第　百　四　號
(昭和九年三月發行)

抄　　錄

軟鋼梁の均一曲げによる降伏について

所員　工學博士　中西不二夫
技手　伊藤正治
技手　北村菊男

軟鋼の梁の斷面の形が對稱的でないものに均一曲げモーメントをかけると、降伏點が二つ出て來る。梁は第一の降伏點に達するまでは殆んど彈性的であつて、この範圍では荷重と撓みとの關係は一つの直線になる。第一の降伏點で荷重撓み曲線は急に曲つて、それ以後は荷重を增すと共に撓みも增して來る。かくて第二の降伏點に達すると荷重撓み曲線はまた急に曲り、それ以後は梁の全體の長さが降伏してしまふまで水平な直線になる。今梁の斷面の形は對稱でないので、中立軸の一方の側は内力が比較的大きく、他の側は小さい。第一の降伏點はこの一方の降伏を起し易い方の側が降伏を始める點であり、第二の降伏點は他の側が降伏を起す點である。

普通には最大内力を受けてゐる點の狀態が、ある狀態に達すると降伏を起すものと考へてゐるが、これは正しくない。均一内力を受けてゐるときと、然らざるときとは降伏するときの最大内力の値は違ふのである。これは降伏が一種の安定の問題であるためであつて、降伏には全體の内力分布を考へなければならないのである。内力分布を考へに入れると第一降伏點の條件は次のやうになる。

$$\int \sigma y dS = \sigma_y \int y dS \, ,$$

玆に σ は斷面に働く垂直内力、σ_y は降伏した部分の内力であつて材料によつて定つた値である。y は斷面の面素 dS の中立軸からの距離で、積分は中立軸の一方の初めに降伏を起す方の側だけについて行ふ。この關係は實驗とよく一致する。

第二降伏點も同じやうな條件で與へられる。これも實驗とよく一致する。また第二降伏點の撓みと第一降伏點の撓みとの比も計算することが出來るが、これも實驗とよく一致する。

No. 104.

(Published March, 1934.)

On the Yield Points of Mild Steel Beams under Uniform Bending.

By

Fujio NAKANISHI, *Kôgakuhakusi*,

Member of the Institute.

Masaharu ITÔ and Kikuo KITAMURA.

Assistants in the Institute.

Abstract.

It was found that when mild steel beams of cross-section asymmetrical about the neutral axes are bent, there are two yield points. Generally one side of the neutral axis yields before the other. The first yield point corresponds to the bending moment under which one side begins to yield, while the second yield point corresponds to that under which the other side begins to yield.

The conditions of yielding were investigated. The first yield point is given by

$$\int \sigma y dS = \sigma_y \int y dS\,,$$

where σ is the normal stress across the cross-section, σ_y the known particular stress for that material, y the distance of the elemental area dS on the cross-section from the neutral axis. The integration is effected over one side of the neutral axis. The condition of the second yield point is also given by a similar formula.

1. Introduction.

In an earlier paper "On the Yield Point of Mild Steel"(1), the yield points under torsion, bending, and tension were explained. In that case, however, the yielding under bending was investigated only with beams of cross-section symmetrical about the neutral axes.

In the present paper are investigated the yield points under uniform bending of mild steel beams of cross-section asymmetrical about the neutral axes.

In the previous investigation the following facts were established with reference to mlid steel beams of symmetrical cross-section:

1. The material is elastic almost up to the yield point.
2. At the yield point a certain part of the beam yields suddenly.
3. The yielding on the tension and compression sides occurs simultaneously.

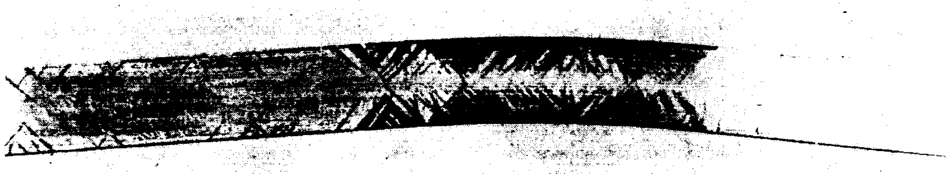

Fig. 1.

(1) Report of the Aeron. Research Inst., No. 72 (1931), pp. 83–140.

