Detail of the Leith Type Third-order of Upwind Scheme and Application to Viscous Incompressible Unbounded Flows for Re ≥ 1000*

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ABSTRACT

In this paper we give a detailed description of the Leith type three-order upwind finite difference schemes indispensable to compute numerical solutions of incompressible unbounded flows for *Re* ≥ 1,000. To test the effectiveness of this scheme, we define three problems: the backward-facing step, the blunt based body and the rectangular cylinder obstacle; give a detailed description of finite difference approximations of initial conditions, boundary conditions and sharp corners for each problem; and give a detailed description of finite difference approximations for the four investigated open boundary conditions. The results of numerical experiments showed that this scheme is stable and accurate as was expected, and also that there are large differences among the four open boundary conditions in flows in the domain near the open boundary, when the problem becomes more complicated.

Keywords: Leith type 3d upwind scheme, open boundary condition, Sommerfeld radiation condition, backward-facing step flow, blunt based body flow, rectangular cylinder obstacle flow

概　　要

安定でかつ精度の良いスキームの開発は、Re ≥ 1,000 に対する非圧縮・非有界流れの数値解を計算する上で必須なことである。この目的のためには、Leith タイプの３次精度の upwind 有限差分スキームが大変有望である。

本稿では、このスキームに対する詳細な記述を与えている。また、この開発したスキームを数値実験するために、Backward-facing step problem, Blunt based body problem 及び Rectangular cylinder obstacle problem の３つの問題を定義し、各々の問題に対する初期条件、境界条件及び鋭い角の有限差分近似表現の詳細な記述を与え、かつ研究に用いた４つの open boundary conditions に対する有限差分近似表現の詳細を与える。

実験結果は、このスキームが予想通りの安定かつ精度の良いものであることを示した。また、実験結果は問題が複雑化するに従い、open boundary の近傍に於いて、４つの open boundary conditions の間で、流れに無視し得ない差異を生じることを示した。従って、この現象は詳しく検討されなければならないが、これは本稿の目的ではなかった。詳細なる検討は稿を改めて発表する。

1. INTRODUCTION

Development of a stable and accurate scheme is indispensable to compute numerical solutions of incompressible unbounded flows for *Re* ≥ 1,000. For this purpose, the Leith type three-order scheme is very promising. This scheme was firstly proposed by Leonard (1979) in case of the one-dimensional scheme [1]. After that, Davis and Moore (1982) extended this scheme from one-dimensional to two-dimensional [2]. However they do not at all give detail of the two-dimensional Leith type third-order finite difference

* **Received 1 July 1998**
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approximations.

In this paper we give detailed description for the Leith type third-order upwind finite difference schemes in Section 3. Moreover to test effectiveness of this scheme, we define three problems of the backward-facing step, the blunt based body and the rectangular cylinder obstacle, give detailed description of finite difference approximations of initial conditions, boundary conditions and sharp corners for each problem in Section 4, and give detailed description of finite difference approximations for the four investigated open boundary conditions in Section 5. Then in Sections 6, 7 and 8 we discuss the results of numerical experiments for each problem briefly.

2. BASIC EQUATIONS

The two-dimensional viscous incompressible flow is governed by the following equations: The vorticity (ζ) transport equation in conservation form in case that *Re* is the Reynolds number is given by

$$\zeta_t + (u\zeta)_x + (\upsilon\zeta)_y = \frac{1}{Re}(\zeta_{xx} + \zeta_{yy}),\ \Gamma = \frac{1}{Re}, \quad (1)$$

and the Poisson equation for the stream-function (ψ) by

$$\psi_{xx} + \psi_{yy} = -\zeta, \quad (2)$$

where t is the time, x and y the axtial and normal coordinates, respectively. The subscripts t, x and y refer to partial derivatives with respect to t, x and y, respectively. The x and y components of the velocity (u, υ) are given by

$$\psi_y = u,\ \psi_x = -\upsilon. \quad (3)$$

3. FINITE DIFFERENCE SCHEMES

We derive the Leith type third-order upwind finite difference scheme for equation (1) on the assumption for simplicity as follows:

(a) Γ and u are constants.
(b) The spatial finite-difference approximations to be used contain fourth derivatives in their leading truncation errors.
(c) The generally small fourth- and higher-spatial-derivative terms are omitted.
(d) The fourth- and higher-spatial-derivative terms being multiplied by the generally small Γ are omitted.
(e) The generally small spatial-cross-derivatives are omitted.

