Role of CFD in Aeronautical Engineering (No.14) —AUSM Type Upwind Schemes—

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Abstract

Upwind scheme is an one of the most important parts of modern numerical scheme for gas dynamics. No scheme, however, is perfect on accuracy, robustness and efficiency. Among a lot of schemes, AUSM developed by Liou and Steffen satisfy many requirement . Today, there are many practical scheme which can be written in the same form with AUSM. We call these scheme AUSM type schemes. Some AUSM type schemes are shown and characteristics of these schemes are investigated in this report. Several improvements of SHUS, which is one of the AUSM type schemes and has unique character in them, are also shown.

1 Introduction

Upwind schemes for gas dynamics have been widely used as the basis of high resolution schemes. FDS (Flux Difference Splitting) [1][2] and FVS (Flux Vector Splitting) [3][4][5] were developed as extension of an upwind scheme for liner equations and have achieved great success. It is known that FDS and FVS, however, have some weak points. A FDS sometime blows up at high Mach number and suffer from the serious carbuncle phenomenon. A FVS's numerical diffusion is too big for the viscous flow problems. HLLE (Harten-Lax-van Leer-Einfeldt) scheme and HLLEM (Harten-Lax-van Leer-Einfeldt Modified) scheme was developed [6] to improve FDS, but HLLE's numerical viscosity is as large as FVS and HLLEM suffers from the carbuncle too.

On the other hand, by simplifying FVS, Liou & Steffen [7] invented AUSM (Advection Upstream Splitting Method). AUSM is very simple, robust for strong shock and accurate for boundary layer, however, show overshoot at shock front. Inspired by AUSM, many schemes have been proposed. Jameson [8][9] showed CUSP (Convective Upstream Split Pressure) which is similar to AUSM but is expressed in combination of central difference and numerical diffusion. Jounouchi et. al. [10] showed SFS (Simplified Flux vector Splitting method) in the similar form with AUSM and improved overshoot at a shock. Wada and Liou [11] showed AUSMDV (AUSM with flux Difference splitting and flux Vector splitting) as an improvement from AUSM and showed precise research on their scheme and others. Shima and Jounouchi [12] showed that many schemes can be made which should be called AUSM type scheme in the common form with AUSM and introduced Uni-particle upwind schemes to AUSM type schemes and exhibited SHUS (Simple High-resolution Upwind Scheme) for an example. Jounouchi et. al. [13] pointed out the physical interpretation and theoretical background of AUSM type schemes and then showed that SFS is applicable to the two-phase flow. On the other hand, Nakamori and Nakamura proposed new FVS (NNFVS hereafter) as the improvement of Steger-Warming's FVS for accuracy of viscous flow. Liou [15] also presented AUSM$^+$ and AUSM$^+$-W to improve AUSM.

Although some of these schemes have been developed independently, these schemes can be written in the common form. These scheme can called as AUSM type schemes. Furthermore, some of them also are the member of Uni-particle upwind schemes which are in a smaller group in AUSM type schemes.

Since AUSM type schemes are simple, robust and accurate enough for practical application, they have been used for many application especially for hyper sonic viscous problems already.

The common notations of AUSM type scheme and Uni-particle upwind scheme are exhibited and characteristic of some schemes are analyzed in this study.

2 Formulation of AUSM type schemes

2.1 The common form of AUSM type schemes

We show the formulation of AUSM type schemes for two dimensional Euler equation. Extension to three dimension is straight forward. Two dimensional Euler equation can be written in the integral form as follows.

$$\int \mathbf{Q} dv + \int \tilde{\mathbf{F}} ds = 0 \qquad (1)$$

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$$\mathbf{Q} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix} \quad (2)$$

$$\tilde{\mathbf{F}} = m\mathbf{\Phi} + p\mathbf{N} \ , \ \mathbf{\Phi} = \begin{pmatrix} 1 \\ u \\ v \\ h \end{pmatrix} \ , \ \mathbf{N} = \begin{pmatrix} 0 \\ x_n \\ y_n \\ 0 \end{pmatrix} \quad (3)$$