4. As shwon in Fig. 1, the form of the part that has yielded resembles a wedge, its point being at the neighbourhood of the neutral axis.

5. Such parts appear one after the other under constant bending moment.

6. Sometimes, the bending moment at first attains a certain higher value, the so called *oberefliessgrenze.* But for the yield point we must take the constant bending moment under which the yielding spreads.

7. The stress distribution in the yielded region, where there are many parts that have yielded as in the middle part of the beam in Fig. 1, may be regarded as being uniform, while the stress in the elastic region, free of such yielded parts, as in the right hand part of the beam in Fig. 1, may be considered to be proportional to the distance from the neutral axis.

8. Accordingly, the bending moment at the yield point may be expressed by the formula

$$M_y = \int \sigma y dS = \sigma_y \int |y| \, dS , \quad \dots\dots\dots\dots\dots \quad (1)$$

where M_y = bending moment at the yield point,

σ = normal stress across the cross-section in the elastic region,

σ_y = normal stress in the yielded region, which is the known particular stress for that material,

y = distance of the elemental area dS from the neutral axis, the integration being effected for over the entire cross-section.

2. Experiments with Beams of Asymmetrical Cross-Section.

To beams of cross-sections, as shown in Fig. 2, cut from the same mild steel bar, uniform bending moment was applied very gradually and the amounts of deflection δ (shown in Fig. 3) measured. The dimensions of the test pieces together with the results of the experiments will be found in Table I. Figs. 4–8 are examples of diagrams showing the relation between bending moment and deflection.

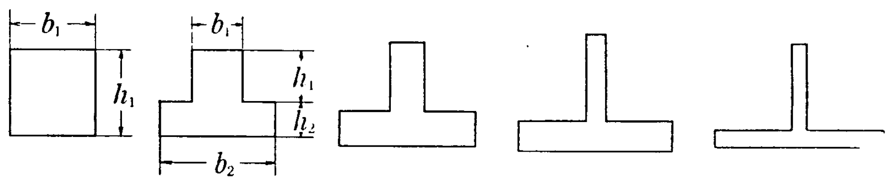

Fig. 2.

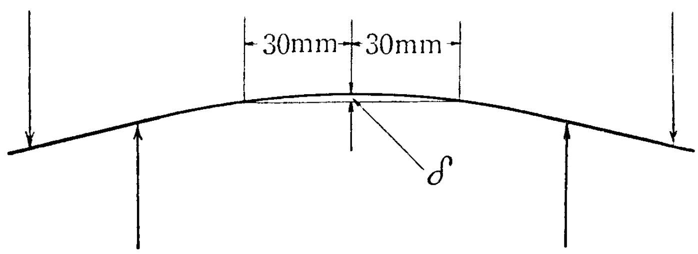

Fig. 3.

