the critical MACH number for bodies of practical interest but of mathematically simple shapes such as the circular cylinder, the elliptic cylinder, the symmetrical JOUKOWSKI aerofoil and the sphere.

In the case of a circular cylinder, IMAI has found the values 0·409

(1) J. STACK, The Compressibility Burble. N. A. C. A. Technical Note No. 543 (1935).
J. STACK, W. F. LINDSEY and R. E. LITTELL, The Compressibility Burble and the Effect of Compressibility on Pressures and Forces acting on an Airfoil. N. A. C. A. Report No. 646 (1938).
W. F. LINDSEY, Drag of Cylinders of Simple Shapes. N. A. C. A. Report No. 619 (1938).

and 0·406 for the critical MACH number M_{crit}, using respectively the second and the third approximate solutions obtained by him[1]. The junior author[2] of the present paper has applied the hodograph method to the compressible fluid flow past a circular cylinder and obtained, among other things, the result that $M_{crit} = 0{\cdot}400$ approximately.

In his electrolytic tank experiment with a circular cylinder, TAYLOR[3] obtained a definite flow pattern when the MACH number M was 0·4, but no definite flow patterns were obtained when M was 0·5 and 0·6. Thus, TAYLOR has been led to the conclusion that in the case of irrotational flow past a circular cylinder the compressibility burble occurs when $M = 0{\cdot}45$ approximately. It is found that this result is in fairly good agreement with the theoretical result mentioned above.

The values of the critical MACH number for the elliptic cylinder and for the symmetrical JOUKOWSKI aerofoil have been calculated by KAPLAN[4] in the special case when the angle of attack is zero. Also, the similar values have been found by IMAI and AIHARA[5] for the elliptic cylinder in case when its minor axis is parallel to the direction of the undisturbed flow.

Further, the case of the sphere has been discussed, as mentioned already, by the junior author of the present paper, and using the

(1) I. IMAI, loc. cit.

(2) K. TAMADA, loc. cit.

(3) G. I. TAYLOR and C. F. SHARMAN, A Mechanical Method for solving Problems of Flow in Compressible Fluids. Proc. Roy. Soc., London, A 121 (1928), 194–217; or, British A. R. C. Reports and Memoranda No. 1195 (1928).

G. I. TAYLOR, Reports on Progress during 1927–28 in Calculation of Flow of Compressible Fluid, and Suggestions for Further Work. British A. R. C. Reports and Memoranda No. 1196 (1928).

G. I. TAYLOR, Recent Work on the Flow of Compressible Fluids. Journ. London Math. Soc., 5 (1930), 224–240.

G. I. TAYLOR, The Flow round a Body moving in a Compressible Fluid. Proc. Third Intern. Congr. Appl. Mech., Stockholm, 1 (1930), 263–274.

G. I. TAYLOR, Well Established Problems in High Speed Flow. Proc. Fifth VOLTA Congress, Rome (1935), 198–214.

(4) C. KAPLAN, loc. cit. (N. A. C. A. Report Nos. 621 and 624).

(5) I. IMAI and T. AIHARA, loc. cit.

second approximate solution obtained the critical MACH number for the sphere has been calculated to be 0·574[1]. It has been found that this theoretical result is in good agreement with the experimental result of PASQUALINI[2], who has found that in the case of the sphere, the compressibility burble occurs when the value of the MACH number exceeds 0·55.

Up to the present time no calculations have yet been carried out to find the values of the critical MACH number for the elliptic cylinder as well as for the symmetrical JOUKOWSKI aerofoil placed at an arbitrary angle of attack. However, it seems to be of practical interest to calculate the values of the critical MACH number for such bodies in the general case when the angle of attack is different from zero.

The present writers have therefore re-investigated the irrotational subsonic flow of a compressible fluid past an elliptic cylinder placed at an arbitrary inclination to the direction of the undisturbed flow, with a special intention of studying the manner in which the value of the critical MACH number for the elliptic cylinder varies with the angle of attack and with the thickness ratio of the ellipse. For this purpose we have employed the method of POGGI, as in KAPLAN's papers. Although KAPLAN has confined his attention chiefly to the state of affairs on the surface of the elliptic cylinder, yet we have generalised the analysis to some extent, by obtaining first the velocity potential at any point in the field of flow and then proceeding to the discussion of the state of affairs on the surface of the body. Various detailed numerical calculations have been carried out, and thus the values of the critical MACH number have been found as functions of both the angle of attack and the thickness ratio of the ellipse. Also, we have calculated the moment about the centre of the ellipse of the fluid pressures acting on

(1) K. TAMADA, loc. cit.
(2) C. PASQUALINI, loc. cit.

its surface, and performing numerical calculations we have discussed the effect of compressibility upon the moment of the elliptic cylinder. The object of the present paper is to describe the results of our investigations.

II. Outline of Poggi's Method.

§4. When a compressible fluid flows steadily and irrotationally in two dimensions, the equation of motion is expressed in the form:

$$\Delta\phi = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial v^2}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial v^2}{\partial y}\right), \qquad (1)$$

where (x, y) are the rectangular coordinates, ϕ the velocity potential for the flow(1), c the local velocity of sound so that $c = \sqrt{dp/d\rho}$, assuming the pressure p to be a function of the density ρ only. Also, v denotes the magnitude of the fluid velocity at any point in the field of flow so that

$$v^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2, \qquad (2)$$

and further the symbol Δ stands, as usual, for the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

We assume that the fluid obeys the adiabatic law $p\rho^{-\gamma} = \text{const.}$ Then, if we denote by c_0 the velocity of sound in the undisturbed flow at infinity moving with constant velocity U, BERNOULLI's theorem gives immediately

$$c^2 = c_0^2\left\{1 - \frac{1}{2c_0^2}(\gamma - 1)(v^2 - U^2)\right\}, \qquad (3)$$

where γ denotes the ratio of the specific heats of the fluid, the value of γ for the air, for example, being 1·405.

---

(1) In the present paper, the velocity potential ϕ is defined as $\boldsymbol{v} = \text{grad}\phi$, where $\boldsymbol{v}$ is the velocity vector of a fluid element at any point.

Thus, combining (1) and (3), we have

$$\Delta\phi = \frac{\dfrac{\partial\phi}{\partial x}\dfrac{\partial v^2}{\partial x}+\dfrac{\partial\phi}{\partial y}\dfrac{\partial v^2}{\partial y}}{2c_0^2\left\{1-\dfrac{1}{2c_0^2}(\gamma-1)(v^2-U^2)\right\}}. \tag{4}$$

To solve this differential equation we now assume that ϕ can be expanded in a series of the form:

$$\phi = \phi_0+\phi_1+\phi_2+\cdots, \tag{5}$$

where ϕ_0 is the velocity potential for the incompressible fluid flow, ϕ_1 a first order correction to the velocity potential due to compressibility of the fluid and ϕ_2 a second order correction, and so on. It may be assumed that ϕ_1 is proportional to M^2 and ϕ_2 is proportional to M^4 and so on, where M is, as before, the MACH number formed with the velocity U of the undisturbed flow and the speed of sound c_0 associated with this undisturbed flow so that $M = U/c_0$.

Putting (5) in (4) and equating terms of the same powers of M on both sides, we obtain the differential equations for determining ϕ_0, ϕ_1, ϕ_2, $\cdots$. Only the first two will be given here. We have

$$\Delta\phi_0 = 0, \tag{6}$$

$$\Delta\phi_1 = \frac{1}{2c_0^2}\left(\frac{\partial\phi_0}{\partial x}\frac{\partial v_0^2}{\partial x}+\frac{\partial\phi_0}{\partial y}\frac{\partial v_0^2}{\partial y}\right), \tag{7}$$

where v_0 is the magnitude of the fluid velocity in the incompressible fluid flow so that

$$v_0^2 = \left(\frac{\partial\phi_0}{\partial x}\right)^2+\left(\frac{\partial\phi_0}{\partial y}\right)^2. \tag{8}$$

First, we solve equation (6) subject to suitable boundary conditions, obtaining the velocity potential ϕ_0 for the incompressible fluid flow.

Then, if we insert the expression for ϕ_0 thus obtained in the right-hand side of equation (7), we shall obtain a differential equation of the form $\Delta\phi_1 = f(x,y)$, with a known function $f(x,y)$.

The equation of this form may be solved in various ways. One of the important powerful methods seems to be that of POGGI[1], which has been successfully employed by several writers[2] and is going to be of use in the present paper too. The essential feature of POGGI's method is, as already mentioned previously, to replace the compressible fluid flow by an incompressible fluid flow due to a suitable distribution of sources throughout the field of flow. Thus, remembering that $\Delta\phi_1 = \mathrm{div}\,\boldsymbol{v}_1$ where $\boldsymbol{v}_1 = \mathrm{grad}\,\phi_1$, the equation $\Delta\phi_1 = f(x,y)$ is regarded as the equation for determining the velocity potential ϕ_1 for incompressible fluid flow with a continuous distribution of sources, the strength of a source at an element $dxdy$ being given by

$$\frac{1}{2\pi} f(x,y)dxdy\,.$$

Therefore, returning to equation (7), the strength of a source at an element $dxdy$ in the field of continuously distributed sources is given by

$$\frac{1}{4\pi c_0^2}\left(\frac{\partial\phi_0}{\partial x}\frac{\partial v_0^2}{\partial x}+\frac{\partial\phi_0}{\partial y}\frac{\partial v_0^2}{\partial y}\right)dx\,dy\,, \tag{9}$$

and our problem becomes to find the velocity potential for incompressible fluid flow due to such a continuous distribution of sources.

§5. Now, suppose that (ξ,η) and (x,y) are the rectangular co-ordinates of points in the ζ- and the z -planes respectively, where

---

(1) L. POGGI, loc. cit.

(2) C. KAPLAN, loc. cit. (N. A. C. A. Report Nos. **621, 624, 671**) (elliptic cylinder and symmetrical JOUKOWSKI aerofoil); K. TAMADA and Y. SAITŌ, loc. cit (circular cylinder); K. TAMADA, loc. cit. (sphere).

$\zeta=\xi+i\eta$ and $z=x+iy$, and that these two planes are conformally related with each other by a relation:

$$\zeta = F(z)\,, \tag{10}$$

where $F(z)$ is an analytic function of z. Let the ζ-plane be the plane of the profile such as the ellipse and let the z-plane be the plane of the circle into which the profile in the ζ-plane is mapped by the above conformal transformation (10). It is well known that at a pair of corresponding points in the two planes at which ζ and z possess no singularities, a source at one such point corresponds with a source of equal strength at the other; in other words, as far as a pair of correrponding ordinary points is concerned, the strength of a source is unaltered by the conformal transformation[1].

Then, it follows that at corresponding elements

$$\frac{1}{4\pi c_0^2}\left(\frac{\partial\phi_0}{\partial\xi}\frac{\partial v_0^2}{\partial\xi}+\frac{\partial\phi_0}{\partial\eta}\frac{\partial v_0^2}{\partial\eta}\right)d\xi d\eta$$

$$=\frac{1}{4\pi c_0^2}\left(\frac{\partial\phi_0}{\partial x}\frac{\partial v_0^2}{\partial x}+\frac{\partial\phi_0}{\partial y}\frac{\partial v_0^2}{\partial y}\right)dx\,dy\,, \tag{11}$$

where, in the expression on the right-hand side, ϕ_0 is the velocity potential for the incompressible fluid flow in the z-plane, while v_0 is the magnitude of the velocity in the incompressible fluid flow in the ζ-plane.

In polar coordinates (r,θ) the strength of a source at an element $dxdy$ of the z-plane is

$$\frac{1}{4\pi c_0^2}\left(\frac{\partial\phi_0}{\partial r}\frac{\partial v_0^2}{\partial r}+\frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial v_0^2}{\partial\theta}\right)r\,dr\,d\theta\,. \tag{12}$$

With the continuous distribution of sources known in the z-plane of the circle and given by (12), the velocity potential for the induced

(1) See, for example, S. L. GREEN, Hydro- and Aerodynamics. (1937), 35.

flow may be calculated as follows. It is evident that the velocity potential so calculated is the required first order correction ϕ_1 to the velocity potential for the compressible fluid flow past the profile in the ζ-plane.

Now, consider a unit source situated at a point $Q(=re^{i\theta})$ of the z-plane. Then, in the presence of a circular boundary of radius R as shown in Fig. 1, the complex velocity potential w_* for the induced flow at any point $P(=z)$ external to or on the boundary is given by

$$\begin{aligned} w_* &= \log(z-z_Q)+\log(z-z_S)-\log z \\ &= \log(z-re^{i\theta})+\log\left(z-\frac{R^2}{r}e^{i\theta}\right)-\log z, \end{aligned} \tag{13}$$

where S denotes the inverse point of the point Q with respect to the circle $|z|=R$ and its complex coordinate is $(R^2/r)e^{i\theta}$.

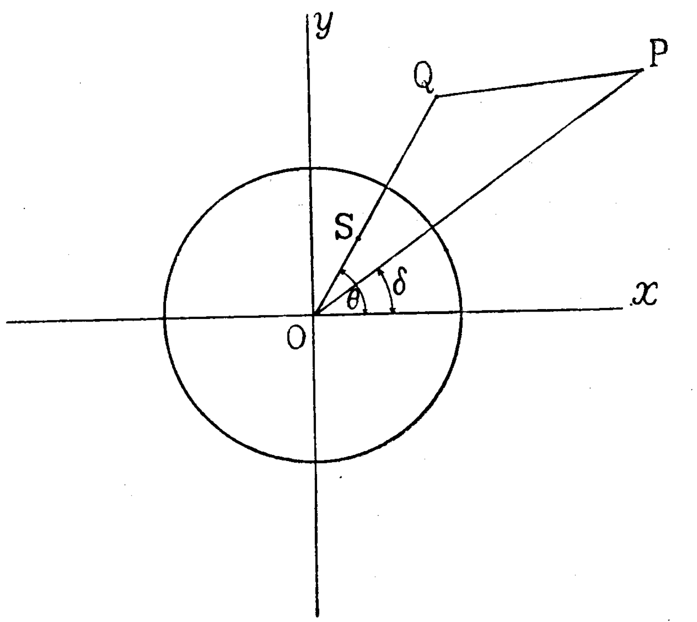

Fig. 1. z-plane.

We put $z = r_* e^{i\delta}$ so that the polar coordinates of the point P are (r_*, δ). Then, inserting this in the right-hand side of (13) and separating real and imaginary parts on both sides, we get the corresponding velocity potential ϕ_* in the form:

$$\phi_* = \Re(w_*) = \frac{1}{2}\log\left\{r^2 - 2rr_*\cos(\theta-\delta) + r_*^2\right\}$$
$$+\frac{1}{2}\log\left\{1 - 2\frac{R^2}{rr_*}\cos(\theta-\delta) + \left(\frac{R^2}{rr_*}\right)^2\right\}. \quad (14)$$

Thus, using this result and remembering that, in the continuous distribution of sources in the outside of the circle, the strength of a source at the element $rdrd\theta$ is given by (12), we get the first order correction ϕ_1 to the velocity potential for the compressible fluid in the form:

$$\phi_1 = \frac{1}{4\pi c_0^2}\int_R^\infty\int_0^{2\pi}\left(\frac{\partial\phi_0}{\partial r}\frac{\partial v_0^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial v_0^2}{\partial\theta}\right)\phi_* r\,dr\,d\theta\,. \quad (15)$$

For the sake of convenience we introduce a new variable λ defined as:

$$\lambda = \frac{R}{r}. \quad (16)$$

Also, we put

$$\lambda_* = \frac{R}{r_*}, \quad (17)$$

and

$$v_r = \frac{\partial\phi_0}{\partial r}, \qquad v_\theta = \frac{1}{r}\frac{\partial\phi_0}{\partial\theta}. \quad (18)$$

Then, using these quantities, the expression for ϕ_1 becomes:

$$\phi_1 = -\frac{R}{4\pi c_0^2}\int_0^1\int_0^{2\pi}\left(v_r\frac{\partial v_0^2}{\partial\lambda} - \frac{v_\theta}{\lambda}\frac{\partial v_0^2}{\partial\theta}\right)\phi_*\frac{d\lambda}{\lambda}d\theta\,, \quad (19)$$

with

$$\phi_* = \frac{1}{2}\log\left\{\frac{1}{\lambda^2} - 2\frac{1}{\lambda\lambda_*}\cos(\theta-\delta) + \frac{1}{\lambda_*^2}\right\}$$

$$+\frac{1}{2}\log\left\{1 - 2\lambda\lambda_*\cos(\theta-\delta) + \lambda^2\lambda_*^2\right\}, \qquad (20)$$

the additive constant $\log R$ in ϕ_* having been neglected.

Thus, if we know the magnitude v_0 of the velocity in the field of flow of an incompressible fluid in the ζ-plane of the profile and the velocity potential ϕ_0 for the incompressible fluid flow in the z-plane of the circle, into which the profile is transformed by the conformal transformation $\zeta = F(z)$, the first order correction ϕ_1 to the velocity potential for the irrotational flow of a compressible fluid in the ζ-plane can be calculated by the formula (19), in conjunction with (20).