$$m = \rho V_n \ , \ V_n = x_n u + y_n v \quad (4)$$

where $\rho, u, v, e, p, h = (e+p)/\rho$ represent density, velocity in x-y direction, total energy per unit volume and total enthalpy respectively. $x_n andy_n$ show unit normals of the surface. This form means that Euler flux can be divided into the advection term and the pressure term. AUSM is based on the fact that the advection term and the pressure term can be upwinded separately. AUSM and also other AUSM type schemes can be written in following form,

$$\tilde{\mathbf{F}} = \frac{m+|m|}{2}\mathbf{\Phi}_+ + \frac{m-|m|}{2}\mathbf{\Phi}_- + \tilde{p}\mathbf{N}, \quad (5)$$

where subscript $\pm$ show physical value at left(+) and right(−) side of cell boundary, and $\tilde{p}$ mixing of pressure using Mach number of left and right state which is defined by,

$$\tilde{p} = \beta_+ p_+ + \beta_- p_- + p', \quad (6)$$

$$\beta_\pm = \frac{1}{4}(2 \mp M_\pm)(M_\pm \pm 1)^2, if \ \ |M_\pm| \le 1 \quad (7)$$

and $\beta_\pm$ are smoothly switched to 1 or 0 for supersonic case. These function are the simplest smooth function which satisfy consistency, however other functions are possible.

2.2 Pressure correction term

Wada and Liou [11] indicated that the pressure term without the pressure correction is sufficient for an usual case, but it cause over shoot at a strong propagating shock like a supersonic colliding jet. They pointed out that the use of normal momentum flux of FVS solve this problem.

Let $\tilde{\mathbf{F}}$ be an uncorrected flux and $\tilde{\mathbf{F}}_{\mathbf{FVS}}$ be a flux of FVS whose normal momentum flux is used. This correction can be written in the form of pressure correction as,

$$\tilde{\mathbf{F}}_{\mathbf{corrected}} = \tilde{\mathbf{F}} + p'\mathbf{N} \quad (8)$$

$$p' = (\tilde{\mathbf{F}}_{\mathbf{FVS}} - \tilde{\mathbf{F}}, \mathbf{N}) \quad (9)$$

If the mass flux of Hänel's FVS is used , p' is written as,

$$p' = \frac{1}{2}(m_+ - m_- - |m|)(V_{n+} - V_{n-}). \quad (10)$$

Because Hänel's FVS is given by,

$$\tilde{\mathbf{F}}_{\mathbf{FVS}} = m_+\mathbf{\Phi}_+ + m_-\mathbf{\Phi}_- + (\beta_+ p_+ + \beta_- p_-)\mathbf{N}. \quad (11)$$

Note that, this correction is only needed for a strong propagating shock and that no correction was used in examples of this report.

2.3 Selection of mass flux

In original AUSM, the mass flux is calculated from a simple switching using Mach number (see equation(A)), however, various schemes can be made replacing mass flux. Formulations of several AUSM type schemes are shown in appendix.

SFS and AUSMDV use variations of van Leer's FVS for their mass flux. Shima and Jounouchi [12] showed that any mass flux of an approximate Riemann solver can be used with an AUSM type scheme and derived SHUS using Roe schemes mass flux. On the other hand, NNFVS was originally designed as improvement of Steger-Warming's FVS , however, it also can be written as an AUSM type scheme.

2.4 Mass flux of a Uni-particle upwind scheme

The essence of Uni-particle formulation is the use of mass flux of an approximate Riemann flux such as FDS or FVS for the AUSM type scheme. SFS, AUSMDV, SHUS and NNFVS are categorized to Uni-particle upwind scheme in this sense. FDS and FVS are started from the approximate solution of Euler equation, so they owe the nature of Euler equation.

The difference in the behavior between Uni-particle upwind schemes and other AUSM type schemes can be clearly understood considering the response of the mass flux to the pressure difference. As the mass flux and the energy flux of an other AUSM type scheme such as AUSM and CUSP are zero if the convective velocity is zero. On the other hand, the mass flux proportional to the pressure difference exists in Uni-particle upwind schemes. For example, SFS and SHUS give following mass flux when convection velocity is zero.

SFS

$$m = \frac{\gamma}{4\bar{c}}(p_+ - p_-) \quad (12)$$

SHUS

$$m = \frac{1}{2\bar{c}}(p_+ - p_-) \quad (13)$$

Other Uni-particle upwind schemes give similar mass flux.

