A New High-Accuracy Method–Finite Spectral Method

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Abstract

The conception of finite spectral method is given in this paper. The method of non-periodic Fourier transform and two finite spectral schemes are presented. Numerical tests of a wave propagation problem and a shock tube problem are performed.

1 Introduction

The advance in computational fluid dynamics enables almost all problems in fluid dynamics to be solved numerically. However, there are still many difficulties to make their results reliable. Two approaches to real results are considered: one is the construction of proper mathematical models, and another is the development of accurate numerical methods. For instance, the numerical research on turbulence just relies on these two keys. Spectral methods have taken an important role in numerical studies on turbulence due to their high-accuracy, while their applications have been restricted to simple geometry and uniform variations. Although the spectral element method [1] and the uniform high-order spectral methods [2] have been proposed, their flexibility is not satisfactory to deal with arbitrary geometry and their algorithm is complicated for programming. To review the progress of spectral methods, all efforts have been made to overcome the contradiction between the global properties of spectral methods and the local properties of flow fields[3]. Therefore, it becomes necessary to construct local spectral methods as flexible as finite difference methods and finite element methods.

Finite spectral method is a new conception defined as local spectral schemes based on non-periodic Fourier transform with finite order and finite points[4]. The non-periodic Fourier transform owns both non-periodicity and uniform mesh, which makes it possible to construct spectral schemes pointwise or cellwise. In this paper, we propose two finite spectral schemes and give their applications to the wave propagation problem and the shock tube problem.

2 Non-periodic Fourier Transform

To change the conventional point of view, we find that non-periodic functions are approximated by Fourier series in arbitrary sub-intervals of the whole periodic interval. Using a truncated Fourier series with periodicity on $[-1,\ 1]$ to interpolate $u(\xi)$, a non-periodic function on $[-l,\ l]$ $(-1 < -l < l < 1)$, we have the following relations at the $2N+1$ uniform discrete points $\xi_j = jl/N$ $(j = -N, -N+1, \ldots, N)$:

$$u(\xi_j) = \sum_{n=-N}^{N} \hat{u}_n e^{i\pi n\xi_j}. \tag{1}$$

The $2N+1$ unknown coefficients $\hat{u}_n$ are obtained by solving the above system.

3 Finite Spectral Method

3.1 Central Spectral Scheme

This is the simplest scheme in finite spectral method, which is similar to the central difference scheme. We introduce $\tilde{u}(\xi^i)$, a local Fourier series centered on x_i, to approximate a smooth function $u(x)$ as

$$\tilde{u}(\xi^i_j) = \sum_{n=-N}^{N} \hat{u}^i_n e^{i\pi n\xi^i_j}, \tag{2}$$

where ξ^i is the local coordinate given by

$$\xi^i = (x - x_i)\frac{l}{N\Delta x}, \tag{3}$$

Δx is the increment of x, l and N have the same definition with eq.(1). $\tilde{u}$ interpolates u at the stencil points

$$\tilde{u}(\xi^i_j) = u(x_{i+j}), \qquad (j = -N,\ -N+1, \ldots, N). \tag{4}$$

Thus the spatial derivatives of u near x_i can be expressed as

$$u^i_x(x) = \xi_x \tilde{u}_\xi(\xi^i) = \frac{i\pi l}{N\Delta x}\sum_{n=-N}^{N} n\hat{u}^i_n e^{i\pi n\xi^i}, \tag{5}$$

$$u^i_{xx}(x) = (\xi_x)^2 \tilde{u}_{\xi\xi}(\xi^i) = -\left(\frac{\pi l}{N\Delta x}\right)^2 \sum_{n=-N}^{N} n^2\hat{u}^i_n e^{i\pi n\xi^i}. \tag{6}$$

The higher order derivatives can be derived similarly. The above equations are reduced to the following simple forms at the center x_i

$$u^i_x(x_i) = -\frac{\pi l}{N\Delta x}\sum_{n=-N}^{N} n\hat{u}^i_{n,i}, \tag{7}$$

$$u^i_{xx}(x_i) = -\left(\frac{\pi l}{N\Delta x}\right)^2 \sum_{n=-N}^{N} n^2\hat{u}^i_{n,r}, \tag{8}$$

where $\hat{u}^i_{n,r}$ and $\hat{u}^i_{n,i}$ are the real part and the imaginary part of $\hat{u}^i_n$ respectively.