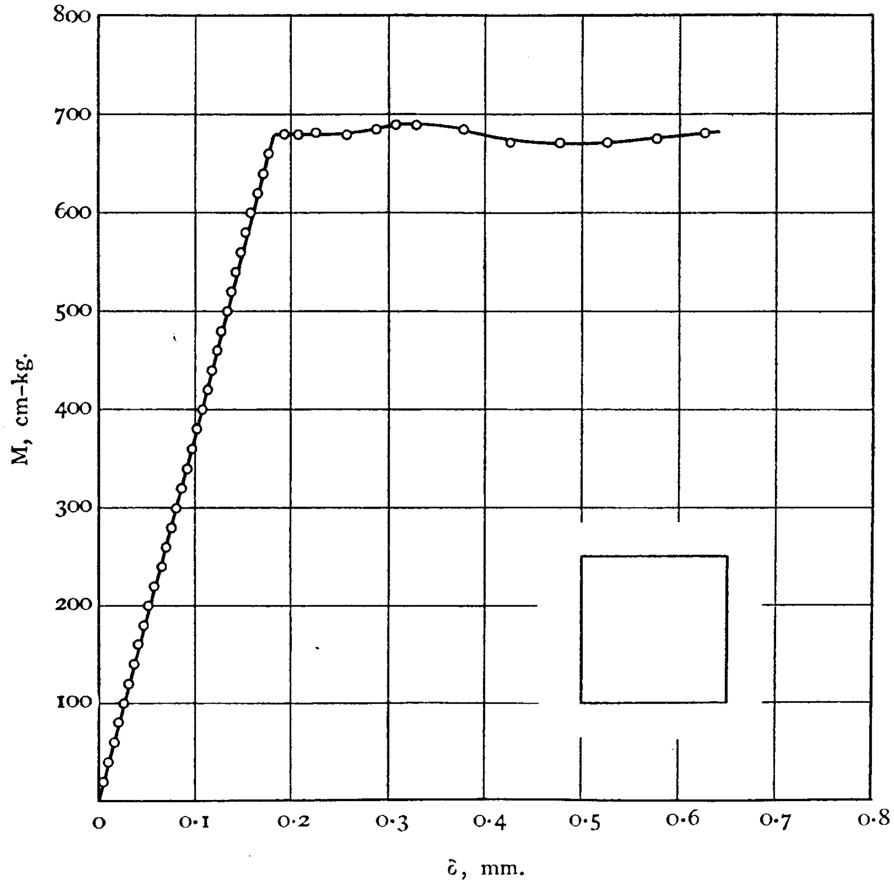

Fig. 4.

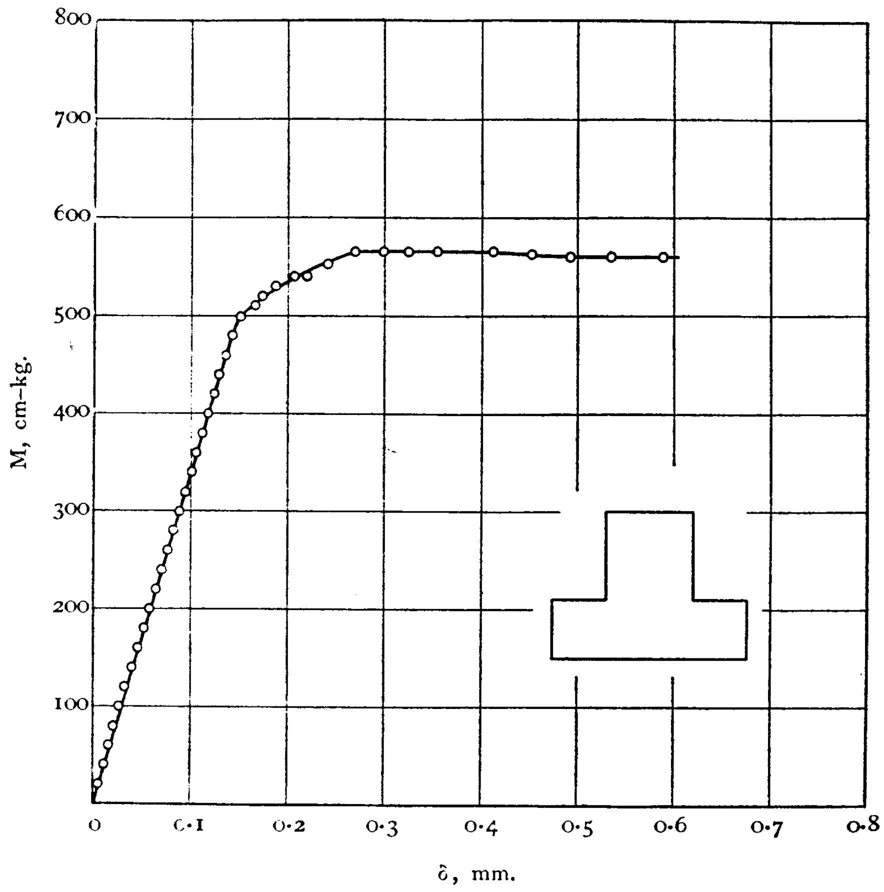

Fig. 5.