Let's consider the first step of shock tube problem. The initial convective velocity is zero everywhere, so the mass flux and the energy flux of an other AUSM type scheme is zero. As the momentum flux due to

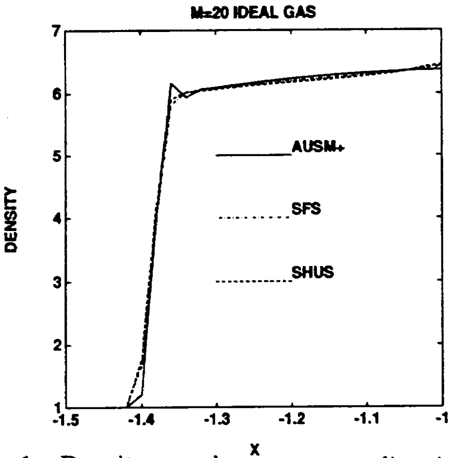

Figure 1: Density on the symmetry line in front of cylinder at Mach=20 (Ideal Gas): Small over shoot is found in the solution by AUSM$^+$.

pressure term exist, velocity at next step is not zero at some points. As a result, pressure must be decrease or constant everywhere and it is not real situation. This is the reason why over shoot at shock front occurs on other AUSM type scheme.

On the other hand, this problem is avoided by Uni-particle upwind scheme because the mass flux and the energy flux proportional to pressure difference exist (See figure 1).

Over shoot of this kind is also found as noise at oblique shock created by a wedge in figure 2.

3 Improvements of SHUS

The most of AUSM type schemes except for SHUS are based on FVS and inherit many common features from FVS.

On the other hand, SHUS is partially based on Roe's FDS, thus it inherits Roe scheme's character and is different from others. It behaves like a more robust Roe scheme, then SHUS is suitable for alternative of Roe scheme.

The mass flux of SHUS is given by Roe scheme using the finite difference of primitive variables as,

$$\begin{aligned} m &= \frac{1}{2}\{(\rho V_n)_+ + (\rho V_n)_- - |\bar{V}_n|\Delta\rho \\ &\quad - \frac{|\bar{M}+1| - |\bar{M}-1|}{2}\bar{\rho}\Delta V_n \\ &\quad - \frac{|\bar{M}+1| + |\bar{M}-1| - 2|\bar{M}|}{2}\frac{\Delta p}{\bar{c}}\} \end{aligned} \quad (14)$$

$$\Delta q = q_- - q_+ \text{、} \bar{M} = \bar{V}_n/\bar{c} \quad (15)$$

where $\bar{V}_n, \bar{\rho}, \bar{c}$ are arithmetic average of normal velocity, density and sound speed.

We show how we can avoid the weak points of Roe scheme with SHUS in the following two sections.

3.1 SHUS for the real gas

One of the benefit of an AUSM type scheme is the easiness of extension to the real gas, because they has no Jacobian.

SHUS needs part of the Jacobian, however, we can show that SHUS can be used for general state equation without any change.

Let's see the Roe scheme for the general state equation where pressure is defined by the general function of density and internal energy per unit volume as,

$$p = p(\rho, e_i) \quad (16)$$

and sound velocity c is given by,

$$c = \sqrt{\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial e_i}\frac{e_i + p}{\rho}} \quad (17)$$

. If we use the finite differences of density, velocity and internal energy to construct the mass flux, we get,

$$\begin{aligned} m &= \frac{1}{2}\{(\rho V_n)_+ + (\rho V_n)_- - |\bar{V}_n|\Delta\rho \\ &\quad - \frac{|\bar{M}+1| - |\bar{M}-1|}{2}\bar{\rho}\Delta V_n \\ &\quad - \frac{|\bar{M}+1| + |\bar{M}-1| - 2|\bar{M}|}{2\bar{c}}(\frac{\partial p}{\partial \rho}\Delta\rho + \frac{\partial p}{\partial e_i}\Delta e_i)\} \end{aligned}$$

(19)

If the last term is replaced with difference of pressure using differential relation such as,

$$dp = \frac{\partial p}{\partial \rho}d\rho + \frac{\partial p}{\partial e_i}de_i, \quad (20)$$

this form is exactly same as the form for ideal gas.