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3.2 Finite Spectral ENO Scheme

The ENO schemes can be divided into two processes: estimation of the smoothness and interpolation of the solution. Although there are some differences between them, the existing ENO schemes all use divided differences for the former and Newton interpolation for the latter[2]. Here we introduce a new scheme to the second process, that is, using non-periodic Fourier interpolation instead of Newton interpolation. The algorithm of the finite spectral ENO scheme is as follows:

1. Primitive function

$$H(x_{i+1/2}) = \Delta x \sum_{k=1}^{i} f(x_k). \tag{9}$$

2. Divided differences

$$\begin{aligned} & H[x_{i-1/2}, \ldots, x_{i+k-1/2}] & (10)\\ = & \{H[x_{i+1/2}, \ldots, x_{i+k-1/2}] - H[x_{i-1/2}, \ldots, x_{i+k-3/2}]\} \\ & /(x_{i+k-1/2} - x_{i-1/2}) \\ & (k = 1, 2, \ldots, 2N). \end{aligned}$$

3. Propagation speed. For the Euler equations, the characteristic speeds $\bar{c}_{i+1/2}$ $(\bar{u}_{i+1/2}, \bar{u}_{i+1/2} \pm \bar{a}_{i+1/2})$ are calculated by using the Roe average[5].

4. Smoothest stencil. Let $j_{\rm local}$ be the point on the local coordinate corresponding to the cell boundary point $i + 1/2$ on the global coordinate,
$i_{\rm min} = i, \quad j_{\rm local} = -N$
if $\bar{c}_{i+1/2} > 0$, then $i_{\rm min} = i_{\rm min} - 1, \quad j_{\rm local} = j_{\rm local} + 1$.
if $|H[x_{i_{\rm min}-1/2}, \ldots x_{i_{\rm min}+l-1/2}]| >$
$|H[x_{i_{\rm min}-3/2}, \ldots x_{i_{\rm min}+l-3/2}]|$,
then $i_{\rm min} = i_{\rm min} - 1, \quad j_{\rm local} = j_{\rm local} + 1$,
(l=2, 3, ..., 2N).

5. Error Limitation. Define that $R^{(l)}(x)$ be the lth error term of Newton interpolation of $f = dH/dx$, then

$$R^{(l)}(x) = H[x_{i_{\rm min}-1/2}, \ldots x_{i_{\rm min}+l-1/2}] \tag{11}$$

$$\prod_{i=i_{\rm min}, i\neq j+1}^{i_{\rm min}+l-1} (x - x_{i-1/2}). \tag{12}$$

Since $R^{(l)}(x_{j+1/2})$ is the approximation of the error, we shorten the stencil when $R^{(l)} > R^{(l-1)}$.

6. Non-periodic Fourier interpolation. The values of the primitive function are replaced to the local coordinate as

$$\tilde{H}(\xi_j^{i+1/2}) = H(x_{i_{\rm min}+1/2+N+j}), \quad (j = -N, -N+1, \ldots, N). \tag{13}$$

The non-periodic Fourier transform is done on the smoothest stencil with

$$\tilde{H}(\xi_j^{i+1/2}) = \sum_{n=-N}^{N} \hat{u}_n^{i+1/2} e^{i\pi n \xi_j^{i+1/2}}. \tag{14}$$

7. Reconstruction of flux

$$\bar{f}_{i+1/2} = \left.\frac{dH(x)}{dx}\right|_{i+1/2} = \frac{i\pi l}{N\Delta x} \sum_{n=-N}^{N} n \hat{u}_n^{i+1/2} e^{i\pi n \xi_{j_{\rm local}}^{i+1/2}}. \tag{15}$$

4 Results

The complex form of LU decomposition method is used for solving eq.(1). Once the LU decomposition matrix is obtained, it can be used repeatedly. The coefficients are calculated in $M(M+1)/2$ $(M = 2N+1)$ operations.

First we choose the wave equation

$$u_t + u_x = 0 \tag{16}$$

as a test problem. The central spectral scheme and the central difference scheme are employed for calculating the spatial derivative, and the Euler scheme is used for the time-integration. Figure 1 shows the exact solution and the numerical results after one period of propagation starting from the initial condition

$$u(x) = \sum_{n=0}^{Nt} \{\cos[2\pi n(x+0.5) + \sin[2\pi n(x+0.5)]\}. \tag{17}$$

We set $l = 0.5$, $N = 3$, $Nt = 8$, $\Delta t = 0.0001$ and the total grid number $Nx = 32$. The result of the central difference method departs from the exact solution considerably, and one peak is lost. In contrast the central spectral method gives good accuracy. This is because that the difference schemes are inherently accompanied by phase errors, i.e. the components with different frequencies propagate in different speeds, while the spectral schemes are characterized by no phase error occuring.