III. The Subsonic Irrotational Flow of a Compressible Fluid past an Elliptic Cylinder.

§6. It is well known that the Joukowski transformation:

$$\zeta = z + \frac{a^2}{z}, \qquad (21)$$

where a is a real positive constant, transforms the circle of radius $R(>a)$ in the z-plane into an ellipse in the ζ-plane, whose semi-major and semi-minor axes are $R+a^2/R$ and $R-a^2/R$ respectively. The thickness ratio t of the ellipse, defined as the ratio of the semi-minor axis to the semi-major axis, then becomes:

$$t = \frac{R-\dfrac{a^2}{R}}{R+\dfrac{a^2}{R}}, \qquad (22)$$

or, writing for convenience,

$$\sigma = \frac{a}{R}, \tag{23}$$

we have

$$t = \frac{1-\sigma^2}{1+\sigma^2}. \tag{24}$$

This gives immediately

$$\sigma^2 = \frac{1-t}{1+t}. \tag{25}$$

Now we denote by w_0 the complex velocity potential for the incompressible fluid flow in the z-plane in case when a uniform stream of velocity U impinges on the circle of radius R from left to right in a direction which makes an acute angle α with the positive direction of the x-axis. Then, we have

$$w_0 = Ue^{-i\alpha}\left(z + \frac{R^2 e^{2i\alpha}}{z}\right). \tag{26}$$

Making use of this, together with (12), the conjugate complex velocity in the ζ-plane is then given by

$$\frac{dw_0}{d\zeta} = \frac{dw_0}{dz}\frac{dz}{d\zeta}$$

$$= Ue^{-i\alpha}\frac{z^2 - R^2 e^{2i\alpha}}{z^2 - a^2}. \tag{27}$$

It will easily be seen that in the ζ-plane, the magnitude of the undisturbed stream at infinity is also U and its direction makes the angle α with the positive direction of the ξ-axis, which itself coincides with the major axis of the ellipse.

When $\lambda = R/r$ and $\sigma = a/R$ are introduced, it follows that

$$v_0^2 = \left|\frac{dw_0}{d\zeta}\right|^2 = U^2\frac{1-2\lambda^2\cos 2(\theta-\alpha)+\lambda^4}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4}. \tag{28}$$

Following POGGI's procedure, we expand v_0^2/U^2 in a FOURIER series. Making use of the expansion :

$$\frac{1}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4}=\frac{1}{1-\sigma^4\lambda^4}\left\{1+2\sum_{n=1}^{\infty}(\sigma\lambda)^{2n}\cos 2n\theta\right\},\quad (\sigma\lambda<1) \tag{29}$$

we have

$$\frac{v_0^2}{U^2}=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_{2n}\cos 2n\theta+b_{2n}\sin 2n\theta)\ , \tag{30}$$

where

$$a_0=2\frac{1+(1-2\sigma^2\cos 2\alpha)\lambda^4}{1-\sigma^4\lambda^4}\ , \tag{31a}$$

and for $n\geqq 1$

$$\left.\begin{aligned} a_{2n}&=2\frac{\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\alpha}{\sigma^2(1-\sigma^4\lambda^4)}(\sigma\lambda)^{2n}\ ,\\ b_{2n}&=-\frac{2\sin 2\alpha}{\sigma^2}(\sigma\lambda)^{2n}. \end{aligned}\right\} \tag{31b}$$

For later use we shall here introduce the following notations for $n\geqq 0$, namely :

$$\left.\begin{aligned} \bar{a}_{2n}&=2\frac{\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\alpha}{\sigma^2(1-\sigma^4\lambda^4)}(\sigma\lambda)^{2n}\ ,\\ \bar{b}_{2n}&=-\frac{2\sin 2\alpha}{\sigma^2}(\sigma\lambda)^{2n}. \end{aligned}\right\} \tag{32}$$

It will be noted that when $n\geqq 1$, $\bar{a}_{2n}=a_{2n}$ and $\bar{b}_{2n}=b_{2n}$, but $\bar{a}_0$ is not equal to a_0 .

Next, taking real parts on both sides of (26) we get the velocity potential ϕ_0 for the incompressible fluid flow in the z-plane in the form:

$$\phi_0=U\left(r+\frac{R^2}{r}\right)\cos(\theta-\alpha)\ . \tag{33}$$

This gives immediately

$$\left.\begin{aligned} v_r &= \frac{\partial \phi_0}{\partial r} = U\left(1-\frac{R^2}{r^2}\right)\cos(\theta-\alpha)\ , \\ v_\theta &= \frac{1}{r}\frac{\partial \phi_0}{\partial \theta} = -U\left(1+\frac{R^2}{r^2}\right)\sin(\theta-\alpha)\ , \end{aligned}\right\} \tag{34}$$

or, using $\lambda = R/r$,

$$\left.\begin{aligned} v_r &= U(1-\lambda^2)\cos(\theta-\alpha)\ , \\ v_\theta &= -U(1+\lambda^2)\sin(\theta-\alpha)\ . \end{aligned}\right\} \tag{35}$$

Also, we may put these in the following forms:

$$\left.\begin{aligned} v_r &= U(c_1\cos\theta+c_2\sin\theta)\ , \\ v_\theta &= U(d_1\cos\theta+d_2\sin\theta)\ , \end{aligned}\right\} \tag{36}$$

where

$$\left.\begin{aligned} c_1 &= (1-\lambda^2)\cos\alpha\ , \\ c_2 &= (1-\lambda^2)\sin\alpha\ ; \\ d_1 &= (1+\lambda^2)\sin\alpha\ , \\ d_2 &= -(1+\lambda^2)\cos\alpha\ . \end{aligned}\right\} \tag{37}$$

Further, from (30) we have

$$\left.\begin{aligned} \frac{\partial v_0^2}{\partial \lambda} &= U^2\left\{\frac{1}{2}a_0'+\sum_{n=1}^{\infty}(a_{2n}'\cos 2n\theta + b_{2n}'\sin 2n\theta)\right\}\ , \\ \frac{\partial v_0^2}{\partial \theta} &= U^2\sum_{n=1}^{\infty}2n(b_{2n}\cos 2n\theta - a_{2n}\sin 2n\theta)\ , \end{aligned}\right\} \tag{38}$$

where the dashes denote differentiation with respect to λ, and combining these with (36), we get

$$v_r\frac{\partial v_0^2}{\partial\lambda} = \frac{1}{2}U^3 a_0'(c_1\cos\theta + c_2\sin\theta)$$

$$+\frac{1}{2}U^3\sum_{n=1}^{\infty}\Big[a_{2n}'c_1\Big\{\cos(2n-1)\theta + \cos(2n+1)\theta\Big\}$$

$$+b_{2n}'c_1\Big\{\sin(2n-1)\theta + \sin(2n+1)\theta\Big\}$$

$$+a_{2n}'c_2\Big\{-\sin(2n-1)\theta + \sin(2n+1)\theta\Big\}$$

$$+b_{2n}'c_2\Big\{\cos(2n-1)\theta - \cos(2n+1)\theta\Big\}\Big], \quad (39)$$

and

$$\frac{v_\theta}{\lambda}\frac{\partial v_0^2}{\partial\theta} = -\frac{U^3}{2\lambda}\sum_{n=1}^{\infty}2n\Big[a_{2n}d_1\Big\{\sin(2n-1)\theta + \sin(2n+1)\theta\Big\}$$

$$-b_{2n}d_1\Big\{\cos(2n-1)\theta + \cos(2n+1)\theta\Big\}$$

$$+a_{2n}d_2\Big\{\cos(2n-1)\theta - \cos(2n+1)\theta\Big\}$$

$$+b_{2n}d_2\Big\{\sin(2n-1)\theta - \sin(2n+1)\theta\Big\}\Big]. \quad (40)$$

If we substitute these expressions for $v_r\partial v_0^2/\partial\lambda$ and $(v_\theta/\lambda)\partial v_0^2/\partial\theta$ into the right-hand side of the formula (19) and perform the integrations with respect to θ and λ, we shall obtain the expression for ϕ_1.

To do this, we first divide the range of integration with respect to λ into two parts in such a way that λ ranges from 0 to λ_* in one part and from λ_* to 1 in the other. Thus, we write the formula (19) in the form :

$$\phi_1 = -\frac{R}{4\pi c_0^2}\Big[\int_0^{\lambda_*}\int_0^{2\pi}\Big(v_r\frac{\partial v_0^2}{\partial\lambda} - \frac{v_\theta}{\lambda}\frac{\partial v_0^2}{\partial\theta}\Big)\phi_*\frac{d\lambda}{\lambda}d\theta$$

$$+\int_{\lambda_*}^{1}\int_0^{2\pi}\Big(v_r\frac{\partial v_0^2}{\partial\lambda} - \frac{v_\theta}{\lambda}\frac{\partial v_0^2}{\partial\theta}\Big)\phi_*\frac{d\lambda}{\lambda}d\theta\Big], \quad (41)$$

and remembering that $\lambda/\lambda_* < 1$ in the inner part where $0 \leqq \lambda \leqq \lambda_*$ and $\lambda_*/\lambda < 1$ in the outer part where $\lambda_* \leqq \lambda \leqq 1$, we express ϕ_* given by (20) in different forms: namely, in the inner part $0 \leqq \lambda \leqq \lambda_*$ we write ϕ_* in the form:

$$\phi_* = \frac{1}{2}\log\left\{1 - 2\frac{\lambda}{\lambda_*}\cos(\theta-\delta) + \left(\frac{\lambda}{\lambda_*}\right)^2\right\} - \log\lambda$$
$$+\frac{1}{2}\log\left\{1 - 2\lambda\lambda_*\cos(\theta-\delta) + \lambda^2\lambda_*^2\right\}, \qquad (42\text{a})$$

while in the outer part $\lambda_* \leqq \lambda \leqq 1$ we write

$$\phi_* = \frac{1}{2}\log\left\{1 - 2\frac{\lambda_*}{\lambda}\cos(\theta-\delta) + \left(\frac{\lambda_*}{\lambda}\right)^2\right\} - \log\lambda_*$$
$$+\frac{1}{2}\log\left\{1 - 2\lambda\lambda_*\cos(\theta-\delta) + \lambda^2\lambda_*^2\right\}. \qquad (42\text{b})$$

Then, inserting the foregoing expressions (39) and (40) for $v_r \partial v_0^2/\partial\lambda$ and $(v_\theta/\lambda)\partial v_0^2/\partial\theta$, we perform the integrations with respect to θ with the aid of the following two well-known integral formulae:

$$\left.\begin{aligned} &\int_0^{2\pi}\log\left\{1 - 2\mu\cos(\theta-\delta) + \mu^2\right\}\cos n\theta\, d\theta = -\frac{2\pi}{n}\mu^n\cos n\delta\,, \\ &\qquad\qquad (n \geqq 1\,,\quad 0 < \mu \leqq 1) \\ &\int_0^{2\pi}\log\left\{1 - 2\mu\cos(\theta-\delta) + \mu^2\right\}\sin n\theta\, d\theta = -\frac{2\pi}{n}\mu^n\sin n\delta\,. \\ &\qquad\qquad (n \geqq 1\,,\quad 0 < \mu \leqq 1) \end{aligned}\right\} \qquad (43)$$

Thus, rewriting various terms in convenient forms and arranging, we have, after replacing the derivatives a_0', a_{2n}', b_{2n}' $(n \geqq 1)$ by a_0, a_{2n}, b_{2n} $(n \geqq 1)$ by means of partial integrations,

$$\phi_1 = -\frac{RU^3}{4c_0^2}\Bigg[\frac{1}{2}(a_0)_{\lambda=0}\left(\lambda_* + \frac{1}{\lambda_*}\right)\cos(\delta-\alpha)$$

$$-\cos(\delta-\alpha)\left\{\int_0^{\lambda_*} a_0\frac{\lambda}{\lambda_*}d\lambda + \int_{\lambda_*}^1 a_0\frac{\lambda_*}{\lambda^3}d\lambda + \int_0^1 a_0\lambda\lambda_*\,d\lambda\right\}$$

$$+\sum_{n=1}^{\infty}\cos\left\{(2n-1)\delta+\alpha\right\}\left\{\int_0^{\lambda_*}\frac{a_{2n}}{\lambda^2}\left(\frac{\lambda}{\lambda_*}\right)^{2n-1}d\lambda + \int_{\lambda_*}^1 a_{2n}\left(\frac{\lambda_*}{\lambda}\right)^{2n-1}d\lambda\right.$$

$$\left.+\int_0^1\frac{a_{2n}}{\lambda^2}(\lambda\lambda_*)^{2n-1}d\lambda\right\}$$

$$+\sum_{n=1}^{\infty}\sin\left\{(2n-1)\delta+\alpha\right\}\left\{\int_0^{\lambda_*}\frac{b_{2n}}{\lambda^2}\left(\frac{\lambda}{\lambda_*}\right)^{2n-1}d\lambda + \int_{\lambda_*}^1 b_{2n}\left(\frac{\lambda_*}{\lambda}\right)^{2n-1}d\lambda\right.$$

$$\left.+\int_0^1\frac{b_{2n}}{\lambda^2}(\lambda\lambda_*)^{2n-1}d\lambda\right\}$$

$$-\sum_{n=1}^{\infty}\cos\left\{(2n+1)\delta-\alpha\right\}\left\{\int_0^{\lambda_*}a_{2n}\left(\frac{\lambda}{\lambda_*}\right)^{2n+1}d\lambda + \int_{\lambda_*}^1\frac{a_{2n}}{\lambda^2}\left(\frac{\lambda_*}{\lambda}\right)^{2n+1}d\lambda\right.$$

$$\left.+\int_0^1 a_{2n}(\lambda\lambda_*)^{2n+1}d\lambda\right\}$$

$$-\sum_{n=1}^{\infty}\sin\left\{(2n+1)\delta-\alpha\right\}\left\{\int_0^{\lambda_*}b_{2n}\left(\frac{\lambda}{\lambda_*}\right)^{2n+1}d\lambda + \int_{\lambda_*}^1\frac{b_{2n}}{\lambda^2}\left(\frac{\lambda_*}{\lambda}\right)^{2n+1}d\lambda\right.$$

$$\left.+\int_0^1 b_{2n}(\lambda\lambda_*)^{2n+1}d\lambda\right\}\Bigg]. \quad (44)$$

§7. We next proceed to the evaluation of various integrals in (44). For this purpose, we put, for the sake of simplicity,

$$I_1 = \sum_{n=1}^{\infty} \cos\left\{(2n-1)\delta+\alpha\right\} \int_0^{\lambda_*} \frac{a_{2n}}{\lambda^2}\left(\frac{\lambda}{\lambda_*}\right)^{2n-1} d\lambda , \tag{45}$$

$$I_2 = \sum_{n=1}^{\infty} \cos\left\{(2n-1)\delta+\alpha\right\} \int_{\lambda_*}^{1} a_{2n}\left(\frac{\lambda_*}{\lambda}\right)^{2n-1} d\lambda , \tag{46}$$

$$I_3 = \sum_{n=1}^{\infty} \cos\left\{(2n-1)\delta+\alpha\right\} \int_0^{1} \frac{a_{2n}}{\lambda^2}(\lambda\lambda_*)^{2n-1} d\lambda , \tag{47}$$

$$I_4 = \sum_{n=1}^{\infty} \sin\left\{(2n-1)\delta+\alpha\right\} \int_0^{\lambda_*} \frac{b_{2n}}{\lambda^2}\left(\frac{\lambda}{\lambda_*}\right)^{2n-1} d\lambda , \tag{48}$$

$$I_5 = \sum_{n=1}^{\infty} \sin\left\{(2n-1)\delta+\alpha\right\} \int_{\lambda_*}^{1} b_{2n}\left(\frac{\lambda_*}{\lambda}\right)^{2n-1} d\lambda , \tag{49}$$

$$I_6 = \sum_{n=1}^{\infty} \sin\left\{(2n-1)\delta+\alpha\right\} \int_0^{1} \frac{b_{2n}}{\lambda^2}(\lambda\lambda_*)^{2n-1} d\lambda , \tag{50}$$

$$I_7 = \sum_{n=1}^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_0^{\lambda_*} a_{2n}\left(\frac{\lambda}{\lambda_*}\right)^{2n+1} d\lambda , \tag{51}$$

$$I_8 = \sum_{n=1}^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_{\lambda_*}^{1} \frac{a_{2n}}{\lambda^2}\left(\frac{\lambda_*}{\lambda}\right)^{2n+1} d\lambda , \tag{52}$$

$$I_9 = \sum_{n=1}^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_0^{1} a_{2n}(\lambda\lambda_*)^{2n+1} d\lambda , \tag{53}$$

$$I_{10} = \sum_{n=1}^{\infty} \sin\left\{(2n+1)\delta-\alpha\right\} \int_0^{\lambda_*} b_{2n}\left(\frac{\lambda}{\lambda_*}\right)^{2n+1} d\lambda , \tag{54}$$

$$I_{11} = \sum_{n=1}^{\infty} \sin\left\{(2n+1)\delta-\alpha\right\} \int_{\lambda_*}^{1} \frac{b_{2n}}{\lambda^2}\left(\frac{\lambda_*}{\lambda}\right)^{2n+1} d\lambda , \tag{55}$$

$$I_{12} = \sum_{n=1}^{\infty} \sin\left\{(2n+1)\delta-\alpha\right\} \int_0^{1} b_{2n}(\lambda\lambda_*)^{2n+1} d\lambda . \tag{56}$$