On the other hand, the mass flux of FVS, i.e. essentially the mass flux of AUSMDV or SFS, for general state equation is just an analogy of ideal gas's and have some inconsistency although it can be used directly. This leads small overshoot at a shock of AUSMDV's and SFS's result for equilibrium real gas flow. See figure 3.

3.2 Robustness for strong expansion

Treatment of the strong expansion where sometimes vacuum is found in the exact solution of Euler equation is a key to improve robustness of an upwind scheme. For Example, Roe scheme gives unphysical numerical flux and it blows up the computation.

On the other hand, FVS gives 0 flux for the super sonic expansion and its behavior is consistent with the exact solution. Many AUSM type schemes inherit this virtue from FVS and are more robust than FDS.

Consider a supersonic asymmetric expansion. The common form of AUSM type schemes clearly indicates that all fluxes vanish if the mass flux is zero.

On the other hand, the mass flux of Roe scheme is not zero for the asymmetric expansion and this gives the unphysical numerical flux. SHUS inherits this

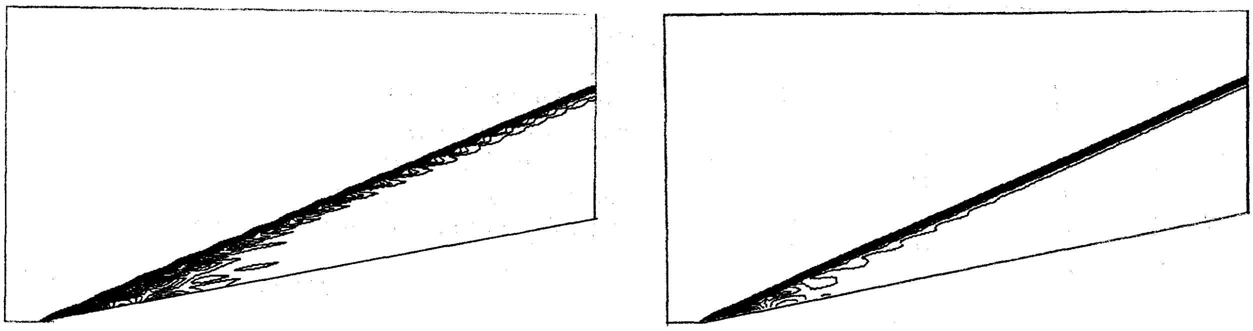

Figure 2: Shock induced by wedge:Pressure contour by AUSM$^+$(LEFT) and SHUS(RIGHT). More noise is found in the solution of AUSM$^+$ than SHUS

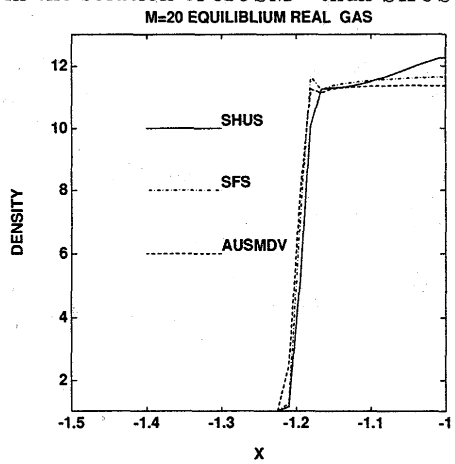

Figure 3: Density on the symmetry line in front of cylinder at Mach=20 (Equilibrium real gas): Small oscillation is found in the solution by SFS and AUSMDV

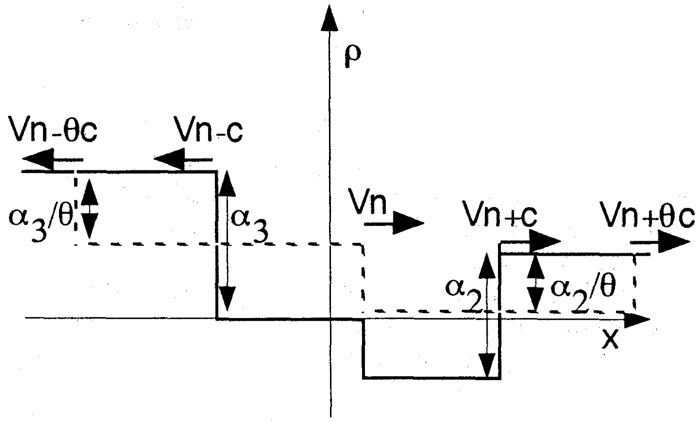

Figure 4: The solution of the linearized Riemann problem: The negative density at an intermediate state are cured by increasing the numerical sound speed.

weak point from Roe scheme. This SHUS's weak point can overcome by the simple procedure stated below.