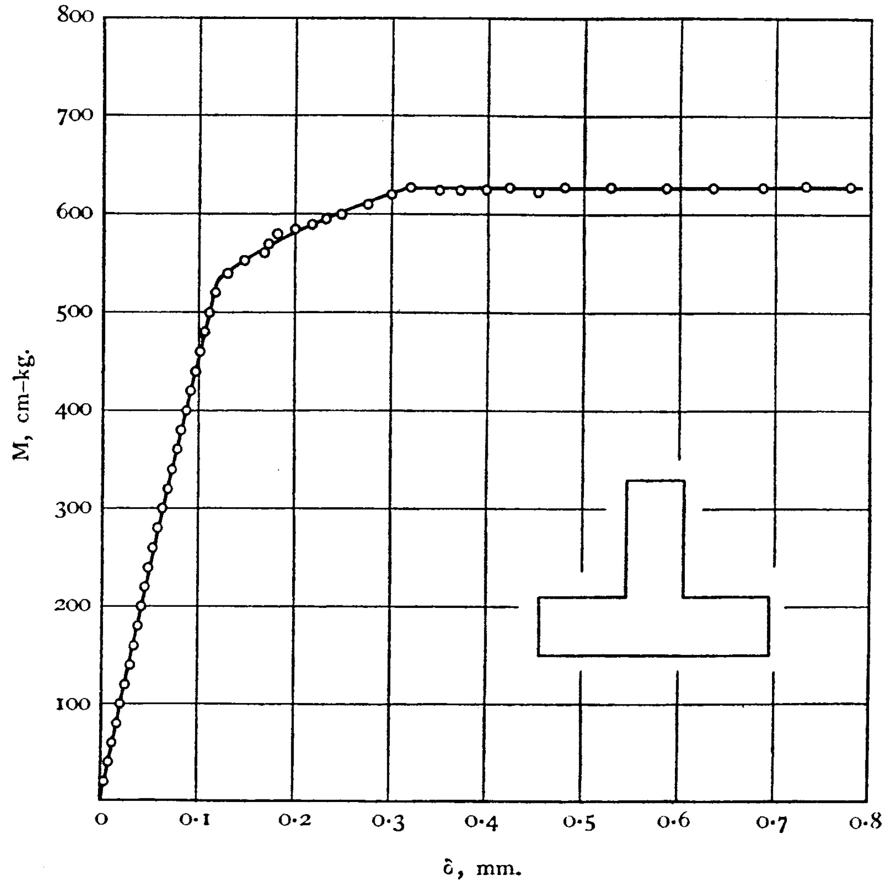

Fig. 6.

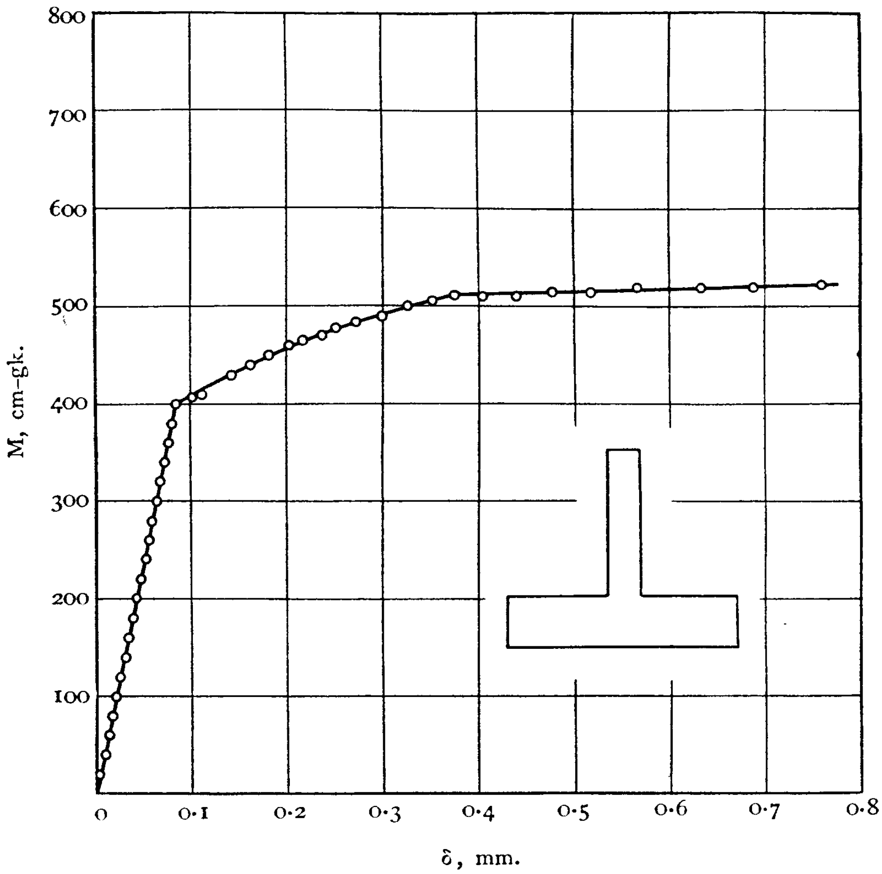

Fig. 7.