Then, rearranging some of the terms, the expression for ϕ_1 may be written in a simpler form as follows:

$$\phi_1 = -\frac{RU^3}{4c_0^2}\Bigg[\frac{1}{2}(a_0)_{\lambda=0}\left(\lambda_* + \frac{1}{\lambda_*}\right)\cos(\delta-\alpha)$$

$$-\cos(\delta-\alpha)\left\{\int_0^{\lambda_*} a_0\frac{\lambda}{\lambda_*}d\lambda + \int_{\lambda_*}^1 a_0\frac{\lambda_*}{\lambda^3}d\lambda + \int_0^1 a_0\lambda\lambda_* d\lambda\right\}$$

$$+(I_1-I_7)+(I_2-I_8)+(I_3-I_9)$$

$$+(I_4-I_{10})+(I_5-I_{11})+(I_6-I_{12})\Bigg]. \qquad (57)$$

In the first place, we shall deform the above twelve integrals I_1 to I_{12}. We begin with the first integral I_1 defined by (45), which may be written in the form:

$$I_1 = \Re\left[\sum_{n=1}^{\infty} e^{i\alpha}\int_0^{\lambda_*}\frac{a_{2n}}{\lambda^2}\left(\frac{\lambda}{\lambda_*}\right)^{2n-1} e^{i(2n-1)\delta}d\lambda\right], \qquad (58)$$

where, in general, $\Re(z)$ means the real part of z. If, therefore, we insert the value of a_{2n} given by (31b), we have, after a little modification,

$$I_1 = \Re\left[e^{i(\delta+\alpha)}\frac{1}{\lambda_*}\int_0^{\lambda_*}\frac{\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\alpha}{1-\sigma^4\lambda^4}\sum_{n=0}^{\infty}\left(\sigma\frac{\lambda^2}{\lambda_*}\right)^{2n} e^{i2n\delta}\,2\lambda d\lambda\right]. \qquad (59)$$

Further, writing $\lambda^2=\tau$ for purposes of integration only and making use of the formula:

$$\sum_{n=0}^{\infty}\left(\sigma\frac{\tau}{\lambda_*}\right)^{2n} e^{i2n\delta} = \left\{1-\left(\sigma\frac{\tau}{\lambda_*}\right)^2 e^{2i\delta}\right\}^{-1},$$

which, together with other formulae of similar forms, will also be used in later lines, we have

$$I_1 = \Re\left[e^{i(\delta+\alpha)} \frac{1}{\lambda_*} \int_0^{\lambda_*^2} \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{(1-\sigma^4\tau^2)(1-\sigma^2\lambda_*^{-2}\tau^2 e^{2i\delta})} d\tau \right]. \quad (60)$$

This integral can be evaluated without difficulty. However, the evaluation of this integral as well as of other integrals which follow will be postponed to later pages.

Next, we deal with the second integral I_2 given by (46). Proceeding as before, we have

$$I_2 = \Re\left[e^{i(\delta+\alpha)} \frac{\lambda_*}{1-\sigma^2\lambda_*^2 e^{2i\delta}} \int_{\lambda_*^2}^{1} \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{1-\sigma^4\tau^2} d\tau \right]. \quad (61)$$

The third integral I_3 defined as (47) can be deformed in a similar manner, and we have

$$I_3 = \Re\left[e^{i(\delta+\alpha)} \lambda_* \int_0^{1} \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{(1-\sigma^4\tau^2)(1-\sigma^2\lambda_*^2\tau^2 e^{2i\delta})} d\tau \right]. \quad (62)$$

In the next place, we deal with the three integrals I_4, I_5 and I_6 which contain b_{2n}. Inserting the value of b_{2n} given by (31b) and proceeding as before, we get

$$I_4 = \Re\left[i e^{i(\delta+\alpha)} \frac{1}{\lambda_*} \sin 2\alpha \int_0^{\lambda_*^2} \frac{d\tau}{1-\sigma^2\lambda_*^{-2}\tau^2 e^{2i\delta}} \right]. \quad (63)$$

The integral I_5 can be evaluated immediately, giving

$$I_5 = \Re\left[i e^{i(\delta+\alpha)} \lambda_* \sin 2\alpha \sum_{n=0}^{\infty} (\sigma\lambda_*)^{2n} e^{i2n\delta} \int_{\lambda_*^2}^{1} d\tau \right]$$

$$= \Re\left[i e^{i(\delta+\alpha)} \frac{\lambda_*(1-\lambda_*^2)}{1-\sigma^2\lambda_*^2 e^{2i\delta}} \sin 2\alpha \right]. \quad (64)$$

Also, the integral I_6 becomes:

$$I_6 = \Re\left[ie^{i(\delta+\alpha)}\lambda_* \sin 2\alpha \int_0^1 \frac{d\tau}{1-\sigma^2\lambda_*^2\tau^2 e^{2i\delta}}\right]. \qquad (65)$$

Next, we consider the integral I_7 given by (51), which, using $\bar{a}_{2n}$ $(n \geqq 0)$ defined as (32), may be written in the form:

$$I_7 = \sum_{n=0}^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_0^{\lambda_*} \bar{a}_{2n}\left(\frac{\lambda}{\lambda_*}\right)^{2n+1} d\lambda - \cos(\delta-\alpha) \int_0^{\lambda_*} \bar{a}_0 \frac{\lambda}{\lambda_*} d\lambda . \qquad (66)$$

For brevity we put

$$I_7' = \sum_{n=0}^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_0^{\lambda_*} \bar{a}_{2n}\left(\frac{\lambda}{\lambda_*}\right)^{2n+1} d\lambda . \qquad (67)$$

Then, we have

$$I_7 = I_7' - \cos(\delta-\alpha) \int_0^{\lambda_*} \bar{a}_0 \frac{\lambda}{\lambda_*} d\lambda . \qquad (68)$$

Using the value of $\bar{a}_{2n}$ given by (32), the integral I_7' can be deformed in the following form:

$$I_7' = \Re\left[e^{i(\delta-\alpha)} \frac{1}{\sigma^2\lambda_*} \int_0^{\lambda_*^2} \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{(1-\sigma^4\tau^2)(1-\sigma^2\lambda_*^{-2}\tau^2 e^{2i\delta})} d\tau\right]. \qquad (69)$$

In like manner, the integral I_8 given by (52) can be written as:

$$I_8 = \sum_{n=0}^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_{\lambda_*}^1 \frac{\bar{a}_{2n}}{\lambda^2}\left(\frac{\lambda_*}{\lambda}\right)^{2n+1} d\lambda - \cos(\delta-\alpha) \int_{\lambda_*}^1 \bar{a}_0 \frac{\lambda_*}{\lambda^3} d\lambda . \qquad (70)$$

Thus, putting

$$I_8' = \sum^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_{\lambda_*}^1 \frac{\bar{a}_{2n}}{\lambda^2}\left(\frac{\lambda_*}{\lambda}\right)^{2n+1} d\lambda , \qquad (71)$$

we have

$$I_8 = I_8' - \cos(\delta-\alpha)\int_{\lambda_*}^{1} \bar{a}_0 \frac{\lambda_*}{\lambda^3} d\lambda\ , \tag{72}$$

and

$$I_8' = \Re\left[e^{i(\delta-\alpha)} \frac{\lambda_*}{\sigma^2(1-\sigma^2\lambda_*^2 e^{2i\delta})} \int_{\lambda_*^2}^{1} \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{\tau^2(1-\sigma^4\tau^2)} d\tau \right] . \tag{73}$$

Further, we consider the integral I_9 given by (53). This may be put in the form:

$$I_9 = \sum_{n=0}^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_0^1 \bar{a}_{2n}(\lambda\lambda_*)^{2n+1} d\lambda - \cos(\delta-\alpha)\int_0^1 \bar{a}_0 \lambda\lambda_* d\lambda\ . \tag{74}$$

Therefore, writing

$$I_9' = \sum_{n=0}^{\infty} \cos\left\{(2n+1)\delta-\alpha\right\} \int_0^1 \bar{a}_{2n}(\lambda\lambda_*)^{2n+1} d\lambda\ , \tag{75}$$

we get

$$I_9 = I_9' - \cos(\delta-\alpha)\int_0^1 \bar{a}_0 \lambda\lambda_* d\lambda\ , \tag{76}$$

and

$$I_9' = \Re\left[e^{i(\delta-\alpha)} \frac{\lambda_*}{\sigma^2} \int_0^1 \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{(1-\sigma^4\tau^2)(1-\sigma^2\lambda_*^2\tau^2 e^{2i\delta})} d\tau \right] . \tag{77}$$

Lastly, we deal with the remaining three integrals I_{10}, I_{11} and I_{12}. We first consider I_{10} given by (54). Using $\bar{b}_{2n}$ defined by (32), we have

$$I_{10} = \sum_{n=0}^{\infty} \sin\left\{(2n+1)\delta-\alpha\right\} \int_0^{\lambda_*} \bar{b}_{2n}\left(\frac{\lambda}{\lambda_*}\right)^{2n+1} d\lambda - \sin(\delta-\alpha)\int_0^{\lambda_*} \bar{b}_0 \frac{\lambda}{\lambda_*} d\lambda\ , \tag{78}$$

and therefore

$$I_{10} = \Re\left[ie^{i(\delta-\alpha)}\frac{1}{\sigma^2\lambda_*}\sin 2\alpha\int_0^{\lambda_*^2}\frac{d\tau}{1-\sigma^2\lambda_*^{-2}\tau^2 e^{2i\delta}}\right]$$

$$+\frac{1}{\sigma^2\lambda_*}\sin 2\alpha\sin(\delta-\alpha)\int_0^{\lambda_*^2}d\tau\ ,$$

or,

$$I_{10} = \Re\left[ie^{i(\delta-\alpha)}\frac{1}{\sigma^2\lambda_*}\sin 2\alpha\int_0^{\lambda_*^2}\frac{d\tau}{1-\sigma^2\lambda_*^{-2}\tau^2 e^{2i\delta}}\right]$$

$$+\frac{\lambda_*}{\sigma^2}\sin 2\alpha\sin(\delta-\alpha)\ . \tag{79}$$

Similarly, the integral I_{11} can be written as:

$$I_{11} = \sum_{n=0}^{\infty}\sin\left\{(2n+1)\delta-\alpha\right\}\int_{\lambda_*}^{1}\frac{\bar{b}_{2n}}{\lambda^2}\left(\frac{\lambda_*}{\lambda}\right)^{2n+1}d\lambda-\sin(\delta-\alpha)\int_{\lambda_*}^{1}\bar{b}_0\frac{\lambda_*}{\lambda^3}d\lambda\ . \tag{80}$$

Thus,

$$I_{11} = \Re\left[ie^{i(\delta-\alpha)}\frac{\lambda_*}{\sigma^2(1-\sigma^2\lambda_*^2 e^{2i\delta})}\sin 2\alpha\int_{\lambda_*^2}^{1}\frac{d\tau}{\tau^2}\right]$$

$$+\frac{\lambda_*}{\sigma^2}\sin 2\alpha\sin(\delta-\alpha)\int_{\lambda_*^2}^{1}\frac{d\tau}{\tau^2}\ ,$$

or,

$$I_{11} = \Re\left[ie^{i(\delta-\alpha)}\frac{1-\lambda_*^2}{\sigma^2\lambda_*(1-\sigma^2\lambda_*^2 e^{2i\delta})}\sin 2\alpha\right]$$

$$+\frac{1-\lambda_*^2}{\sigma^2\lambda_*}\sin 2\alpha\sin(\delta-\alpha)\ . \tag{81}$$

Further, proceeding in a similar manner, we can deform the integral I_{12} in the form:

$$I_{12} = \sum_{n=0}^{\infty} \sin\left\{(2n+1)\delta-\alpha\right\} \int_0^1 \bar{b}_{2n}(\lambda\lambda_*)^{2n+1} d\lambda - \sin(\delta-\alpha)\int_0^1 \bar{b}_0 \lambda\lambda_* d\lambda$$

$$= \Re\left[ie^{i(\delta-\alpha)}\frac{\lambda_*}{\sigma^2}\sin 2\alpha \int_0^1 \frac{d\tau}{1-\sigma^2\lambda_*^2\tau^2 e^{2i\xi}}\right] + \frac{\lambda_*}{\sigma^2}\sin 2\alpha \sin(\delta-\alpha) \, . \tag{82}$$

Finally, using the above results, the expression for ϕ_1 can be put in the following form:

$$\phi_1 = -\frac{RU^3}{4c_0^2}\left[\frac{1}{2}(a_0)_{\lambda=0}\frac{1+\lambda_*^2}{\lambda_*}\cos(\delta-\alpha)\right.$$

$$+\cos(\delta-\alpha)\left\{\int_0^{\lambda_*}(\bar{a}_0-a_0)\frac{\lambda}{\lambda_*}d\lambda + \int_{\lambda_*}^1 (\bar{a}_0-a_0)\frac{\lambda_*}{\lambda^3}d\lambda\right.$$

$$\left.+\int_0^1 (\bar{a}_0-a_0)\lambda\lambda_* d\lambda\right\}$$

$$+(I_1-I_7')+(I_2-I_8')+(I_3-I_9')$$

$$\left.+(I_4-I_{10})+(I_5-I_{11})+(I_6-I_{12})\right] . \tag{83}$$

§ 8. Now, we consider an indefinite integral J_1 defined as:

$$J_1 = \int \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{(1-\sigma^4\tau^2)(1-\sigma^2 k^2\tau^2 e^{2i\xi})} d\tau \, , \tag{84}$$

where k is a constant. The integral of this form with $k=\lambda_*^{-1}$ and with $(0, \lambda_*^2)$ as the limits occurs in I_1 and I_7' given respectively by (60) and (69), and also the similar integral with $k=\lambda_*$ and with (0, 1) as the limits occurs in I_3 and I_9' given by (62) and (77) respectively. It is to be noted here that since $\sigma<1$ and $\lambda_*<1$, the value of the quantity $\sigma k\tau$ is less than unity not only in the former integral in which $k=\lambda_*^{-1}$ and $0\leqq\tau\leqq\lambda_*^2$ but also in the latter in which $k=\lambda_*$ and $0\leqq\tau\leqq 1$. We remember also that $\sigma^2\tau$ is always less than unity.

Expanding the above integrand in partial fractions and performing simple integrations, we obtain

$$J_1 = \frac{1-2\sigma^2\cos 2\alpha+\sigma^4}{2\sigma^2(\sigma^2-k^2e^{2i\delta})}\log\frac{1+\sigma^2\tau}{1-\sigma^2\tau} + \frac{(\cos 2\alpha-\sigma^2)k^2e^{2i\delta}+\sigma^2\cos 2\alpha-1}{2\sigma k\, e^{i\delta}(\sigma^2-k^2e^{2i\delta})}\log\frac{1+\sigma k\tau e^{i\delta}}{1-\sigma k\tau e^{i\delta}}\,. \tag{85}$$

Next, we consider another indefinite integral J_2 defined as:

$$J_2 = \int\frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{1-\sigma^4\tau^2}d\tau\,. \tag{86}$$

The integral of this form with $(\lambda_*^2, 1)$ as the limits occurs in I_2 given by (61). Expanding, as before, the integrand in partial fractions and carrying out simple integrations, we get

$$J_2 = \frac{\sigma^2\cos 2\alpha-1}{\sigma^2}\tau + \frac{1-2\sigma^2\cos 2\alpha+\sigma^4}{2\sigma^4}\log\frac{1+\sigma^2\tau}{1-\sigma^2\tau}\,. \tag{87}$$

It is to be noted that since the integral J_2 is a limiting form of J_1 when k tends to zero, the result (87) can also be obtained directly from (85) by considering the limit when $k\to 0$.