Roe scheme also produces a non-physical solution even for a symmetric expansion since it gives a non-physical momentum flux. SHUS, however, gives no flux for this case then SHUS is more robust than Roe scheme. Because the symmetric expansion is found at the back of the body at the initial state of impulsive start where the computation frequently blow up.

The finite difference of density and mass flux can be splitted in to wave components as,

$$\Delta\rho = \sum_{i=1}^{3} \alpha_i \tag{21}$$

$$\Delta(\rho V_n) = \sum_{i=1}^{3} \lambda_i \alpha_i, \tag{22}$$

where eigen values and corresponding wave components are given by,

$$\lambda_1 = \bar{V_n}, \alpha_1 = \Delta\rho - \frac{\Delta p}{\bar{c}^2} \tag{23}$$

$$\lambda_2 = \bar{V_n} + \bar{c}, \alpha_2 = \frac{\rho\Delta V_n}{2\bar{c}} + \frac{\Delta p}{2\bar{c}^2} \tag{24}$$

$$\lambda_3 = \bar{V_n} - \bar{c}, \alpha_3 = -\frac{\rho\Delta V_n}{2\bar{c}} + \frac{\Delta p}{2\bar{c}^2} \tag{25}$$

Mass flux can be also written using the eigen values and the wave components as,

$$m = \frac{1}{2}((\rho V_n)_+ + (\rho V_n)_- - \sum_{i=1}^{3} |\lambda_i| \alpha_i). \tag{26}$$

This wave decomposition is illustrated by the solution of linearized Riemann problem. It can be presented as figure 4. A non-physical solution is produced if the density at an intermediate state is negative. Einfeldt et al [6] pointed out that this phenomenon happens because the width of intermediate states are too narrow. We follow their spirit but the procedure is different.

We can reduce the variation due to wave α_2 and $alpha_3$ by dividing by a common scaler θ. The width of intermediate states are broadened by multiplying θ to the sound velocity to keep conservation. Following this procedure, the eigen values and the wave components are modified as,

$$\lambda_2' = \bar{V_n} + \theta\bar{c}, \alpha_2' = \alpha_2/\theta \tag{27}$$

$$\lambda_3' = \bar{V_n} - \theta\bar{c}, \alpha_3' = \alpha_3/\theta, \tag{28}$$

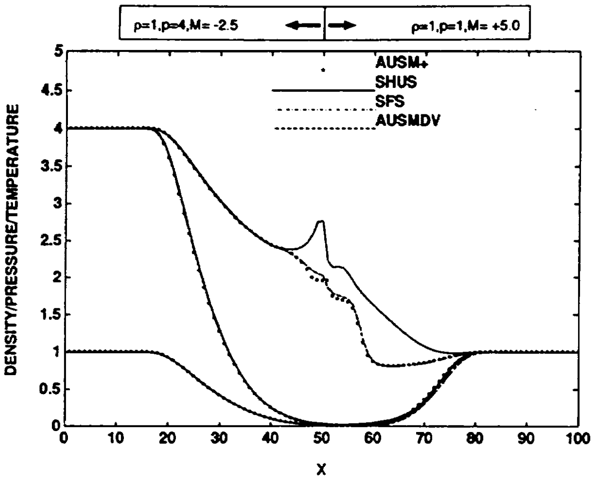

Figure 5: Solution by several 1st order AUSM type schemes on the asymmetric expansion .

where θ is given as below in order to satisfy positivity condition.

$$\rho_- + \alpha_3' \geq 0 \tag{29}$$

$$\rho_+ - \alpha_2' \geq 0 \tag{30}$$

Another component is also changed to satisfy equation (21) as,

$$\lambda_1' = \bar{V}_n, \alpha_1' = \Delta\rho - \frac{\Delta p}{\theta \bar{c}^2} \tag{31}$$

These eigen values and wave components also satisfy equation (22).