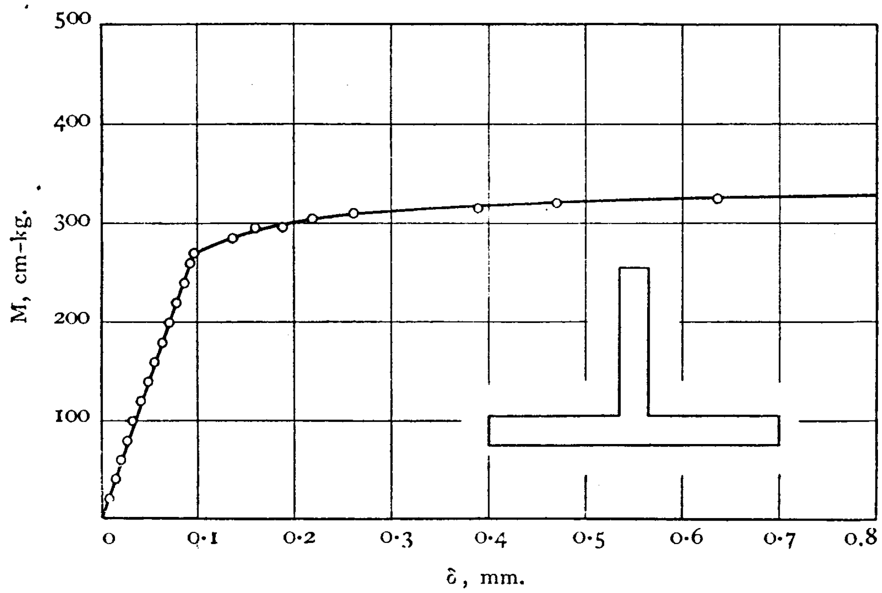

Fig. 8.

TABLE I.

I. Test Piece	II. Dimensions in mm				III. First Yield Point in cm-kg	IV. σ_y in kg/mm²	V. Second Yield Point in cm-kg	VI. σ_y in kg/mm²	VII. Ratio of Deflections at the Two Yield Points	
	b_1	h_1	b_2	h_2					Experiment	Calculation
No. 251	10·03	10·04			680	27·0		27·0		
252	10·01	10·02			680	27·1		27·1		
253	6·04	6·02	13·51	4·02	525	28·8	570	26·8	1·80	1·74
254	6·04	6·01	13·55	4·02	500	27·5	565	26·7	1·79	1·74
255	4·04	8·04	16·00	3·99	540	27·8	645	26·7	2·64	2·63
256	4·03	7·99	16·03	4·03	533	27·5	628	26·0	2·66	2·63
257	2·23	10·00	18·03	3·49	400	26·7	520	26·2	4·34	4·26
258	2·20	9·90	18·04	3·54	385	26·1	515	26·3	4·41	4·26
259	1·81	10·00	20·00	2·02	270	26·5	330	26·3		6·34
260	1·80	10·00	20·03	2·03	269	26·8	330	26·5		6·34

3. The Two Yield Points.

It will be seen from Figs. 5–7 that the slope of the curve changes suddenly at two points, which we shall call the first and second yield points.

There has been no experiment, as far as we can learn, in which these two points have been clearly observed. But, if we apply uniform bending moment very slowly and measure the deflection minutely, we shall always find the existence of these two points. The first yield point especially is very clear, since above this point, there is much time effect on the deflection, while there is almost none below it.

Up to the yield point the material may be considered to be elastic; the normal stress is approximately proportional to the distance from the neutral axis. Since in the present case the shape of the cross-section is not symmetrical about the neutral axis, the magnitude of the stress on one side of the axis is comparatively greater than that on the other. Consequently the two sides do not yield simultaneously. The yielding occurs first on one side, followed by that on the other after the bending moment has somewhat increased. The first yield point corresponds to the bending moment under which the first side begins to yield, and similarly the second yield point corresponds to that under which the second side begins to yield.