Further, we put

$$J_3 = \int\frac{d\tau}{1-\sigma^2k^2\tau^2e^{2i\delta}}\,. \tag{88}$$

The integral of this form with $k=\lambda_*^{-1}$ and with $(0, \lambda_*^2)$ as the limits occurs in I_4 and I_{10} given respectively by (63) and (79). Also, the similar integral with $k=\lambda_*$ and with (0,1) as the limits occurs in I_6 and I_{12} given by (65) and (82) respectively. This integral can be evaluated immediately and we have

$$J_3 = \frac{1}{2\sigma k e^{i\delta}}\log\frac{1+\sigma k\tau e^{i\delta}}{1-\sigma k\tau e^{i\delta}}\,. \tag{89}$$

Lastly, we also consider an indefinite integral J_4 defined as:

$$J_4 = \int \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{\tau^2(1-\sigma^4\tau^2)} d\tau . \tag{90}$$

The integral of this form with $(\lambda_*^2, 1)$ as the limits occurs in I_8' given by (73). Expanding the integrand in partial fractions and performing simple integrations, we get

$$J_4 = \frac{1}{2}(1-2\sigma^2\cos 2\alpha+\sigma^4)\log\frac{1+\sigma^2\tau}{1-\sigma^2\tau} - (\sigma^2-\cos 2\alpha)\frac{1}{\tau}. \tag{91}$$

§9. Using the above results we are now able to calculate the values of the quantities I_1, I_2, I_3, I_4, I_6, I_7', I_8', I_9', I_{10} and I_{12} given respectively by (60), (61), (62), (63), (65), (69), (73), (77), (79) and (82).

First, combining I_1 and I_7' we have

$$I_1 - I_7' = \Re\left[e^{i\delta}\left(e^{i\alpha}-\frac{e^{-i\alpha}}{\sigma^2}\right)\frac{1}{\lambda_*}\int_0^{\lambda_*^2} \frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{(1-\sigma^4\tau^2)(1-\sigma^2\lambda_*^{-2}\tau^2 e^{2i\delta})} d\tau\right]. \tag{92}$$

If, in (85), we put $k=\lambda_*^{-1}$ and take 0 and λ_*^2 as the lower and the upper limits respectively, we can obtain immediately the value of the integral in the above formula, and consequently we have

$$I_1 - I_7' = \Re\left[\frac{e^{i\delta}(\sigma^2 e^{i\alpha}-e^{-i\alpha})(1-2\sigma^2\cos 2\alpha+\sigma^4)\lambda_*}{2\sigma^4(\sigma^2\lambda_*^2-e^{2i\delta})}\log\frac{1+\sigma^2\lambda_*^2}{1-\sigma^2\lambda_*^2}\right.$$
$$+\frac{(\sigma^2 e^{i\alpha}-e^{-i\alpha})\{(\cos 2\alpha-\sigma^2)e^{2i\delta}+\sigma^2\lambda_*^2\cos 2\alpha-\lambda_*^2\}}{2\sigma^3(\sigma^2\lambda_*^2-e^{2i\delta})}$$
$$\left.\times\log\frac{1+\sigma\lambda_* e^{i\delta}}{1-\sigma\lambda_* e^{i\delta}}\right]. \tag{93}$$

Thus, taking the real parts of the quantities on the right-hand side, we get

$$I_1 - I_7' = \frac{1}{2\sigma^4(1-2\sigma^2\lambda_*^2\cos 2\delta+\sigma^4\lambda_*^4)}$$

$$\times\Big\{\lambda_*(1-2\sigma^2\cos 2\alpha+\sigma^4)\Big[(1+\sigma^4\lambda_*^2)\cos(\delta+\alpha)-\sigma^2(1+\lambda_*^2)\cos(\delta-\alpha)\Big]$$

$$\times\log\frac{1+\sigma^2\lambda_*^2}{1-\sigma^2\lambda_*^2}$$

$$+\frac{\sigma}{4}\Big[\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta+3\alpha)-\Big\{\sigma^2(1+2\sigma^4)+2+\sigma^4\Big\}\lambda_*^2\cos(2\delta+\alpha)$$

$$+3\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta-\alpha)-\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta-3\alpha)$$

$$+(1-\sigma^2)(1-\sigma^4\lambda_*^4)\cos 3\alpha+(1-\sigma^2)\Big\{(2-\sigma^2)\lambda_*^4+1-2\sigma^2\Big\}\cos\alpha\Big]$$

$$\times\log\frac{1+2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}{1-2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}$$

$$+\frac{\sigma}{2}\Big[\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta+3\alpha)+\Big\{\sigma^2(1+2\sigma^4)-2-\sigma^4\Big\}\lambda_*^2\sin(2\delta+\alpha)$$

$$+3\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta-\alpha)+\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta-3\alpha)$$

$$+(1+\sigma^2)(1-\sigma^4\lambda_*^4)\sin 3\alpha+(1+\sigma^2)\Big\{\sigma^2(2+\sigma^2)\lambda_*^4-1-2\sigma^2\Big\}\sin\alpha\Big]$$

$$\times\tan^{-1}\frac{2\sigma\lambda_*\sin\delta}{1-\sigma^2\lambda_*^2}\Big\}. \qquad (94)$$

Next, taking λ_*^2 and 1 as the lower and the upper limits respectively in the integral J_2, we calculate, with the aid of the formula (87), the value of the integral in the quantity I_2 given by (61). Then, using the result thus obtained we get

$$I_2 = \Re\Big[\frac{e^{i(\delta+\alpha)}\lambda_*}{1-\sigma^2\lambda_*^2 e^{2i\delta}}\Big\{\frac{\sigma^2\cos 2\alpha-1}{\sigma^2}(1-\lambda_*^2)$$

$$+\frac{1-2\sigma^2\cos 2\alpha+\sigma^4}{2\sigma^4}\log\frac{(1+\sigma^2)(1-\sigma^2\lambda_*^2)}{(1-\sigma^2)(1+\sigma^2\lambda_*^2)}\Big\}\Big]. \qquad (95)$$

Similarly, if we take λ_*^2 and 1 as the lower and the upper limits respectively in (91), we can obtain the value of the integral in I_8' given by (73). Using the result thus obtained we have

$$I_8' = \Re\left[\frac{e^{i(\delta-\alpha)}\lambda_*}{1-\sigma^2\lambda_*^2 e^{2i\delta}}\left\{\frac{\cos 2\alpha-\sigma^2}{\sigma^2}\left(1-\frac{1}{\lambda_*^2}\right)\right.\right.$$
$$\left.\left.+\frac{1-2\sigma^2\cos 2\alpha+\sigma^4}{2\sigma^2}\log\frac{(1+\sigma^2)(1-\sigma^2\lambda_*^2)}{(1-\sigma^2)(1+\sigma^2\lambda_*^2)}\right\}\right]. \quad (96)$$

Combining I_2 and I_8' we get

$$I_2-I_8' = \Re\left[\frac{\lambda_*}{\sigma^2(1-\sigma^2\lambda_*^2 e^{2i\delta})}\left\{e^{i(\delta+\alpha)}(\sigma^2\cos 2\alpha-1)(1-\lambda_*^2)\right.\right.$$
$$\left.\left.+e^{i(\delta-\alpha)}(\cos 2\alpha-\sigma^2)\frac{1-\lambda_*^2}{\lambda_*^2}\right\}\right].$$
$$+\frac{\lambda_*(1-2\sigma^2\cos 2\alpha+\sigma^4)}{2\sigma^4(1-\sigma^2\lambda_*^2 e^{2i\delta})}(e^{i(\delta+\alpha)}-\sigma^2 e^{i(\delta-\alpha)})\log\frac{(1+\sigma^2)(1-\sigma^2\lambda_*^2)}{(1-\sigma^2)(1+\sigma^2\lambda_*^2)}\Bigg].$$
$$(97)$$

Therefore, we have

$$I_2-I_8' = \frac{1}{2\sigma^4(1-2\sigma^2\lambda_*^2\cos 2\delta+\sigma^4\lambda_*^4)}$$
$$\times\left\{\lambda_*(1-2\sigma^2\cos 2\alpha+\sigma^4)\left[(1+\sigma^4\lambda_*^2)\cos(\delta+\alpha)-\sigma^2(1+\lambda_*^2)\cos(\delta-\alpha)\right]\right.$$
$$\times\log\frac{(1+\sigma^2)(1-\sigma^2\lambda_*^2)}{(1-\sigma^2)(1+\sigma^2\lambda_*^2)}$$
$$+\frac{\sigma^2(1-\lambda_*^2)}{\lambda_*}\left[\left\{1-\sigma^4\lambda_*^4-2(1-\sigma^4)\lambda_*^2\right\}\cos(\delta+\alpha)\right.$$
$$\left.\left.-2\sigma^2(1-\lambda_*^4)\cos(\delta-\alpha)+(1-\sigma^4\lambda_*^4)\cos(\delta-3\alpha)\right]\right\}. \quad (98)$$

Next, from (62) and (77) we have

$$I_3-I_9' = \Re\left[e^{i\delta}\left(e^{i\alpha}-\frac{e^{-i\alpha}}{\sigma^2}\right)\lambda_*\int_0^1\frac{\sigma^2(1+\tau^2)-(1+\sigma^4\tau^2)\cos 2\alpha}{(1-\sigma^4\tau^2)(1-\sigma^2\lambda_*^2\tau^2 e^{2i\delta})}d\tau\right].$$
$$(99)$$

The value of the integral on the right-hand side can easily be obtained with the aid of (84) and (85), by putting $k=\lambda_*$ and taking 0 and 1 as the lower and the upper limits. Using the result thus obtained we get

$$I_3-I_9' = \Re\Bigg[\frac{e^{i\delta}(\sigma^2 e^{i\alpha}-e^{-i\alpha})(1-2\sigma^2\cos 2\alpha+\sigma^4)\lambda_*}{2\sigma^4(\sigma^2-\lambda_*^2 e^{2i\delta})}\log\frac{1+\sigma^2}{1-\sigma^2}$$
$$+\frac{(\sigma^2 e^{i\alpha}-e^{-i\alpha})\{(\cos 2\alpha-\sigma^2)\lambda_*^2 e^{2i\delta}+\sigma^2\cos 2\alpha-1\}}{2\sigma^3(\sigma^2-\lambda_*^2 e^{2i\delta})}$$
$$\times\log\frac{1+\sigma\lambda_* e^{i\delta}}{1-\sigma\lambda_* e^{i\delta}}\Bigg], \qquad (100)$$

and consequently

$$I_3-I_9' = \frac{1}{2\sigma^4(\sigma^4-2\sigma^2\lambda_*^2\cos 2\delta+\lambda_*^4)}$$
$$\times\Big\{\lambda_*(1-2\sigma^2\cos 2\alpha+\sigma^4)\Big[(\sigma^4+\lambda_*^2)\cos(\delta+\alpha)-\sigma^2(1+\lambda_*^2)\cos(\delta-\alpha)\Big]$$
$$\times\log\frac{1+\sigma^2}{1-\sigma^2}$$
$$+\frac{\sigma}{4}\Big[\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta+3\alpha)-\big\{\sigma^2(1+2\sigma^4)+2+\sigma^4\big\}\lambda_*^2\cos(2\delta+\alpha)$$
$$+3\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta-\alpha)-\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta-3\alpha)$$
$$-(1-\sigma^2)(\sigma^4-\lambda_*^4)\cos 3\alpha+(1-\sigma^2)\big\{(1-2\sigma^2)\lambda_*^4+\sigma^2(2-\sigma^2)\big\}\cos\alpha\Big]$$
$$\times\log\frac{1+2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}{1-2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}$$
$$+\frac{\sigma}{2}\Big[\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta+3\alpha)+\big\{\sigma^2(1+2\sigma^4)-2-\sigma^4\big\}\lambda_*^2\sin(2\delta+\alpha)$$
$$+3\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta-\alpha)+\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta-3\alpha)$$
$$-(1+\sigma^2)(\sigma^4-\lambda_*^4)\sin 3\alpha-(1+\sigma^2)\big\{(1+2\sigma^2)\lambda_*^4-\sigma^2(2+\sigma^2)\big\}\sin\alpha\Big]$$
$$\times\tan^{-1}\frac{2\sigma\lambda_*\sin\delta}{1-\sigma^2\lambda_*^2}\Big\}. \qquad (101)$$

Also, combining I_4 and I_{10} given by (63) and (79) respectively, we get

$$I_4 - I_{10} = \Re\left[ie^{i\delta}\left(e^{i\alpha} - \frac{e^{-i\alpha}}{\sigma^2}\right)\frac{1}{\lambda_*}\sin 2\alpha \int_0^{\lambda_*^2} \frac{d\tau}{1-\sigma^2\lambda_*^{-2}\tau^2 e^{2i\delta}}\right] - \frac{\lambda_*}{\sigma^2}\sin 2\alpha \sin(\delta-\alpha) \,. \qquad (102)$$

The value of the integral on the right-hand side can be readily obtained, with the aid of (88) and (89), by putting $k=\lambda_*^{-1}$ and taking 0 and λ_*^2 as the lower and the upper limits respectively. Using the result so obtained we have

$$I_4 - I_{10} = \Re\left[\frac{i}{2\sigma^3}(\sigma^2 e^{i\alpha} - e^{-i\alpha})\sin 2\alpha \log\frac{1+\sigma\lambda_* e^{i\delta}}{1-\sigma\lambda_* e^{i\delta}}\right] - \frac{\lambda_*}{\sigma^2}\sin 2\alpha \sin(\delta-\alpha) \,, \qquad (103)$$

and therefore

$$I_4 - I_{10} = -\frac{1+\sigma^2}{4\sigma^3}\sin\alpha \sin 2\alpha \log\frac{1+2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}{1-2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2} + \frac{1-\sigma^2}{2\sigma^3}\cos\alpha\sin 2\alpha \tan^{-1}\frac{2\sigma\lambda_*\sin\delta}{1-\sigma^2\lambda_*^2} - \frac{\lambda_*}{\sigma^2}\sin 2\alpha\sin(\delta-\alpha) \,. \qquad (104)$$

Further, by (64) and (81), we have

$$I_5 - I_{11} = \Re\left[ie^{i\delta}(\sigma^2\lambda_*^2 e^{i\alpha} - e^{-i\alpha})\frac{1-\lambda_*^2}{\sigma^2\lambda_*(1-\sigma^2\lambda_*^2 e^{2i\delta})}\sin 2\alpha\right] - \frac{1-\lambda_*^2}{\sigma^2\lambda_*}\sin 2\alpha\sin(\delta-\alpha) \,, \qquad (105)$$

and therefore

$$I_5 - I_{11} = \frac{(1-\lambda_*^2)(1-\sigma^4\lambda_*^4)}{\sigma^2\lambda_*(1-2\sigma^2\lambda_*^2\cos 2\delta+\sigma^4\lambda_*^4)}\sin 2\alpha \sin(\delta-\alpha)$$

$$-\frac{1-\lambda_*^2}{\sigma^2\lambda_*}\sin 2\alpha \sin(\delta-\alpha)\ . \qquad (106)$$

Lastly, combining I_6 and I_{12} given by (65) and (82) respectively, we have

$$I_6 - I_{12} = \Re\left[ie^{i\delta}(\sigma^2 e^{i\alpha} - e^{-i\alpha})\frac{\lambda_*}{\sigma^2}\sin 2\alpha \int_0^1 \frac{d\tau}{1-\sigma^2\lambda_*^2\tau^2 e^{2i\delta}}\right]$$

$$-\frac{\lambda_*}{\sigma^2}\sin 2\alpha \sin(\delta-\alpha)\ . \qquad (107)$$

The value of the integral in this formula can be readily obtained with the aid of (88) and (89), by putting $k=\lambda_*$ and taking 0 and 1 as the lower and the upper limits respectively. Thus, we get

$$I_6 - I_{12} = \Re\left[\frac{i}{2\sigma^3}(\sigma^2 e^{i\alpha} - e^{-i\alpha})\sin 2\alpha \log\frac{1+\sigma\lambda_* e^{i\delta}}{1-\sigma\lambda_* e^{i\delta}}\right]$$

$$-\frac{\lambda_*}{\sigma^2}\sin 2\alpha \sin(\delta-\alpha)\ , \qquad (108)$$

and consequently

$$I_6 - I_{12} = -\frac{1+\sigma^2}{4\sigma^3}\sin\alpha \sin 2\alpha \log\frac{1+2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}{1-2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}$$

$$+\frac{1-\sigma^2}{2\sigma^3}\cos\alpha \sin 2\alpha \tan^{-1}\frac{2\sigma\lambda_*\sin\delta}{1-\sigma^2\lambda^2_*}$$

$$-\frac{\lambda_*}{\sigma^2}\sin 2\alpha \sin(\delta-\alpha)\ . \qquad (109)$$

Comparing (104) and (109), it will be seen that

$$I_4 - I_{10} = I_6 - I_{12} \,. \tag{110}$$

So far we have calculated the values of the six quantities $I_1 - I'_7$, $I_2 - I'_8$, $I_3 - I'_9$, $I_4 - I_{10}$, $I_5 - I_{11}$ and $I_6 - I_{12}$ in the formula (83). The remaining integrals in (83) can also be evaluated immediately. From (31a) and (32), we have

$$\bar{a}_0 - a_0 = 2\frac{\sigma^2(1+\lambda^4) - (1+\sigma^4\lambda^4)\cos 2\alpha}{\sigma^2(1-\sigma^4\lambda^4)} - 2\frac{1+(1-2\sigma^2\cos 2\alpha)\lambda^4}{1-\sigma^4\lambda^4}$$

$$= -\frac{2\cos 2\alpha}{\sigma^2} \,,$$

and thus, using this result and peforming simple integrations, we have

$$\cos(\delta-\alpha)\left\{\int_0^{\lambda_*}(\bar{a}_0 - a_0)\frac{\lambda}{\lambda_*}d\lambda + \int_{\lambda_*}^1(\bar{a}_0 - a_0)\frac{\lambda_*}{\lambda^3}d\lambda + \int_0^1(\bar{a}_0 - a_0)\lambda\lambda_*\,d\lambda\right\}$$

$$= -\frac{1+\lambda_*^2}{\sigma^2\lambda_*}\cos 2\alpha\cos(\delta-\alpha) \,. \tag{111}$$