As a result, this correction is realized by only replacing the definition of Mach number in the mass flux of equation (14). The corrected Mach number $\bar{\bar{M}}$ is given by,

$$\bar{\bar{M}} = \bar{M}/\theta \tag{32}$$

where

$$\theta = max(1, \frac{1}{2\rho_+}(\frac{\bar{\rho}\Delta V_n}{\bar{c}} - \frac{\Delta p}{\bar{c}^2}), \frac{1}{2\rho_-}(\frac{\bar{\rho}\Delta V_n}{\bar{c}} + \frac{\Delta p}{\bar{c}^2})) \tag{33}$$

Solution of asymmetric super sonic expansion by several AUSM type schemes are shown in figure(5). Unmodified SHUS or Roe scheme can not compute this case.

4 Summary and discussion

The common formulation of an AUSM type scheme is exhibited and it is shown that some recent schemes such as SFS, AUSMDV, SHUS and NNFVS can be expressed as a Uni-particle upwind scheme. Uni-particle upwind schemes are a simple but effective improvements for AUSM type schemes. It is also shown that SHUS has a little different characteristics from other schemes but that it can be as robust and as easy to extend, as other AUSM type schemes.

We can not say which scheme is the best at this point . Nevertheless some character of each scheme can be pointed out. All schemes stated below are robust for strong shocks and expansions, and accurate for stationary contact surfaces and boundary layers.

AUSM is simple and can capture a moving contact but show small overshoot at a shock. **SFS** is simple but can not capture a moving contact. **AUSMDV** is simple and can capture a moving contact but contain an artificial function just tuned for a moving contact. **SHUS** is simple, can capture a moving contact and suitable for replacement of Roe scheme especially with an implicit scheme. However modification stated before is necessary for a strong expansion. **NNFVS** can capture a moving contact but slightly more complex than the other schemes.

Although enough research on other AUSM type schemes such as AUSM$^+$ or CUSP has not been done, it can be said that they are not free from over shoot at a shock and are slightly more complex. Liou[15] reported that AUSM$^+$-W improve the overshoot by introducing the pressure correction term.

The carbuncle phenomenon is a one of the biggest problem for the hypersonic computation. Although AUSM type schemes less frequently encounter the carbuncle than Roe scheme, they are not always free from it. Among AUSM type schemes, the original AUSM, and also possibly AUSM$^+$ and CUSP, is relatively safe. Nevertheless it experiences the shock front instability which is thought to be related to the carbuncle if the computational mesh is fine enough or the higher order scheme is used [16]. The analysis of the mechanism of the carbuncle and the solution are expected.

A Formulation of AUSM type schemes

AUSM

$$m = (\frac{\bar{M} + |\bar{M}|}{2})\rho_+ c_+ + (\frac{\bar{M} - |\bar{M}|}{2})\rho_- c_- \tag{34}$$

$$\bar{M} = M_+ + M_- \tag{35}$$

$$M_\pm = \pm\frac{1}{4}(\frac{V_{n\pm}}{c_\pm} \pm 1)^2, if \ |M_\pm| < 1 \tag{36}$$

Pure upwind side value or 0 will be used for supersonic case.

AUSM$^+$ AUSM$^+$ is very similar to AUSM , but smooth splitting function for average Mach number and pressure are different.

$$M_\pm = \pm\frac{1}{4}(M_\pm \pm 1)^2 \pm \frac{1}{8}(M_\pm^2 - 1)^2, if \ |M_\pm| < 1 \tag{37}$$

$$\beta_\pm = \frac{1}{4}(2 \mp M_\pm)(M_\pm \pm 1)^2 \pm \frac{3}{16}M_\pm(M_\pm^2 - 1)^2, if \ |M_\pm| \leq 1 \tag{38}$$

$$M_\pm = \frac{V_{n\pm}}{\bar{c}} \tag{39}$$

An arithmetic average of sound speed $\bar{c}$ is good for general case.