As explained in the earlier paper mentioned, the condition of yielding cannot be given merely by the state of stress at the point where the stress is greatest. It is affected by the entire stress distribution. That part where the stress is greatest will naturally be the first to yield. This yielding changes the stress distribution and a new distribution results. When the shape of the cross-section is symmetrical about the neutral axis, this new distribution is also unstable and the yielding, under the constant bending moment, spreads immediately from the two sides to the neighbourhood of the neutral axis.

When the shape of cross-section is not symmetrical, the manner of yielding is quite different, since in this case the stress on one side is comparatively great while that on the other is yet small. The part where the stress is greatest will similarly be the first to yield, resulting in a change in the stress distribution. However, the stress on the other side being still small, the new stress distribution does not yet reach that state under which the yeilding would spread: in other words, the new state of stress is yet stable. Thus, for the yielding to spread, the moment must be increased.

Further increase of bending moment will cause the yielding on one side to spread gradually, and the stress on the other side, which is still elastic, becomes greater, until finally the stress distribution on the elastic side will reach that state wherein yielding occurs. This is the second yield point.

As the yielding on one side has already spread considerably, the yielding of the other side will immediately reach the neutral axis. Such yielded regions will then appear gradually along the length of the beam, as a consequence of which the deflection increases. Since the yielding spreads in the manner described, the bending moment after the second yield point will be constant until the yielding spreads to the whole length of the beam. This will be seen from Figs. 5–7, the bending moment-deflection curve being horizontal after the second yield point.

Fig. 9 is a typical diagram showng the relation between the bending moment and the mean curvature of the beam. A and B are respectively the first and second yield points. From B to C the curve is horizontal, and from C the bending moment increases again.

Figs. 10 and 11 are strain figures of the longitudinal sections of beams, the cross-sectional form of which is shown in Fig. 9. Fig. 10 shows the state between A and B in Fig. 9. It will be seen that although the yielding on one side has already spread considerably, there is yet no trace of it on the other. Fig. 11 shows the state between B and C in Fig. 9. It will be seen that there are two regions, that is, a region like C in which the yielding has spread throughout the cross-section, and that like B in which the lower side has not yet yielded. Region B corresponds to state B in Fig. 9, and region C to state C. Under constant bending moment, regions like C spread gradually lengthwise, and the mean curvature increases along the line BC in Fig. 9.

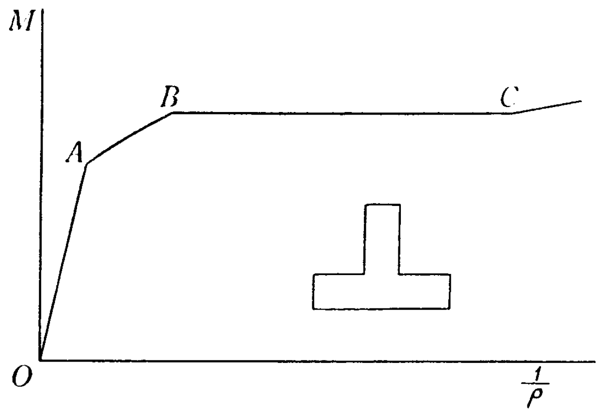

Fig. 9.