Further, we have, from (31a),

$$(a_0)_{\lambda=0} = 2 \,, \tag{112}$$

and therefore

$$\frac{1}{2}(a_0)_{\lambda=0}\frac{1+\lambda_*^2}{\lambda_*}\cos(\delta-\alpha) = \frac{1+\lambda_*^2}{\lambda_*}\cos(\delta-\alpha) \,. \tag{113}$$

Thus, inserting these results, together with the values of the quantities $I_1 - I'_7$, $I_2 - I'_8$, $I_3 - I'_9$, $I_4 - I_{10}$, $I_5 - I_{11}$ and $I_6 - I_{12}$, given by (94), (98), (101), (104), (106) and (109) respectively, in the right-hand side of the formula (83), the expression for ϕ_1 finally becomes, after some reductions,

$$\phi_1 = -\frac{RU^3}{4c_0^2}\Bigg[\frac{1+\lambda_*^2}{\lambda_*}\Big\{\cos(\delta-\alpha)-\frac{1}{\sigma^2}\cos(\delta-3\alpha)\Big\}$$

$$+\frac{F}{2\sigma^4(1-2\sigma^2\lambda_*^2\cos 2\delta+\sigma^4\lambda_*^4)}+\frac{G}{2\sigma^4(\sigma^4-2\sigma^2\lambda_*^2\cos 2\delta+\lambda_*^4)}$$

$$-\frac{1+\sigma^2}{2\sigma^3}\sin\alpha\sin 2\alpha\log\frac{1+2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}{1-2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}$$

$$+\frac{1-\sigma^2}{\sigma^3}\cos\alpha\sin 2\alpha\tan^{-1}\frac{2\sigma\lambda_*\sin\delta}{1-\sigma^2\lambda_*^2}$$

$$+\frac{(1-\lambda_*^2)(1-\sigma^4\lambda_*^4)}{\sigma^2\lambda_*(1-2\sigma^2\lambda_*^2\cos 2\delta+\sigma^4\lambda_*^4)}\sin 2\alpha\sin(\delta-\alpha)\Bigg], \quad (114)$$

where we have put, for the sake of simplicity,

$$F = \lambda_*(1-2\sigma^2\cos 2\alpha+\sigma^4)\Big[(1+\sigma^4\lambda_*^2)\cos(\delta+\alpha)-\sigma^2(1+\lambda_*^2)\cos(\delta-\alpha)\Big]$$

$$\times\log\frac{1+\sigma^2}{1-\sigma^2}$$

$$+\frac{\sigma}{4}\Big[\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta+3\alpha)-\Big\{\sigma^2(1+2\sigma^4)+2+\sigma^4\Big\}\lambda_*^2\cos(2\delta+\alpha)$$

$$+3\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta-\alpha)-\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta-3\alpha)$$

$$+(1-\sigma^2)(1-\sigma^4\lambda_*^4)\cos 3\alpha+(1-\sigma^2)\Big\{\sigma^2(2-\sigma^2)\lambda_*^4+(1-2\sigma^2)\Big\}\cos\alpha\Big]$$

$$\times\log\frac{1+2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}{1-2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}$$

$$+\frac{\sigma}{2}\Big[\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta+3\alpha)+\Big\{\sigma^2(1+2\sigma^4)-2-\sigma^4\Big\}\lambda_*^2\sin(2\delta+\alpha)$$

$$+3\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta-\alpha)+\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta-3\alpha)$$

$$+(1+\sigma^2)(1-\sigma^4\lambda_*^4)\sin 3\alpha+(1+\sigma^2)\Big\{\sigma^2(2+\sigma^2)\lambda_*^4-1-2\sigma^2\Big\}\sin\alpha\Big]$$

$$\times\tan^{-1}\frac{2\sigma\lambda_*\sin\delta}{1-\sigma^2\lambda_*^2}$$

$$+\frac{\sigma^2(1-\lambda_*^2)}{\lambda_*}\Big[\Big\{1-\sigma^4\lambda_*^4-2(1-\sigma^4)\lambda_*^2\Big\}\cos(\delta+\alpha)-2\sigma^2(1-\lambda_*^4)\cos(\delta-\alpha)$$

$$+(1-\sigma^4\lambda_*^4)\cos(\delta-3\alpha)\Big], \quad (115)$$

and

$$G = \lambda_*(1-2\sigma^2\cos 2\alpha+\sigma^4)\Big[(\sigma^4+\lambda_*^2)\cos(\delta+\alpha)-\sigma^2(1+\lambda_*^2)\cos(\delta-\alpha)\Big]$$

$$\times \log\frac{1+\sigma^2}{1-\sigma^2}$$

$$+\frac{\sigma}{4}\Big[\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta+3\alpha)-\Big\{\sigma^2(1+2\sigma^4)+2+\sigma^4\Big\}\lambda_*^2\cos(2\delta+\alpha)$$

$$+3\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta-\alpha)-\sigma^2(1+\sigma^2)\lambda_*^2\cos(2\delta-3\alpha)$$

$$-(1-\sigma^2)(\sigma^4-\lambda_*^4)\cos 3\alpha+(1-\sigma^2)\Big\{(1-2\sigma^2)\lambda_*^4+\sigma^2(2-\sigma^2)\Big\}\cos\alpha\Big]$$

$$\times \log\frac{1+2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}{1-2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}$$

$$+\frac{\sigma}{2}\Big[\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta+3\alpha)+\Big\{\sigma^2(1+2\sigma^4)-2-\sigma^4\Big\}\lambda_*^2\sin(2\delta+\alpha)$$

$$+3\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta-\alpha)+\sigma^2(1-\sigma^2)\lambda_*^2\sin(2\delta-3\alpha)$$

$$-(1+\sigma^2)(\sigma^4-\lambda_*^4)\sin 3\alpha-(1+\sigma^2)\Big\{(1+2\sigma^2)\lambda_*^4-\sigma^2(2+\sigma^2)\Big\}\sin\alpha\Big]$$

$$\times \tan^{-1}\frac{2\sigma\lambda_*\sin\delta}{1-\sigma^2\lambda_*^2}. \qquad (116)$$

Using this general formula for ϕ_1 we can calculate, when necessary, the additional velocity components due to compressibility at any point in the z-plane of the circle as well as in the ζ-plane of the ellipse.

§ 10. Before proceeding further, it will be of some interest to show here that the known expression for the first order correction to the velocity potential for the irrotational flow of a compressible fluid past a circular cylinder can be easily obtained as a limiting form of the preceding general expression (114) for ϕ_1 by allowing $\sigma(=a/R)$ to approach zero, remembering that in the limit $\sigma\to 0$, the ellipse degenerates into the circle of radius R.

When σ is sufficiently small, we have the following expansions:

$$\frac{1}{1-2\sigma^2\lambda_*^2\cos 2\delta+\sigma^4\lambda_*^4} = 1+2\sigma^2\lambda_*^2\cos 2\delta+O(\sigma^4)\,,$$

$$\frac{1}{\sigma^4-2\sigma^2\lambda_*^2\cos 2\delta+\lambda_*^4} = \frac{1}{\lambda_*^4}\left\{1+\frac{2\sigma^2}{\lambda_*^2}\cos 2\delta+O(\sigma^4)\right\}\,,$$

$$\log\frac{1+2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2}{1-2\sigma\lambda_*\cos\delta+\sigma^2\lambda_*^2} = 4\sigma\lambda_*\cos\delta+\frac{4}{3}\sigma^3\lambda_*^3\cos 3\delta+O(\sigma^5)\,,$$

$$\tan^{-1}\frac{2\sigma\lambda_*\sin\delta}{1-\sigma^2\lambda_*^2} = 2\sigma\lambda_*\sin\delta+\frac{2}{3}\sigma^3\lambda_*^3\sin 3\delta+O(\sigma^5)\,,$$

$$\log\frac{1+\sigma^2}{1-\sigma^2} = 2\sigma^2+O(\sigma^6)\,.$$

Making use of these expansions and performing various tedious but straightforward calculations, we get, from (114),

$$\phi_1 = -\frac{RU^3}{4c_0^2}\left[\left(-\frac{13}{3}\lambda_*+2\lambda_*^3-\frac{1}{3}\lambda_*^5\right)\cos(\delta-\alpha) + \left(\lambda_*-\frac{1}{3}\lambda_*^3\right)\cos 3(\delta-\alpha)+O(\sigma^2)\right]. \quad (117)$$

Thus, we have

$$\lim_{\sigma\to 0}\phi_1 = -\frac{RU^3}{4c_0^2}\left[\left(-\frac{13}{3}\lambda_*+2\lambda_*^3-\frac{1}{3}\lambda_*^5\right)\cos(\delta-\alpha) + \left(\lambda_*-\frac{1}{3}\lambda_*^3\right)\cos 3(\delta-\alpha)\right], \quad (118)$$

or, introducing the MACH number $M=U/c_0$ for the undisturbed stream and remembering that $\lambda_*=R/r_*$,

$$\lim_{\sigma\to 0}\phi_1 = RM^2U\left[\left\{\frac{13}{12}\frac{R}{r_*}-\frac{1}{2}\left(\frac{R}{r_*}\right)^3+\frac{1}{12}\left(\frac{R}{r_*}\right)^5\right\}\cos(\delta-\alpha) + \left\{-\frac{1}{4}\frac{R}{r_*}+\frac{1}{12}\left(\frac{R}{r_*}\right)^3\right\}\cos 3(\delta-\alpha)\right]. \quad (119)$$

It will easily be seen that except for some differences in notations[1] this limiting form agrees with the expression for the first order correction to the velocity potential for the irrotational flow of a compressible fluid past a circular cylinder, which, as mentioned already, has been first obtained by JANZEN and by the late LORD RAYLEIGH and has been confirmed later by POGGI, IMAI, KAPLAN, TAMADA and SAITŌ by employing various different methods.

§ 11. We next proceed to the discussion of the state of affairs on the surface of the elliptic cylinder at which $\lambda_* = 1$.

First, we shall obtain the velocity potential at the surface of the body. This can immediately be obtained by putting $\lambda_* = 1$ in (114). Thus, denoting it by $(\phi_1)_{\lambda_*=1}$, we have, after some reductions,

$$
\begin{aligned}
(\phi_1)_{\lambda_*=1} = -\frac{RU^3}{2c_0^2}\Bigg[&\cos(\delta-\alpha)-\frac{1}{\sigma^2}\cos(\delta-3\alpha)+\frac{1}{1-2\sigma^2\cos 2\delta+\sigma^4} \\
&\times\Bigg\{\frac{1-2\sigma^2\cos 2\alpha+\sigma^4}{2\sigma^4}\Big[(1-\sigma^2)^2\cos\alpha\cos\delta-(1+\sigma^2)^2\sin\alpha\sin\delta\Big] \\
&\qquad\times\log\frac{1+\sigma^2}{1-\sigma^2} \\
&+\frac{1+\sigma^2}{4\sigma^3}\Big[(1-\sigma^2)^2\cos\alpha(\cos 2\alpha-\cos 2\delta) \\
&\qquad+(1-2\sigma^2\cos 2\alpha+\sigma^4)\sin\alpha\sin 2\delta\Big]\log\frac{1+2\sigma\cos\delta+\sigma^2}{1-2\sigma\cos\delta+\sigma^2} \\
&+\frac{1-\sigma^2}{2\sigma^3}\Big[(1+\sigma^2)^2\sin\alpha(\cos 2\alpha-\cos 2\delta) \\
&\qquad-(1-2\sigma^2\cos 2\alpha+\sigma^4)\cos\alpha\sin 2\delta\Big]\tan^{-1}\frac{2\sigma\sin\delta}{1-\sigma^2}\Bigg\} \\
&-\frac{1+\sigma^2}{4\sigma^3}\sin\alpha\sin 2\alpha\log\frac{1+2\sigma\cos\delta+\sigma^2}{1-2\sigma\cos\delta+\sigma^2} \\
&+\frac{1-\sigma^2}{2\sigma^3}\cos\alpha\sin 2\alpha\tan^{-1}\frac{2\sigma\sin\delta}{1-\sigma^2}\Bigg]. \qquad (120)
\end{aligned}
$$

(1) It is to be noted here that in some of the papers of the previous writers the radius of the circular cylinder was taken to be unity so that in our notation $R = 1$.

Next, let Δv denote the additional velocity due to compressibility at any point on the surface of the circular boundary in the z-plane. Then, remembering that the fluid flows from left to right, we have

$$\Delta v = -\frac{1}{R}\frac{\partial}{\partial\delta}(\phi_1)_{\lambda_*=1}\,. \qquad (121)$$

Thus, differentiating $(\phi_1)_{\lambda_*=1}$ given above by δ partially, we get

$$\begin{aligned}
\frac{\Delta v}{U} = \frac{M^2}{2}\Bigg[&-\sin(\delta-\alpha)+\frac{1}{\sigma^2}\sin(\delta-3\alpha)-\frac{1}{(1-2\sigma^2\cos 2\delta+\sigma^4)^2}\\
&\times\Bigg\{\frac{1-2\sigma^2\cos 2\alpha+\sigma^4}{2\sigma^4}\Big[(1-\sigma^2)^2\cos\alpha\big\{(1+3\sigma^2+\sigma^4)\sin\delta+\sigma^2\sin 3\delta\big\}\\
&\qquad+(1+\sigma^2)^2\sin\alpha\big\{(1-3\sigma^2+\sigma^4)\cos\delta+\sigma^2\cos 3\delta\big\}\Big]\\
&\qquad\times\log\frac{1+\sigma^2}{1-\sigma^2}\\
&-\frac{1+\sigma^2}{2\sigma^3}(1-2\sigma^2\cos 2\alpha+\sigma^4)\Big[(1-\sigma^2)^2\cos\alpha\sin 2\delta\\
&\qquad+\big\{(1+\sigma^4)\cos 2\delta-2\sigma^2\big\}\sin\alpha\Big]\log\frac{1+2\sigma\cos\delta+\sigma^2}{1-2\sigma\cos\delta+\sigma^2}\\
&-\frac{1-\sigma^2}{\sigma^3}(1-2\sigma^2\cos 2\alpha+\sigma^4)\Big[(1+\sigma^2)^2\sin\alpha\sin 2\delta\\
&\qquad-\big\{(1+\sigma^4)\cos 2\delta-2\sigma^2\big\}\cos\alpha\Big]\tan^{-1}\frac{2\sigma\sin\delta}{1-\sigma^2}\\
&+\frac{1}{\sigma^2}\Big[(1-4\sigma^2+\sigma^4)\cos\alpha+(1+\sigma^4)\cos 3\alpha\Big]\\
&\qquad\times\Big[(1+\sigma^2+\sigma^4)\sin\delta-\sigma^2\sin 3\delta\Big]\\
&+\frac{1}{\sigma^2}\Big[(1+4\sigma^2+\sigma^4)\sin\alpha-(1+\sigma^4)\sin 3\alpha\Big]\\
&\qquad\times\Big[(1-\sigma^2+\sigma^4)\cos\delta-\sigma^2\cos 3\delta\Big]\Bigg\}\Bigg]. \qquad (122)
\end{aligned}$$

If, in particular, we put $\alpha = 0°$ in this formula, we may obtain the expression for the additional velocity due to compressibility on the surface of the circular boundary in the z-plane in a special case when the undisturbed stream flows parallel to the major axis of the ellipse. We thus have

$$\left(\frac{\Delta v}{U}\right)_{\alpha=0°} = \frac{M^2}{2}\frac{1-\sigma^2}{\sigma^2}\Bigg[\sin\delta - \frac{1-\sigma^2}{(1-2\sigma^2\cos 2\delta+\sigma^4)^2}$$
$$\times\Bigg\{\frac{(1-\sigma^2)^2}{2\sigma^2}\Big[(1+3\sigma^2+\sigma^4)\sin\delta+\sigma^2\sin 3\delta\Big]\log\frac{1+\sigma^2}{1-\sigma^2}$$
$$-\frac{(1+\sigma^2)(1-\sigma^2)^2}{2\sigma}\sin 2\delta\log\frac{1+2\sigma\cos\delta+\sigma^2}{1-2\sigma\cos\delta+\sigma^2}$$
$$+\frac{1-\sigma^2}{\sigma}\Big[(1+\sigma^4)\cos 2\delta-2\sigma^2\Big]\tan^{-1}\frac{2\sigma\sin\delta}{1-\sigma^2}$$
$$+2\Big[(1+\sigma^2+\sigma^4)\sin\delta-\sigma^2\sin 3\delta\Big]\Bigg\}\Bigg]. \quad (123)$$

It will be seen that except for slight difference in notation this formula agrees with what has already been obtained by KAPLAN(1) by considering the said particular case from the beginning.