SFS

$$m = m_+ + m_- \tag{40}$$

$$m_\pm = \pm\frac{\rho_\pm c_\pm^2}{4\bar{c}}(M_\pm \pm 1)^2, \text{if} \quad |M_\pm| \le 1 \tag{41}$$

$$M_\pm = \frac{V_n}{\tilde{c}_\pm}\ ,\ \tilde{c}_\pm = \frac{c_\pm^2}{\bar{c}}\ ,\ c_\pm^2 = \gamma\frac{p_\pm}{\rho_\pm} \tag{42}$$

where $\bar{c}$ is arithmetic average of sound speed and pure upwind side value or 0 of $M_\pm$will be used for supersonic case.

AUSMDV

$$m = \rho_+\tilde{V}_+ + \rho_-\tilde{V}_- \tag{43}$$

$$\begin{aligned}\tilde{V}_{n\pm} &= \alpha_\pm(\pm\bar{c}\frac{(M_\pm \pm 1)^2}{4} - \frac{V_{n\pm} \pm |V_{n\pm}|}{2}) \\ &+\frac{V_{n\pm} \pm |V_{n\pm}|}{2}, if \quad |M_\pm| < 1\end{aligned} \tag{44}$$

$$\alpha_\pm = \frac{2(p/\rho)_\pm}{(p/\rho)_+ + (p/\rho)_-}, M_\pm = \frac{V_{n\pm}}{\bar{c}},\ \bar{c} = max(c_+, c_-) \tag{45}$$

Pure upwind side value or 0 will be used for supersonic case. The pressure correction term is introduced as,

$$p' = \frac{1}{2}(\frac{1}{2}+s)(\rho_+V_{n+} - \rho_-V_{n-} - |m|)(V_{n+} - V_{n-}) \tag{46}$$

$$s = \frac{1}{2}min(1, 10\frac{|p_+ - p_-|}{min(p_+, p_-)}) \tag{47}$$

SHUS See section 3.

NNFVS

$$m = m_+ + m_- \tag{48}$$

$$m_\pm = \rho_\pm\lambda_1^\pm + \frac{p_\pm}{\bar{c}^2}(-2\lambda_1^\pm + \lambda_2^\pm + \lambda_3^\pm) \tag{49}$$

with average sound velocity,modified Mach number and smoothed eigen values such as,

$$\bar{c} = \frac{c_+ + c_-}{2},\ \ \bar{M}_\pm = \frac{V_{n\pm}}{\bar{c}} \tag{50}$$

$$\lambda_1^\pm = \begin{cases} \frac{\bar{c}(\bar{M}\pm|\bar{M}|)}{2} & |\bar{M}| \ge \varepsilon \\ \frac{\bar{c}(\bar{M}\pm\varepsilon)^2}{2} & |\bar{M}| < \varepsilon \end{cases} \tag{51}$$

$$\varepsilon = 0.2\frac{min(p_+, p_-)}{max(p_+, p_-)} \tag{52}$$

$$\lambda_2^+ = \begin{cases} \frac{\bar{c}}{8}(\bar{M}_+ + 1)^2(\bar{M}_+^2 - 4\bar{M}_+ + 7) & |\bar{M}_+| < 1 \\ \frac{\bar{c}}{2}(\bar{M}_+ + 1 + |\bar{M}_+ + 1|) & |\bar{M}_+| \ge 1 \end{cases} \tag{53}$$

$$\lambda_3^+ = \begin{cases} \frac{\bar{c}}{8}(\bar{M}_+ - 1)(\bar{M}_+ + 1)^3 & |\bar{M}_+| < 1 \\ \frac{\bar{c}}{2}(\bar{M}_+ - 1 + |\bar{M}_+ - 1|) & |\bar{M}_+| \ge 1 \end{cases} \tag{54}$$

$$\lambda_2^-(\bar{M}) = \lambda_3^+(-\bar{M}) \tag{55}$$

$$\lambda_3^-(\bar{M}) = \lambda_2^+(-\bar{M}) \tag{56}$$

The use of one side upwind density is recommended if $|\bar{M}| < \varepsilon$. The pressure correction term is given by,

$$p' = \frac{1}{2}(m_+ - m_- - |m|)(V_{n+} - V_{n-}) \tag{57}$$

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