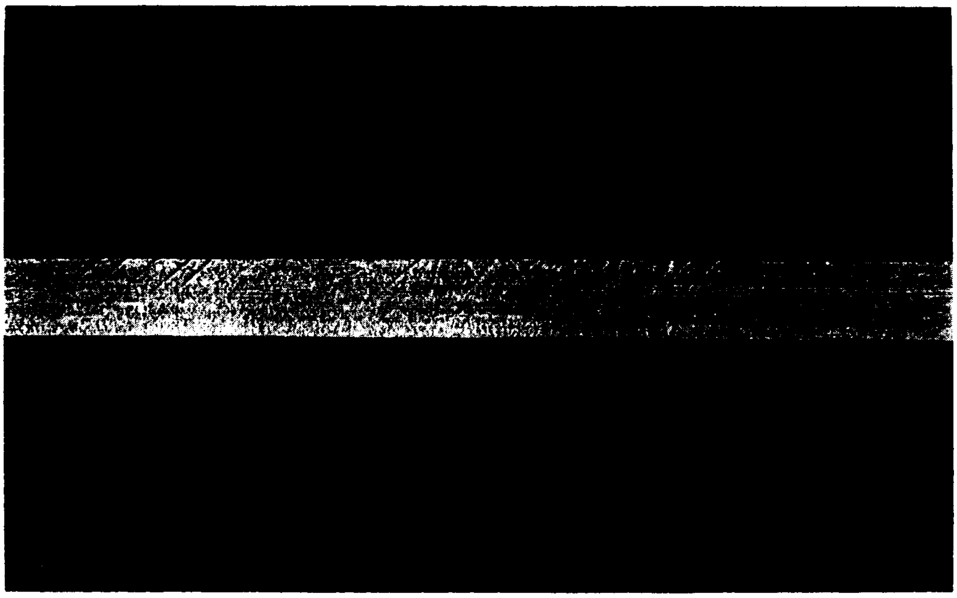

Fig. 10.

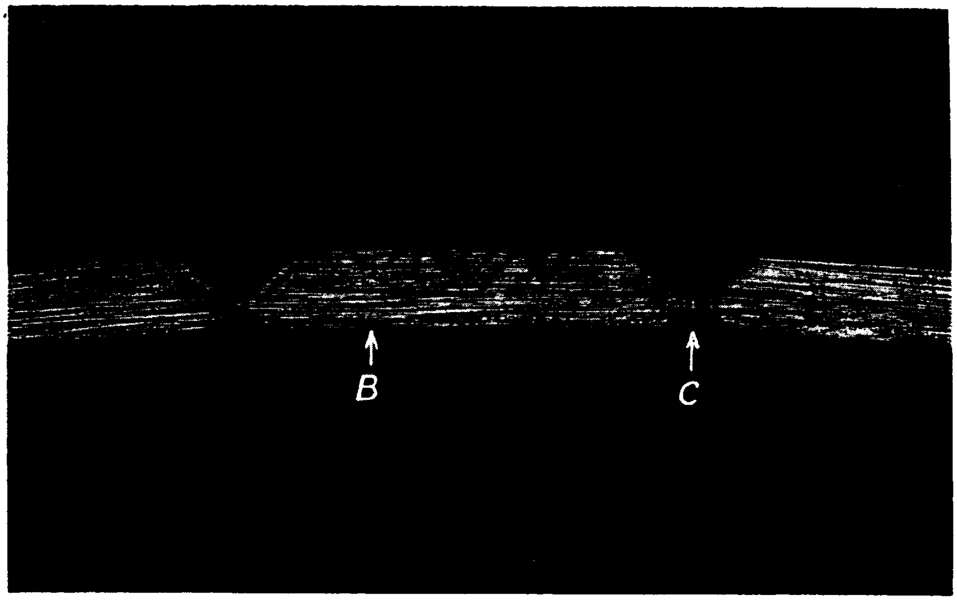

Fig. 11.

The surface of such parts as where yielding has occurred is somewhat rough, and visible without etching if viewed obliquely. Therefore, the manner in which yielding spreads can be seen by increasing the bending moment gradually and looking obliquely at the surface of the test piece.

4. The First Yield Point.

As explained in the earlier paper, the condition of yielding of beams of symmetrical cross-section under uniform bending is expressed by equation (1) or by

$$\int \sigma y dS = \sigma_y \int |y| dS ,$$

where the integration is to be effected for over the whole cross-section. In this case the yielding takes place from both sides of the neutral axis. But the beginning of the yielding cannot be regarded as being strictly simultaneous. Probably one side yields first, the consequent changes in stress distribution causing the opposite side to yield immediately after it. If so, the above equation is the condition of yielding of one side of the axis. And yet the integration is effected for over the entire cross-section, which probably is because the form of cross-section is symmetrical about the neutral axis and the condition of yielding is the same whether the integration is effected for over one side of the neutral axis or over the entire cross-section.

In general, however, in beams of asymmetrical cross-section the condition of yielding is expressed by

$$\int \sigma y dS = \sigma_y \int y dS \,, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where the integration is to be effected over the side of the neutral axis that is liable to yield. The bending moment at the first yield point, M_1, is

$$M_1 = \int \sigma y dS \,, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

the integration being effected over the entire cross-section.