Also, if we put $\alpha = 90°$ we may obtain the similar result for the case when the undisturbed stream flows parallel to the minor axis of the ellipse in the positive direction of the y-axis. We have

$$\left(\frac{\Delta v}{U}\right)_{\alpha=90°} = \frac{M^2}{2}\frac{1+\sigma^2}{\sigma^2}\Bigg[\cos\delta - \frac{1+\sigma^2}{(1-2\sigma^2\cos 2\delta+\sigma^4)^2}$$
$$\times\Bigg\{\frac{(1+\sigma^2)^2}{2\sigma^2}\Big[(1-3\sigma^2+\sigma^4)\cos\delta+\sigma^2\cos 3\delta\Big]\log\frac{1+\sigma^2}{1-\sigma^2}$$
$$-\frac{1+\sigma^2}{2\sigma}\Big[(1+\sigma^4)\cos 2\delta-2\sigma^2\Big]\log\frac{1+2\sigma\cos\delta+\sigma^2}{1-2\sigma\cos\delta+\sigma^2}$$
$$-\frac{(1-\sigma^2)(1+\sigma^2)^2}{\sigma}\sin 2\delta\tan^{-1}\frac{2\sigma\sin\delta}{1-\sigma^2}$$
$$+2\Big[(1-\sigma^2+\sigma^4)\cos\delta-\sigma^2\cos 3\delta\Big]\Bigg\}\Bigg], \quad (124)$$

(1) C. KAPLAN, loc. cit. (Report No. 624).

and it will readily be seen that this agrees with the formula obtained recently by IMAI and AIHARA[1].

The effect of compressibility, i.e., $\Delta v/U$, having been found, it is now a simple matter to find the expression for the total velocity at the surface of the body placed in the compressible fluid flow. Thus, correct to the order of M^2, the total velocity at the circular boundary in the z-plane is given by

$$\left(\frac{v}{U}\right)_{\text{circle}}=2\sin(\delta-\alpha)+\frac{\Delta v}{U}, \qquad (125)$$

with the general expression (122) for $\Delta v/U$, since in the incompressible fluid flow the velocity on the circular boundary is given by $2U\sin(\delta-\alpha)$, and on the surface of the elliptic cylinder in the ζ-plane by

$$\left(\frac{v}{U}\right)_{\text{ellipse}}=\frac{1}{(1-2\sigma^2\cos 2\delta+\sigma^4)^{\frac{1}{2}}}\left(\frac{v}{U}\right)_{\text{circle}}. \qquad (126)$$

IV. Calculation of the Critical Mach Number.

§ 12. Without causing any confusion, we shall hereafter write v/U for $(v/U)_{\text{ellipse}}$ for the sake of simplicity. Then, we have, by (125) and (126),

$$\frac{v}{U}=\frac{1}{(1-2\sigma^2\cos 2\delta+\sigma^4)^{\frac{1}{2}}}\left\{2\sin(\delta-\alpha)+\frac{\Delta v}{U}\right\}, \qquad (127)$$

where the expression for $\Delta v/U$ is given by (122) in the general case when the angle of attack α is neither $0°$ nor $90°$.

With the aid of this formula we can calculate the velocity distribution over the surface of an elliptic cylinder with a given thickness ratio and with a given angle of attack, when the MACH number for the undisturbed

(1) I. IMAI and T. AIHARA, loc. cit.

flow is given. Also, the position of maximum velocity as well as the magnitude of maximum velocity itself can be found. Further, application of BERNOULLI's theorem enables us to calculate the pressure distribution over the surface of an elliptic cylinder.

From the practical point of view it is of great importance to calculate the so-called critical MACH number, M_{crit}, at which the local speed of sound is first attained in the field of flow. In effect, it is to be expected, in accordance with the results of recent experiments[1], that when the value of the MACH number $M(=U/c_0)$ for the undisturbed flow exceeds the critical value M_{crit}, the actual flow around the body ceases to resemble even approximately an irrotational flow; in other words, the compressibility burble occurs when the value of M exceeds M_{crit}. We have therefore calculated the values of the critical MACH number for various elliptic cylinders with different thickness ratios in two cases when the angle of attack α is equal to $5°$ and $10°$ respectively.

The maximum velocity in the field of flow occurs evidently at a certain point on the surface of the elliptic cylinder, and the position of the maximum velocity as well as the magnitude of the maximum velocity v_{max} can be found in the manner described just in the above. The critical MACH number M_{crit} can then be calculated by the equation:

$$v_{max} = c\,, \tag{128}$$

which, with the help of (3), may also be put in the form:

$$\left(\frac{v_{max}}{U}\right)^2 = \frac{2}{\gamma+1}\frac{1}{M_{crit}^2} + \frac{\gamma-1}{\gamma+1}\,, \tag{129}$$

or,

$$\frac{1}{M_{crit}^2} = \frac{\gamma+1}{2}\left(\frac{v_{max}}{U}\right)^2 - \frac{\gamma-1}{2}\,. \tag{130}$$

(1) The principal results of recent experimental investigations have been described briefly in the Introduction.

Very elaborate and tedious numerical calculations have been carried out in detail by the junior author and the values of the critical MACH number have been calculated as a function of the thickness ratio of the ellipse in two cases in which the angle of attack α is respectively equal to 5° and 10°, both lying in the practically important range. The results obtained are of considerable interest.

The values of the critical MACH number M_{crit} thus calculated are tabulated in Tables I and II, and they are also shown graphically in Fig. 2, taking the thickness ratio t of the ellipse as the abscissa. The first column of each of the tables gives the values of $\sigma(=a/R)$ and the second column gives the corresponding values of the thickness ratio t, while in the third column with the heading "M_{crit}, compressible fluid flow" are given the values of the critical MACH number M_{crit} in the case of compressible fluid flow. In Fig. 2, the similar values of M_{crit} for the cases in which α is equal to 0° and 90° respectively are also shown for comparison, the values for the case $\alpha=0^\circ$ having been taken from one of KAPLAN's papers(1), while those for $\alpha=90^\circ$ from IMAI and AIHARA's paper(2).

TABLE I. ($\alpha=5^\circ$)

σ	Thickness ratio t	M_{crit}	
		Compressible fluid flow	Incompressible fluid flow
1	0	0	0
0·9	0·105	0·619	0·664
0·8	0·220	0·682	0·734
0·6	0·471	0·582	0·637
0·4	0·724	0·492	0·543
0·2	0·923	0·438	0·485
0	1	0·421	0·466

(1) C. KAPLAN, loc. cit. (Report No. 624).
(2) I. IMAI and T. AIHARA, loc. cit.

TABLE II. ($\alpha = 10^{\circ}$).

σ	Thickness ratio t	M_{crit}	
		Compressible fluid flow	Incompressible fluid flow
1	0	0	0
0·9	0·105	0·399	0·437
0·8	0·220	0·568	0·614
0·6	0·471	0·561	0·611
0·4	0·724	0·484	0·537
0·2	0·923	0·437	0·484
0	1	0·421	0·466

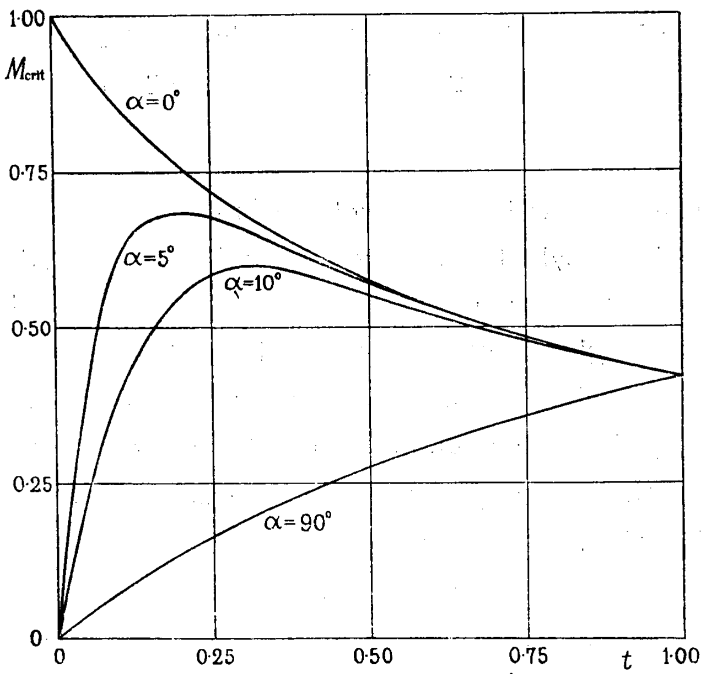

Fig. 2.

From Fig. 2 it will be seen that when the angle of attack α takes any value different from 0° and 90°, the value of M_{crit} becomes a maximum at a certain definite value of the thickness ratio t. Thus, for instance, when $\alpha = 5°$, the maximum value of M_{crit} occurs when $t = 0{\cdot}2$ approximately, and also when $\alpha = 10°$, the maximum of M_{crit} corresponds with $t = 0{\cdot}3$ approximately. This point is of considerable interest and is worthy of special notice(1). It may be remarked here that as can readily be seen from Fig. 2, such a maximum in the value of M_{crit} does not occur when α is equal to $0°$ or $90°$. In fact, when $\alpha = 0°$ so that the undisturbed flow is parallel to the major axis of the ellipse, the value of M_{crit} decreases monotonously with increasing thickness ratio t, while in the case when $\alpha = 90°$ so that the undisturbed stream flows parallel to the minor axis of the ellipse, the value of M_{crit} increases monotonously as t increases.

Thus, we see that when the angle of attack α takes any definite value different from $0°$ and $90°$, there is always an ellipse with a certain appropriate thickness ratio such that the value of M_{crit} for it is greater than those of M_{crit} for all other ellipses with smaller or greater thickness ratios.

§ 13. Using the well-known solution for the incompressible fluid flow past an elliptic cylinder we have also calculated formally the values of M_{crit} for various values of the thickness ratio t of the ellipse in two cases when the angle of attack α is equal to $5°$ and $10°$ respectively. The values thus calculated are given in the fourth column with the heading "M_{crit}, incompressible fluid flow" in each of the preceding two tables. If these values of M_{crit} for the case of incompressible fluid flow are called the zero approximations to the values of M_{crit} for the case of compressible fluid flow, then the values of M_{crit} previously given may be called the first approximations to the true values for the case of compressible fluid flow.

(1) Similar results have also been obtained for the case of a symmetrical JOUKOWSKI aerofoil. See, S. TOMOTIKA and H. UMEMOTO, On the Subsonic Flow of a Compressible Fluid past a Symmetrical JOUKOWSKI Aerofoil, which will be published in this Report in the near future.

In order to compare these zero and first approximate values of M_{crit}, they are plotted against the thickness ratio t in Figs. 3 and 4, where the dotted-line curves give the values of M_{crit} for the incompressible fluid flow, while the full-line curves giving the first approximate values of M_{crit} for the compressible fluid flow have been reproduced from Fig. 2. It will easily be seen that the first approximate values of M_{crit} are generally smaller by about 10 per cent than the corresponding values for the incompressible fluid flow.

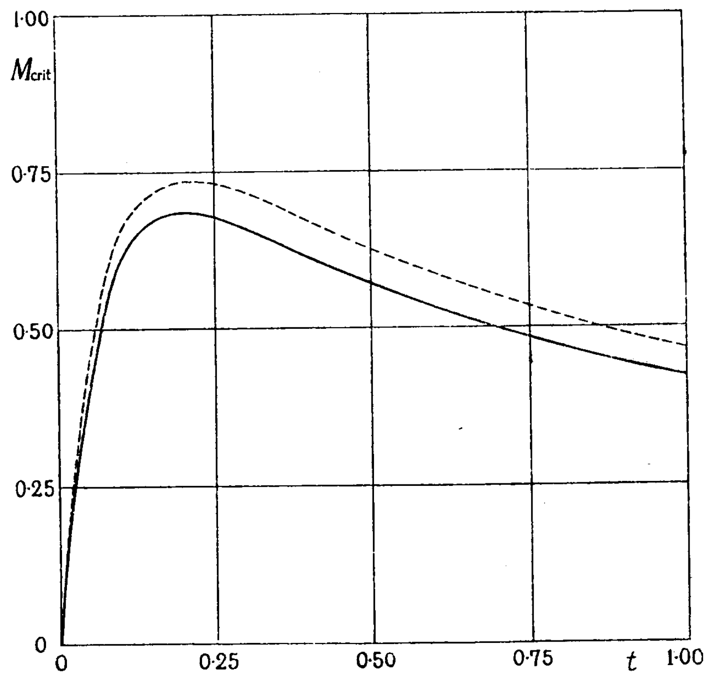

Fig. 3. $\alpha = 5°$.

As mentioned already in the Introduction, the subsonic flow of a compressible fluid past a circular cylinder has been discussed by various writers, and especially IMAI has calculated the values of the critical MACH number M_{crit} using respectively the zero, the first, the second and the third approximate solutions for the problem. When arranged in order of ascending degree of approximation, the values of M_{crit} thus calculated are 0·466, 0·421, 0·409 and 0·404. Also, the junior author of the present paper has obtained the result $M_{crit} = 0{\cdot}400$ by discussing the compressible fluid flow past a circular cylinder by the hodograph method. It seems to be highly probable, therefore, that the true value of M_{crit} for the case of a circular cylinder is approximately 0·400. In this connection it is to be noted that this value 0·400 is smaller, by about 5 per cent, than the first approximate value 0·421.

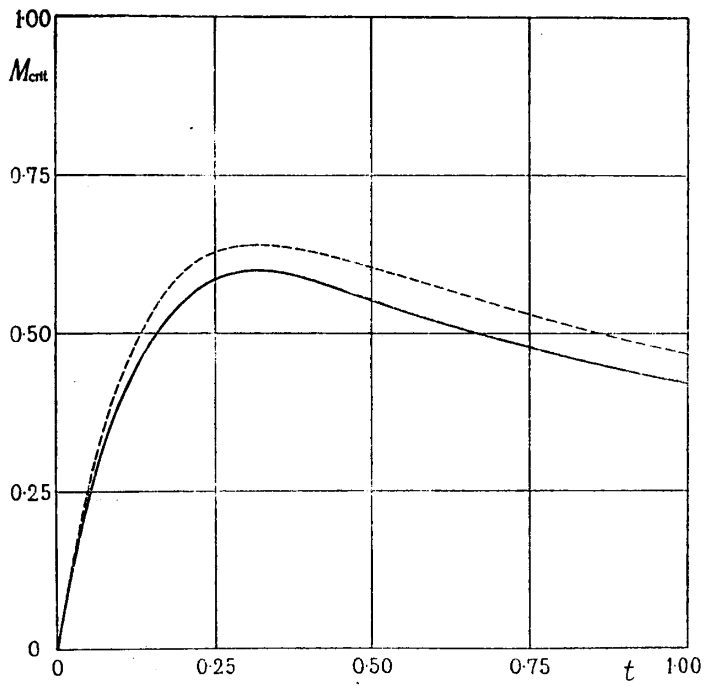

Fig. 4. $\alpha = 10°$.

The results for the circular cylinder may perhaps be extrapolated to the case of an elliptic cylinder, and thus we may conclude, without serious error, that the first approximate values of M_{crit} calculated in the present paper are greater, by about 5 per cent at most, than the true values of M_{crit} for the elliptic cylinder.

V. The Moment of the Fluid Pressures acting on the Elliptic Cylinder.

§ 14. Lastly, we shall calculate the resultant moment about the centre of the ellipse of the fluid pressures acting on the elliptic cylinder.

We shall begin with the general formula for the resultant moment of the fluid pressures acting on a cylindrical body of arbitrary cross-section, which is placed in a two-dimensional irrotational flow of a compressible perfect fluid flowing with a uniform velocity at large distance from the body. Let the plane of fluid motion be taken as the ζ-plane. To avoid confusion with the notation M for the MACH number associated with the undisturbed stream, we shall denote by M_c the resultant moment of the fluid pressures acting per unit length on the surface of the body with respect to the origin of the coordinate axes.

We now assume that we enclose the body by a very large circle Σ of radius R_* with the origin of the coordinate axes as the centre, and we denote the polar coordinates of a point on the circle Σ by (R_*, φ). Also, we denote the components of fluid velocity at any point on the circle Σ in the directions of increasing R_* and φ by v_{R*} and v_φ respectively. Then, by applying the principle of angular momentum or otherwise, we can prove[1] that the moment M_c is given by

$$M_c = -R_*^2 \int_0^{2\pi} \rho v_{R*} v_\varphi d\varphi \,, \tag{131}$$

where ρ is the density of the fluid in the ζ-plane, considering ultimately the limit when $R_* \to \infty$.