Next we apply the finite spectral ENO scheme to the Sod's standard shock tube problem. The one-dimensional Euler equations are written as

$$u_t + f_x = 0, \tag{18}$$

where

$$u = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad f = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ u(e+p) \end{bmatrix}. \tag{19}$$

The flux vector f is divided into three characteristic components as

$$\begin{aligned} f^1 &= \frac{\rho(u-a)}{2\gamma} \begin{bmatrix} 1 \\ u-a \\ h-ua \end{bmatrix}, & (20)\\ f^2 &= \frac{(\gamma-1)\rho u}{\gamma} \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix}, \\ f^3 &= \frac{\rho(u+a)}{2\gamma} \begin{bmatrix} 1 \\ u+a \\ h+ua \end{bmatrix}, \end{aligned}$$

to which the finite spectral ENO scheme is imposed. The time-integration is performed in the following form:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(\bar{f}_{i+\frac{1}{2}} - \bar{f}_{i-\frac{1}{2}}) = u_i^n - \frac{\Delta t}{\Delta x}\sum_{k=1}^{3}(\bar{f}^k_{i+\frac{1}{2}} - \bar{f}^k_{i-\frac{1}{2}}). \tag{21}$$

The initial conditions are $(\rho_L, u_L, P_L) = (1, 0, 1)$, $(\rho_R, u_R, P_R) = (0.125, 0, 0.01)$, and the CFL number, the non-periodic interval and the grid number are set to $CFL = 0.3$, $l = 0.02$, $Nx = 100$. Figure 2 and 3 illustrate respectively the density distribution of the shock tube problem without and with error limitation. The shock wave is captured sharply, and the effect of the error limitation is obvious. The comparison between the numerical result and the exact solution is excellent.

5 Concluding Remarks

Non-periodic Fourier transform enables us to construct spectral schemes pointwise or cellwise. The central spectral scheme and the finite spectral ENO scheme proposed in this paper are two applications of them. These local schemes keep the high-order accuracy of spectral method, and possess the flexibility which has not been existed in conventional ones. The results of the two examples implies that finite spectral method is attractive for solving both unsteady problems and discontinuous problems. Since finite spectral method is a most fundamental methodology, further developments are necessary to be done.

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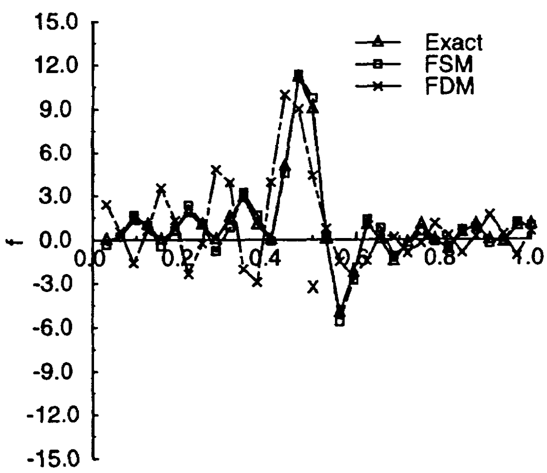

Figure 1 Comparison between exact solution and numerical solutions of wave equation by using 3rd-order central spectral scheme and 3rd-order central difference scheme.

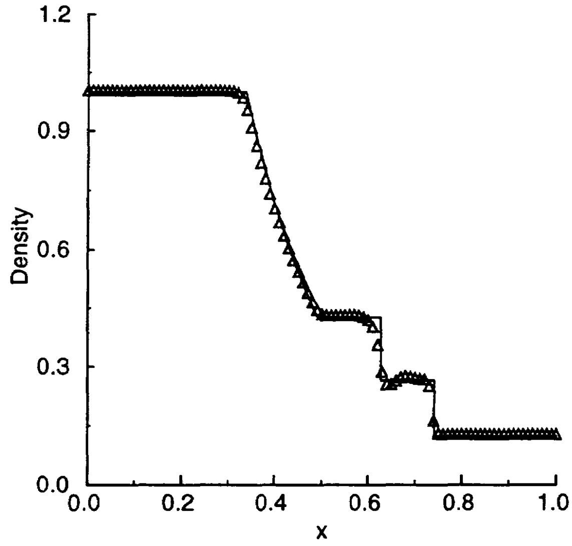

Figure 2 Density distribution of shock tube problem using finite spectral ENO scheme with five-point stencil and without error limitation.

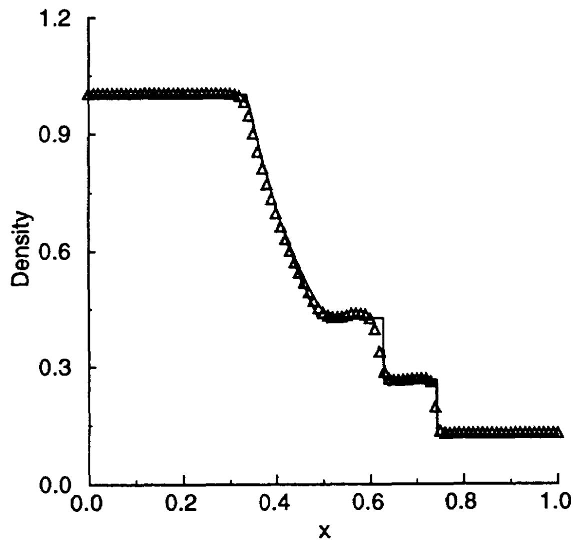

Figure 3 Density distribution of shock tube problem using finite spectral ENO scheme with five-point stencil and with error limitation.