We can obtain the value of σ_y from the results of experiments with rectangular beams and calculate the first yield points of various beams by means of equations (2) and (3); and by comparing these values with the experiments, it is possible to ascertain whether equation (2) is correct or not.

Or, conversely, we can calculate the values of σ_y from the results of experiments of varions beams by equations (3) and (2) and ascertain the correctness of equation (2) by examining whether the values of σ_y thus calculated are constant or not.

Here the latter method has been used. The first yield points obtained by experiments are in column III, Table I, and the values of σ_y calculated by equations (3) and (2) in column IV. The values of σ_y may be regarded as being constant. That is to say the condition of yielding expressed by (2) may be said to agree well with experiments.

5. The Second Yield Point.

By increasing the bending moment after the first yield point has manifested itself, the yielding on one side gradually spreads inwards, with consequent changes in the position of the neutral axis, while the stress in the other side increases gradually, until the second yield point is reached.

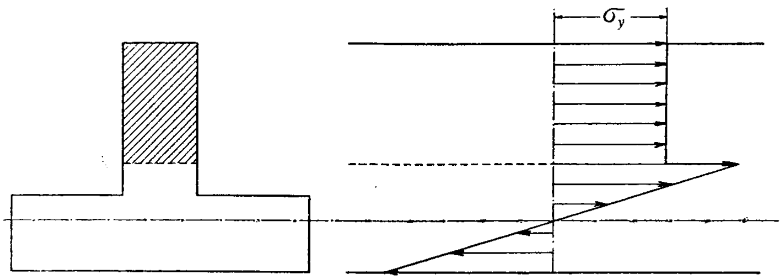

Fig. 12.

Above the first yield point, the wedge-shaped yielded parts are scattered here and there as shown in Fig. 10, and, near the point of the wedge, the stress distribution is probably greatly complicated. But here we assume for brevity that the yielding has spread to a certain depth, and that the stress in that yielded layer is constant and equal to σ_y, whence the stress distribution will be as shown in Fig. 12.

Since the side which has not yet yielded begings to do so at the second yield point, the following relation, exactly the same as equation (2), will hold for this point:

$$\int \sigma y dS = \sigma_y \int y dS \, , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)'$$

the integration being effected for over the side which has not yet yielded.

As for the side in which the yielding has only partly spread, the following relation must always hold after the first yield point,

$$\int \sigma y dS = \sigma_y \int y dS , \qquad (4)$$

where σ = normal stress across the cross-section, which is constant and is equal to σ_y in the layer that is assumed to have yielded. This is proportional to the distance from the neutral axis in the layer assumed to be yet elastic, as shown in Fig. 12, the integrations being effected for over this side of the neutral axis.

By means of equations (2)′ and (4) it is possible to find the positions of the neutral axes and, consequently, the stress distributions at the second yield points of various beams. The calculation may appear somewhat laborious, as the position of the neutral axis gradually changes, but it will not be at all difficult if we use the graphical method.

The values of σ_y, calculated from the experimental results by (2)′ and (4), will be found in column VI, Table I. They may be considered to be constant. These values moreover are practically equal to those in column IV. These facts show that the second yield points can be calculated from equations (2)′ and (4), assuming that the stress distribution after the first yield point is as shown in Fig. 12.

6. Ratio of Deflections at the Two Yeild Points.

The amounts of deflection at the first yield points, δ_1, may be easily calculated. Those at the second yield points, δ_2, can also be calculated from the assumed stress distributions. The ratios δ_2/δ_1 are shown in column VII, Table I. The calculated values agree fairly well with the results of experiments.

7. Summary.

1. If beams of mild steel are bent, there are generally two yield points.

2. When the shape of cross-section is not symmetrical about the neutral axis, one side of the axis generally yields before the other. The first yield point corresponds to the bending moment under which one side begins to yield, and the second yield point to that under which the other side begins to yield.

3. The first yield point may be calculated from equations (2) and (3). The values calculated from these equations agree well with experiments.

4. The second yield point can be calculated from equations (2)′ and (4), assuming that the stress distribution at that point is as shown in Fig. 12. The values thus calculated agree well with experiments.

5. The deflections at the second yield points can be calculated from the assumed stress distribution shown in Fig. 12. The ratios of deflections at the second yield points, calculated in this way, for those of deflections at the first yield points agree well with experiments.