Let the complex velocity potential for the irrotational flow of a compressible fluid past the body under consideration be denoted by w. Then, we readily have, on the circle Σ,

$$\frac{dw}{d\zeta} = (v_{R*} - i v_\varphi) e^{-i\varphi} ,$$

(1) C. KAPLAN, loc. cit. (Report No. 671).

and therefore, remembering that $\zeta = R_* e^{i\varphi}$ on the circle Σ, we have(1)

$$\Re\left[\left(\frac{dw}{d\zeta}\right)^2 \zeta\, d\zeta\right] = 2R_*^2 v_{R*} v_\varphi d\varphi\ .$$

Hence, if we make use of this result the above formula (131) for M_c can be put in the form:

$$M_c = -\frac{1}{2}\Re\left[\oint_\Sigma \rho\left(\frac{dw}{d\zeta}\right)^2 \zeta\, d\zeta\right], \qquad (132)$$

where the integral is taken round the large circle Σ of radius R_* in the counter-clockwise sense.

So far we have dealt with the general case of a cylindrical body with arbitrary cross-section. We next proceed to the calculation of the resultant moment M_c of the pressures acting per unit length on the elliptic cylinder under discussion in the present paper, which is placed in the two-dimensional stream of a compressible fluid. Then, remembering that the plane of fluid motion has been taken as the ζ-plane and that the origin of the coordinate axes has been taken at the centre of the ellipse, the said moment M_c can be calculated by the above formula (132).

The evaluation of the integral in (132) can however be conveniently carried out in the z-plane of the circle of radius R, to which the ellipse in the ζ-plane is transformed conformally by the JOUKOWSKI transformation $\zeta = z + a^2/z$. Thus, remembering that

$$\frac{dw}{d\zeta} = \frac{dw}{dz}\frac{dz}{d\zeta}\ ,$$

the formula for M_c becomes:

$$M_c = -\frac{1}{2}\Re\left[\oint_C \rho\left(\frac{dw}{dz}\right)^2 \frac{dz}{d\zeta}\zeta\, dz\right], \qquad (133)$$

where the integral is taken round the large contour C surrounding the circle in the counter-clockwise sense, which corresponds with the large

(1) $\Re(z)$ means, as before, the real part of z.

circle Σ in the ζ-plane.

The conjugate complex velocity dw/dz for the compressible fluid flow in the z-plane can be written as:

$$\frac{dw}{dz} = \frac{dw_0}{dz} + \Delta\frac{dw}{dz}. \qquad (134)$$

In this equation, w_0 is the complex velocity potential for the incompressible fluid flow past the circle in the z-plane and is therefore given by (26), namely:

$$w_0 = Ue^{-i\alpha}\left(z + \frac{R^2e^{2i\alpha}}{z}\right),$$

which gives

$$\frac{dw_0}{dz} = Ue^{-i\alpha}\left(1 - \frac{R^2e^{2i\alpha}}{z^2}\right). \qquad (135)$$

Also, $\Delta(dw/dz)$ denotes the additional conjugate complex velocity due to compressibility of the fluid, and it is of magnitude proportional to M^2. The approximate expression for $\Delta(dw/dz)$ when z is sufficiently large will be obtained presently.

The combination of BERNOULLI's theorem with the adiabatic law gives immediately

$$\rho = \rho_\infty\left[1 + \frac{\gamma-1}{2}M^2\left(1 - \frac{v^2}{U^2}\right)\right]^{\frac{1}{\gamma-1}}, \qquad (136)$$

where $M(=U/c_0)$ is, as before, the MACH number for the undisturbed flow with velocity U and ρ_∞ the density of the fluid in the undisturbed stream.

Remembering that $v = v_0 + \Delta v$ where v_0 is the velocity in the incompressible fluid flow and Δv the additional velocity proportional to M^2 due to compressibility, we expand the right-hand side of the above formula (136) in a power series of M. We thus have

$$\rho = \rho_\infty\left[1+\frac{1}{2}M^2\left(1-\frac{v_0^2}{U^2}\right)+\cdots\right]. \tag{137}$$

Inserting the expressions for dw/dz and ρ given by (134) and (137) respectively in the right-hand side of (133) and neglecting those terms which contain powers of M higher than the fourth, we have

$$M_c = -\frac{1}{2}\rho_\infty\Re\left[\oint_C\left(\frac{dw}{dz}\right)^2\frac{dz}{d\zeta}\zeta\,dz\right]$$
$$-\frac{1}{4}\rho_\infty M^2\Re\left[\oint_C\left(1-\frac{v_0^2}{U^2}\right)\left(\frac{dw_0}{dz}\right)^2\frac{dz}{d\zeta}\zeta\,dz\right]. \tag{138}$$

It is to be remarked here, however, that in evaluating the integrals in this formula we have to consider the limit when the contour C is removed to infinity. In the following lines, this formula will be used for the calculation of the moment M_c of the fluid pressures acting on the elliptic cylinder with respect to the centre of the ellipse, which coincides with the origin of the coordinate axes.

§15. Next, we shall obtain the approximate expression for the additional conjugate complex velocity $\Delta(dw/dz)$ due to compressibility when z is sufficiently large. For this purpose, we shall first obtain the approximate expression which the first order correction ϕ_1 to the velocity potential for the compressible fluid flow would take when λ_* is very small, i.e., when r_* is very large, the general expression for ϕ_1 being given by (114).

When λ_* is sufficiently small, we have the following expansions:

$$\frac{1}{1-2\sigma^2\lambda_*^2\cos 2\delta+\sigma^4\lambda_*^4} = 1+2\sigma^2\lambda_*^2\cos 2\delta+O(\lambda_*^4)\ ,$$

$$\frac{1}{\sigma^4-2\sigma^2\lambda_*^2\cos 2\delta+\lambda_*^4} = \frac{1}{\sigma^4}\left\{1+\frac{2\lambda_*^2}{\sigma^2}\cos 2\delta+O(\lambda_*^4)\right\}\ ,$$

$$\log \frac{1+2\sigma\lambda_* \cos\delta + \sigma^2\lambda_*^2}{1-2\sigma\lambda_* \cos\delta+\sigma^2\lambda_*^2} = 4\sigma\lambda_* \cos\delta + \frac{4}{3}\sigma^3\lambda_*^3 \cos 3\delta + O(\lambda_*^5)\,,$$

$$\tan^{-1}\frac{2\sigma\lambda_* \sin\delta}{1-\sigma^2\lambda_*^2} = 2\sigma\lambda_* \sin\delta + \frac{2}{3}\sigma^3\lambda_*^3 \sin 3\delta + O(\lambda_*^5)\,.$$

Substituting these expansions in the right-hand side of the general formula (114) for ϕ_1 and then performing various simple approximate calculations, we can obtain the approximate expression for ϕ_1. Thus, retaining only those terms which contain the first power of λ_*, we get approximately

$$\phi_1 = -\frac{RU^3}{4c_0^2}\Big[\frac{1}{2\sigma^6}(1-2\sigma^2\cos 2\alpha+\sigma^4) \times\Big[2\sigma^2\cos(\delta+\alpha)-(1+\sigma^4)\cos(\delta-\alpha)\Big]\log\frac{1+\sigma^2}{1-\sigma^2} + 2\cos(\delta+3\alpha)+\Big(\sigma^2-\frac{3}{\sigma^2}\Big)\cos(\delta+\alpha)+\frac{1}{\sigma^4}\cos(\delta-\alpha) - \frac{1}{\sigma^2}\cos(\delta-3\alpha)-\sigma^2\cos(3\delta-\alpha)+\cos(3\delta-3\alpha)\Big]\lambda_*\,, \quad (139)$$

or, since $\lambda_* = R/r_*$,

$$\phi_1 = -\frac{1}{4}R^2M^2U\Big[\frac{1}{2\sigma^6}(1-2\sigma^2\cos 2\alpha+\sigma^4) \times\Big[2\sigma^2\cos(\delta+\alpha)-(1+\sigma^4)\cos(\delta-\alpha)\Big]\log\frac{1+\sigma^2}{1-\sigma^2} + 2\cos(\delta+3\alpha)+\Big(\sigma^2-\frac{3}{\sigma^2}\Big)\cos(\delta+\alpha)+\frac{1}{\sigma^4}\cos(\delta-\alpha) - \frac{1}{\sigma^2}\cos(\delta-3\alpha)-\sigma^2\cos(3\delta-\alpha)+\cos(3\delta-3\alpha)\Big]\frac{1}{r_*}\,, \quad (140)$$

where the MACH number $M(=U/c_0)$ has been used.

The components of the additional fluid velocity due to compressibility at a point far distant from the body can now be obtained immediately. Thus, if we denote by $\varDelta v_{r*}$ and $\varDelta v_{\delta}$ the radial and the circumferential components of the additional velocity, we have

$$\begin{aligned}\varDelta v_{r*} &= \frac{\partial \phi_1}{\partial r_*}\\ &= \frac{1}{4} R^2 M^2 U\Big[\frac{1}{2\sigma^6}(1-2\sigma^2\cos 2\alpha+\sigma^4)\\ &\qquad\times\Big[2\sigma^2\cos(\delta+\alpha)-(1+\sigma^4)\cos(\delta-\alpha)\Big]\log\frac{1+\sigma^2}{1-\sigma^2}\\ &\qquad+2\cos(\delta+3\alpha)+\Big(\sigma^2-\frac{3}{\sigma^2}\Big)\cos(\delta+\alpha)+\frac{1}{\sigma^4}\cos(\delta-\alpha)\\ &\qquad-\frac{1}{\sigma^2}\cos(\delta-3\alpha)-\sigma^2\cos(3\delta-\alpha)+\cos(3\delta-3\alpha)\Big]\frac{1}{r_*^2},\end{aligned} \tag{141}$$

and

$$\begin{aligned}\varDelta v_{\delta} &= \frac{1}{r_*}\frac{\partial \phi_1}{\partial \delta}\\ &= \frac{1}{4} R^2 M^2 U\Big[\frac{1}{2\sigma^6}(1-2\sigma^2\cos 2\alpha+\sigma^4)\\ &\qquad\times\Big[2\sigma^2\sin(\delta+\alpha)-(1+\sigma^4)\sin(\delta-\alpha)\Big]\log\frac{1+\sigma^2}{1-\sigma^2}\\ &\qquad+2\sin(\delta+3\alpha)+\Big(\sigma^2-\frac{3}{\sigma^2}\Big)\sin(\delta+\alpha)+\frac{1}{\sigma^4}\sin(\delta-\alpha)\\ &\qquad-\frac{1}{\sigma^2}\sin(\delta-3\alpha)-3\sigma^2\sin(3\delta-\alpha)+3\sin(3\delta-3\alpha)\Big]\frac{1}{r_*^2}.\end{aligned} \tag{142}$$

The additional conjugate complex velocity $\varDelta(dw/dz)$ is given by

$$\varDelta\frac{dw}{dz} = (\varDelta v_{r*} - i\varDelta v_{\delta})e^{-i\delta}, \tag{143}$$

and therefore, using the above expressions for $\varDelta v_{r*}$ and $\varDelta v_{\delta}$, we have, correct to the order of $1/r_*^2$,

$$\Delta \frac{dw}{dz} = \frac{1}{4} R^2 M^2 U \Bigg[\Bigg\{ \frac{1}{2\sigma^6}(1-2\sigma^2 \cos 2\alpha + \sigma^4) \times \Big[2\sigma^2 e^{-i\alpha} - (1+\sigma^4)e^{i\alpha}\Big] \log \frac{1+\sigma^2}{1-\sigma^2} + \left(\sigma^2 - \frac{3}{\sigma^2}\right)e^{-i\alpha} + \frac{e^{i\alpha}}{\sigma^4} + 2e^{-3i\alpha} - \frac{e^{3i\alpha}}{\sigma^2} \Bigg\} \frac{e^{-2i\delta}}{r_*^2} + 2(e^{3i\alpha} - \sigma^2 e^{i\alpha})\frac{e^{-4i\delta}}{r_*^2} + (\sigma^2 e^{-i\alpha} - e^{-3i\alpha})\frac{e^{2i\delta}}{r_*^2} \Bigg]. \quad (144)$$

Further, if we put $z = r_* e^{i\delta}$ in this equation, we get

$$\Delta \frac{dw}{dz} = \frac{1}{4} R^2 M^2 U \Bigg[\Bigg\{ \frac{1}{2\sigma^6}(1-2\sigma^2 \cos 2\alpha + \sigma^4) \times \Big[2\sigma^2 e^{-i\alpha} - (1+\sigma^4)e^{i\alpha}\Big] \log \frac{1+\sigma^2}{1-\sigma^2} + \left(\sigma^2 - \frac{3}{\sigma^2}\right)e^{-i\alpha} + \frac{e^{i\alpha}}{\sigma^4} + 2e^{-3i\alpha} - \frac{e^{3i\alpha}}{\sigma^2} \Bigg\} \frac{1}{z^2} + 2(e^{3i\alpha} - \sigma^2 e^{i\alpha})\frac{r_*^2}{z^4} + (\sigma^2 e^{-i\alpha} - e^{-3i\alpha})\frac{z^2}{r_*^4} \Bigg]. \quad (145)$$

The approximate expression for the quantity dw/dz can now be obtained immediately. Thus, by (134), (135) and (145), we have, correct to the order of $1/r_*^2$,

$$\frac{dw}{dz} = U e^{-i\alpha}\left(1 - \frac{R^2 e^{2i\alpha}}{z^2}\right) + \frac{1}{4} R^2 M^2 U \Bigg[\Bigg\{ \frac{1}{2\sigma^6}(1-2\sigma^2 \cos 2\alpha + \sigma^4) \times \Big[2\sigma^2 e^{-i\alpha} - (1+\sigma^4)e^{i\alpha}\Big] \log \frac{1+\sigma^2}{1-\sigma^2} + \left(\sigma^2 - \frac{3}{\sigma^2}\right)e^{-i\alpha} + \frac{e^{i\alpha}}{\sigma^4} + 2e^{-3i\alpha} - \frac{e^{3i\alpha}}{\sigma^2} \Bigg\} \frac{1}{z^2} + 2(e^{3i\alpha} - \sigma^2 e^{i\alpha})\frac{r_*^2}{z^4} + (\sigma^2 e^{-i\alpha} - e^{-3i\alpha})\frac{z^2}{r_*^4} \Bigg]. \quad (146)$$

We are now able to evaluate the two integrals in the formula (138) for M_c. To do this, we shall take, without loss of generality, the large circle of radius r_* as the contour C, so that r_* in (146) will be hereafter considered to be constant.

We begin with the first integral, namely:

$$\oint_C \left(\frac{dw}{dz}\right)^2 \frac{dz}{d\zeta} \zeta \, dz .$$

It is evident however that only the coefficient, denoted by A for the moment, of $1/z$ in the expansion of the integrand in powers of z makes a contribution to this integral.

Remembering that the Joukowski transformation $\zeta = z + a^2/z$ gives immediately

$$\frac{dz}{d\zeta}\zeta = z + \frac{2a^2}{z} + \cdots , \qquad (147)$$

and making use of (146), the required coefficient of $1/z$ in the expansion of the function $(dw/dz)^2(dz/d\zeta)\zeta$ in a power series of z can easily be found to be, in the limit when $r_* \to \infty$,

$$\begin{aligned} A &= 2U^2(a^2 e^{-2i\alpha} - R^2) \\ &+ \frac{1}{2} R^2 M^2 U^2 \left\{ \frac{1}{2\sigma^6}(1 - 2\sigma^2 \cos 2\alpha + \sigma^4) \right. \\ &\qquad \times \left[2\sigma^2 e^{-2i\alpha} - (1+\sigma^4) \right] \log \frac{1+\sigma^2}{1-\sigma^2} \\ &\qquad \left. + \left(\sigma^2 - \frac{3}{\sigma^2}\right) e^{-2i\alpha} + \frac{1}{\sigma^4} + 2e^{-4i\alpha} - \frac{e^{2i\alpha}}{\sigma^2} \right\} . \end{aligned} \qquad (148)$$

The value of the integral under consideration is equal to $2\pi i A$. Thus, we have, in the limit $r_* \to \infty$,

$$\Re\left[\oint_C\left(\frac{dw}{dz}\right)^2\frac{dz}{d\zeta}\zeta\,dz\right]$$

$$= 4\pi a^2U^2\sin 2\alpha$$

$$+\pi R^2M^2U^2\left\{\frac{1}{\sigma^4}(1-2\sigma^2\cos 2\alpha+\sigma^4)\sin 2\alpha\log\frac{1+\sigma^2}{1-\sigma^2}\right.$$

$$\left.+\left(\sigma^2-\frac{3}{\sigma^2}\right)\sin 2\alpha+2\sin 4\alpha+\frac{1}{\sigma^2}\sin 2\alpha\right\}$$

$$= 4\pi a^2U^2\sin 2\alpha\left[1+\frac{1}{2}M^2\left\{\frac{1}{2\sigma^6}(1-2\sigma^2\cos 2\alpha+\sigma^4)\log\frac{1+\sigma^2}{1-\sigma^2}\right.\right.$$

$$\left.\left.-\frac{1}{\sigma^4}(1-2\sigma^2\cos 2\alpha+\sigma^4)+\frac{3}{2}\right\}\right], \tag{149}$$

where the relation $\sigma=a/R$ has been used. Hence,

$$-\frac{1}{2}\rho_\infty\Re\left[\oint_C\left(\frac{dw}{dz}\right)^2\frac{dz}{d\zeta}\zeta\,dz\right]$$

$$= -2\pi\rho_\infty a^2U^2\sin 2\alpha\left[1+\frac{1}{2}M^2\left\{\frac{1}{2\sigma^6}(1-2\sigma^2\cos 2\alpha+\sigma^4)\log\frac{1+\sigma^2}{1-\sigma^2}\right.\right.$$

$$\left.\left.-\frac{1}{\sigma^4}(1-2\sigma^2\cos 2\alpha+\sigma^4)+\frac{3}{2}\right\}\right]. \tag{150}$$

If we denote by M_i the moment about the origin of the fluid pressures acting on the elliptic cylinder when placed, at the angle of attack α, in the stream of an incompressible fluid of density ρ_∞, it is well known that when the undisturbed velocity is denoted by U,

$$M_i = -2\pi\rho_\infty a^2U^2\sin 2\alpha\,. \tag{151}$$

Substituting this in the right-hand side of (150), we have

$$-\frac{1}{2}\rho_\infty \Re\left[\oint_C \left(\frac{dw}{dz}\right)^2 \frac{dz}{d\zeta}\zeta\, dz\right]$$
$$= M_i\left[1+\frac{1}{2}M^2\left\{\frac{1}{2\sigma^6}(1-2\sigma^2\cos 2\alpha+\sigma^4)\log\frac{1+\sigma^2}{1-\sigma^2}\right.\right.$$
$$\left.\left.-\frac{1}{\sigma^4}(1-2\sigma^2\cos 2\alpha+\sigma^4)+\frac{3}{2}\right\}\right]. \quad (152)$$

Next, we consider the second integral in (138), namely:

$$\oint_C \left(1-\frac{v_0^2}{U^2}\right)\left(\frac{dw_0}{dz}\right)^2 \frac{dz}{d\zeta}\zeta\, dz\,.$$

To this integral, only the coefficient of $1/z$ in the expansion of the integrand in a power series of z makes a contribution, and our next problem is therefore to find the said coefficient which will be denoted by B for the present.

With the aid of (28) we have

$$1-\frac{v_0^2}{U^2} = 1-\frac{1-2\lambda_*^2\cos 2(\delta-\alpha)+\lambda_*^4}{1-2\sigma^2\lambda_*^2\cos 2\delta+\sigma^4\lambda_*^4}\,, \quad (153)$$

and therefore, expanding the second term on the right-hand side in powers of λ_*, we get approximately

$$1-\frac{v_0^2}{U^2} = -\left\{2\sigma^2\cos 2\delta-\cos 2(\delta-\alpha)\right\}\lambda_*^2\,, \quad (154)$$

or, since $\lambda_* = R/r_*$,

$$1-\frac{v_0^2}{U^2} = -\left\{\sigma^2(e^{2i\delta}+e^{-2i\delta})-\frac{1}{2}(e^{2i(\delta-\alpha)}+e^{-2i(\delta-\alpha)})\right\}\frac{R^2}{r_*^2}. \quad (155)$$

Thus, if, as before, we put $z=r_* e^{i\delta}$, we have, correct to the order of $1/r_*^2$,

$$1-\frac{v_0^2}{U^2} = -R^2\left\{\left(\frac{\sigma^2}{r_*^4}-\frac{1}{2}\frac{e^{-2i\alpha}}{r_*^4}\right)z^2+\left(\sigma^2-\frac{1}{2}e^{2i\alpha}\right)\frac{1}{z^2}\right\}. \quad (156)$$

The required coefficient B of $1/z$ can now be obtained immediately. Thus, by (135), (147) and (156), we get, in the limit when $r_* \to \infty$,

$$B = -R^2 U^2\left(\sigma^2 e^{-2i\alpha} - \frac{1}{2}\right). \tag{157}$$

The value of the integral under discussion is equal to $2\pi i B$. We have therefore, in the limit $r_* \to \infty$,

$$\begin{aligned}\Re\left[\oint_C \left(1-\frac{v_0^2}{U^2}\right)\left(\frac{dw_0}{dz}\right)^2 \frac{dz}{d\zeta}\zeta\, dz\right] &= -2\pi R^2 \sigma^2 U^2 \sin 2\alpha \\ &= -2\pi a^2 U^2 \sin 2\alpha\,, \end{aligned}\tag{158}$$

where the relation $\sigma = a/R$ has been used.

Therefore, taking (151) into account, we get

$$\begin{aligned}-\frac{1}{4}\rho_\infty M^2 \Re\left[\oint_C \left(1-\frac{v_0^2}{U^2}\right)\left(\frac{dw_0}{dz}\right)^2 \frac{dz}{d\zeta}\zeta\, dz\right] &= \frac{1}{2}\pi \rho_\infty a^2 U^2 M^2 \sin 2\alpha \\ &= -\frac{1}{4} M_i M^2\,. \end{aligned}\tag{159}$$

Finally, inserting the results (152) and (159) in the right-hand side of (138), we obtain the expression for the moment M_c in the form:

$$M_c = M_i\left[1 + \frac{1}{2}M^2\left\{\frac{1}{2\sigma^6}(1-2\sigma^2\cos 2\alpha+\sigma^4)\log\frac{1+\sigma^2}{1-\sigma^2} - \frac{1}{\sigma^4}(1-2\sigma^2\cos 2\alpha+\sigma^4)+1\right\}\right], \tag{160}$$

or,

$$\frac{M_c}{M_i} = 1 + \frac{1}{2}M^2 + \frac{1}{2}M^2(1-2\sigma^2\cos 2\alpha+\sigma^4)$$
$$\times\left(\frac{1}{2\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{\sigma^4}\right). \quad (161)$$

It will be seen that except for some differences in notations, this result agrees with KAPLAN's result(1) for the case of no circulation round the elliptic cylinder.

§ 16. The numerical values of the ratio M_c/M_i for various values of the MACH number $M(=U/c_0)$ ranging from 0 to 1 have been calculated by the above formula (161) for two elliptic cylinders with different thickness ratios. For one elliptic cylinder, we have assumed(2) that $\sigma=0{\cdot}82$ so that the thickness ratio of the ellipse is $t=0{\cdot}196$ approximately, while for the other elliptic cylinder we have taken(3) $\sigma=0{\cdot}73$ which corresponds with $t=0{\cdot}305$ approximately. Also, three different values 5°, 10° and 15° have been assumed for the angle of attack α.

The values of M_c/M_i thus calculated are tabulated in Tables III and IV, and they are also shown graphically in Figs. 5 and 6 by full-line curves, taking the MACH number M as the abscissa. The results for the elliptic cylinder for which $\sigma=0{\cdot}82$ and $t=0{\cdot}196$ are shown in Table III and Fig. 5, while those for the elliptic cylinder for which $\sigma=0{\cdot}73$ and $t=0{\cdot}305$ in Table IV and Fig. 6.

(1) C. KAPLAN, loc. cit. (Report No. 671).

(2) The thickness ratio $t=0{\cdot}196$ corresponds approximately with the point of maximum for the curve of M_{crit} for $\alpha=5°$ shown in Fig. 2.

(3) The thickness ratio $t=0{\cdot}305$ corresponds approximately with the point of maximum for the curve of M_{crit} for $\alpha=10°$ shown in Fig. 2.

TABLE III. ($\sigma = 0{\cdot}82$, $t = 0{\cdot}196$)

M	M_c / M_i		
	$\alpha = 5°$	$\alpha = 10°$	$\alpha = 15°$
0	1	1	1
0·1	1·0053	1·0054	1·0057
0·2	1·0212	1·0218	1·0227
0·3	1·0477	1·0490	1·0511
0·4	1·0848	1·0871	1·0908
0·5	1·1325	1·1361	1·1419
0·6	1·1908	1·1959	1·2043
0·7	1·2597	1·2667	1·2781
0·8	1·3392	1·3483	1·3632
0·9	1·4293	1·4408	1·4597
1	1·5300	1·5442	1·5675

TABLE IV. ($\sigma = 0{\cdot}73$, $t = 0{\cdot}305$)

M	M_c / M_i		
	$\alpha = 5°$	$\alpha = 10°$	$\alpha = 15°$
0	1	1	1
0·1	1·0055	1·0056	1·0057
0·2	1·0219	1·0223	1·0229
0·3	1·0493	1·0501	1·0516
0·4	1·0876	1·0892	1·0917
0·5	1·1369	1·1393	1·1433
0·6	1·1971	1·2006	1·2063
0·7	1·2683	1·2730	1·2808
0·8	1·3504	1·3566	1·3668
0·9	1·4434	1·4513	1·4642
1	1·5475	1·5572	1·5731

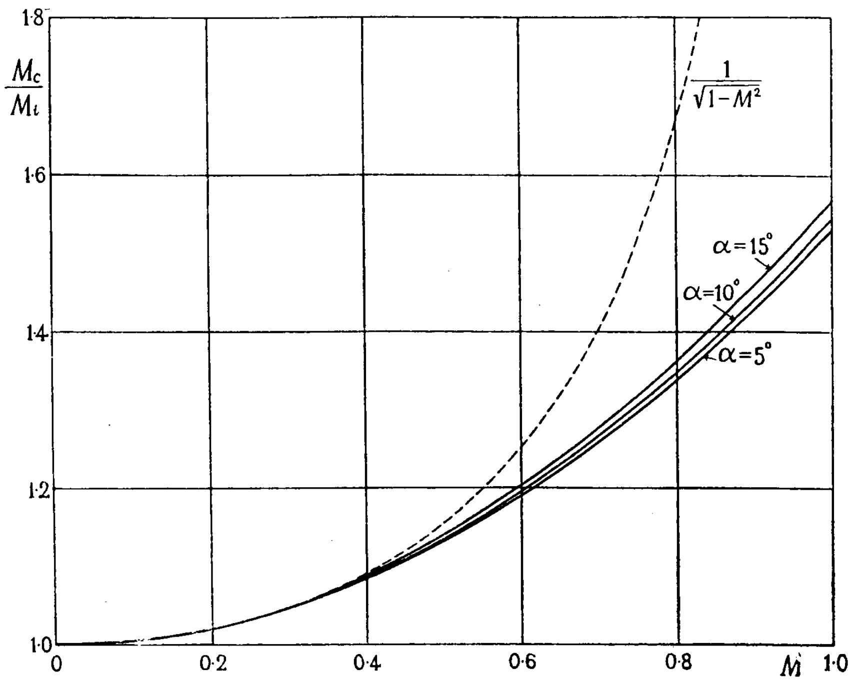

Fig. 5. $\sigma = 0{\cdot}82$, $t = 0{\cdot}196$.

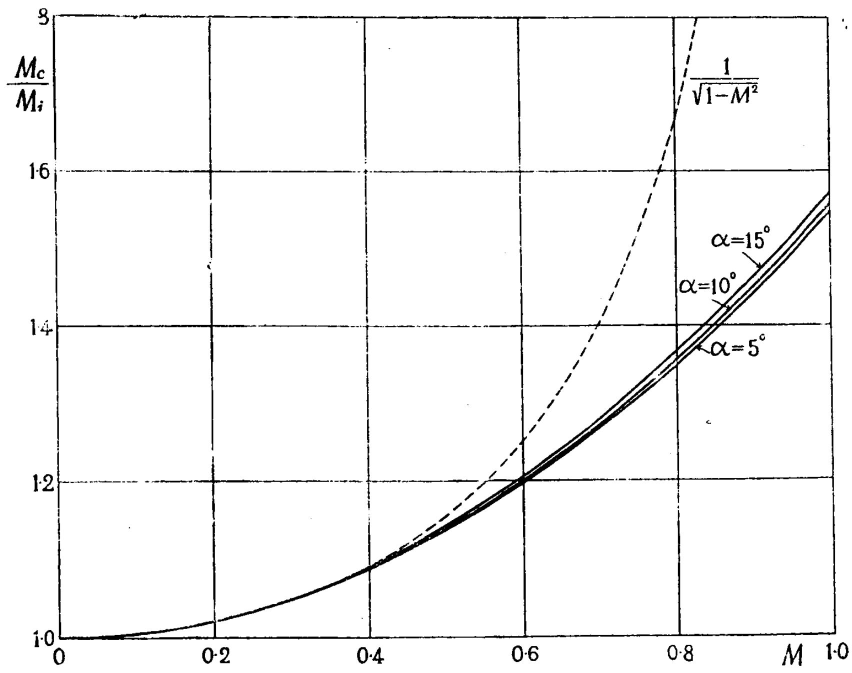

Fig. 6. $\sigma = 0{\cdot}73$, $t = 0{\cdot}305$.

From Figs. 5 and 6 it will readily be seen that the value of M_c/M_i increases as the MACH number M increases and that when the value of the MACH number is fixed, the value of M_c/M_i increases slightly as the angle of attack α increases.

According to GLAUERT-PRANDTL's linear theory[1], on the other hand, the approximate expression for the ratio M_c/M_i for a cylindrical body placed in the stream of a compressible fluid is given by

$$\frac{M_c}{M_i} = \frac{1}{\sqrt{1-M^2}}, \qquad (162)$$

where M is, as before, the MACH number for the undisturbed stream so that $M = U/c_0$.

This formula is expected to hold good especially when the body is thin and its angle of attack is small, since, as mentioned already, GLAUERT-PRANDTL's linear theory is based on the assumptions that the flow is irrotational and the disturbances due to the presence of the body are small in comparison with the velocity of the undisturbed stream.

Thus, it is of great interest to compare the values of M_c/M_i calculated by the above approximate formula (162) with the corresponding values given by the more appropriate and accurate formula for M_c/M_i obtained in the present paper, which is given by (161) and is considered to be valid even for an elliptic cylinder with large thickness ratio placed at no small angle of attack so that the disturbances due to the presence of the body are not at all small in comparison with the undisturbed velocity.

The values of $1/\sqrt{1-M^2}$ have therefore been calculated for various values of M and are shown graphically by a dotted-line curve in each of Figs. 5 and 6.

---

(1) H. GLAUERT, loc. cit.

In each of these figures it will be seen that when the value of M is less than 0·5 approximately, the dotted-line curve representing GLAUERT-PRANDTL's approximate formula (162) nearly coincides with the full-line curves representing the values of M_c/M_i calculated by the more appropriate and accurate formula (161). It is interesting to notice that provided $M<0{\cdot}5$, GLAUERT-PRANDTL's formula gives sufficiently good approximations even when an elliptic cylinder with the thickness ratio $t=0{\cdot}3$ is placed at an angle of attack $\alpha=15°$.

VI. Summary.

§ 17. In the present paper, the two-dimensional irrotational subsonic flow of a compressible fluid past an elliptic cylinder placed at an arbitrary inclination to the direction of the undisturbed stream has been re-investigated, with a special intention of studying the manner in which the value of the so-called critical MACH number for the elliptic cylinder varies with the angle of attack as well as with the thickness ratio of the ellipse. The critical MACH number is, as is well known, the value of the MACH number at which the local speed of sound is first attained in the field of flow, and according to the results of recent experimental investigations, it has a close connection with an important phenomenon called the compressibility burble.

We have employed the method of POGGI, as in KAPLAN's papers. Although KAPLAN has confined his attention chiefly to the state of affairs on the surface of the elliptic cylinder, yet we have generalised the analysis to some extent, by obtaining first the velocity potential at any point in the field of flow and then proceeding to the discussion of the state of affairs on the surface of the body.

Very laborious numerical calculations have been carried out in detail and thus the values of the critical MACH number M_{crit} have been found as functions of both the angle of attack and the thickness ratio of the ellipse. The results are shown graphically in Figs. 2, 3 and 4, where the ordinate gives the values of M_{crit} and the abscissa gives the thickness ratio t of the ellipse. One of the interesting results obtained is that when the angle of attack takes any value different from $0°$ and $90°$, the curve of the critical MACH number plotted against the thickness ratio t has a maximum at a certain definite value of t. This point seems to be worthy of special notice.

Also, we have calculated the moment of the fluid pressures acting on the surface of the elliptic cylinder with respect to the centre of the ellipse, and performing various numerical calculations we have discussed the effect of compressibility upon the moment of the elliptic cylinder. It is found that when the value of the angle of attack is fixed, the value of the ratio M_c/M_i for a definite elliptic cylinder increases as the MACH number increases and also that the value of M_c/M_i for a definite MACH number increases slightly as the angle of attack increases, where M_c denotes the moment which an elliptic cylinder experiences when placed in the compressible fluid flow and M_i the moment which the same elliptic cylinder experiences when placed, at the same angle of attack, in the incompressible fluid flow.

Further, the validity of GLAUERT-PRANDTL's approximate formula for the ratio M_c/M_i has been discussed briefly.

August, 1940.

Physical Institute,
Faculty of Science,
Imperial University of Osaka.