Then we can derive the following equations:

$$\frac{\partial^2\zeta}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\zeta}{\partial t}\right) = \frac{\partial}{\partial t}\left(-u\frac{\partial\zeta}{\partial x} - \upsilon\frac{\partial\zeta}{\partial y} + \Gamma\frac{\partial^2\zeta}{\partial x^2} + \Gamma\frac{\partial^2\zeta}{\partial y^2}\right)$$
$$= -u\frac{\partial}{\partial x}\left(\frac{\partial\zeta}{\partial t}\right) - \upsilon\frac{\partial}{\partial y}\left(\frac{\partial\zeta}{\partial t}\right) + \Gamma\frac{\partial^2}{\partial x^2}\left(\frac{\partial\zeta}{\partial t}\right) + \Gamma\frac{\partial^2}{\partial y^2}\left(\frac{\partial\zeta}{\partial t}\right)$$
$$\simeq u^2\frac{\partial^2\zeta}{\partial x^2} + \upsilon^2\frac{\partial^2\zeta}{\partial y^2} - 2u\Gamma\frac{\partial^3\zeta}{\partial x^3} - 2\upsilon\Gamma\frac{\partial^3\zeta}{\partial y^3}, \quad (4)$$

$$\frac{\partial^3\zeta}{\partial t^3} = \frac{\partial}{\partial t}\left(\frac{\partial^2\zeta}{\partial t^2}\right) \simeq \frac{\partial}{\partial t}\left(u^2\frac{\partial^2\zeta}{\partial x^2} + \upsilon^2\frac{\partial^2\zeta}{\partial^2 y} - 2u\Gamma\frac{\partial^3\zeta}{\partial x^3} - 2\upsilon\Gamma\frac{\partial^3\zeta}{\partial y^3}\right)$$
$$= u^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial\zeta}{\partial t}\right) + \upsilon^2\frac{\partial^2}{\partial y^2}\left(\frac{\partial\zeta}{\partial t}\right)$$
$$= -u^3\frac{\partial^3\zeta}{\partial x^3} - \upsilon^3\frac{\partial^3\zeta}{\partial y^3}. \quad (5)$$

We expand $\zeta_{i,j}$ about time level n to obtain

$$\zeta^{n+1}_{i,j} = \zeta^n_{i,j} + \Delta t\frac{\partial\zeta^n_{i,j}}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2\zeta^n_{i,j}}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3\zeta^n_{i,j}}{\partial t^3}$$
$$+ O(\Delta t^4), \quad (6)$$

where t = nΔt. Here we insert (1), (4) and (5) into (6) to obtain

$$\zeta^{n+1}_{i,j} \cong \zeta^n_{i,j} + \Delta t\left(-u\frac{\partial\zeta^n_{i,j}}{\partial x} - \upsilon\frac{\partial\zeta^n_{i,j}}{\partial y} + \Gamma\frac{\partial^2\zeta^n_{i,j}}{\partial x^2} + \Gamma\frac{\partial^2\zeta^n_{i,j}}{\partial y^2}\right)$$
$$+ \frac{\Delta t^2}{2}\left(u^2\frac{\partial^2\zeta^n_{i,j}}{\partial x^2} + \upsilon^2\frac{\partial^2\zeta^n_{i,j}}{\partial y^2} - 2u\Gamma\frac{\partial^3\zeta^n_{i,j}}{\partial x^3} - 2\upsilon\Gamma\frac{\partial^3\zeta^n_{i,j}}{\partial y^3}\right)$$
$$+ \frac{\Delta t^3}{6}\left(-u^3\frac{\partial^3\zeta^n_{i,j}}{\partial x^3} - \upsilon^3\frac{\partial^3\zeta^n_{i,j}}{\partial y^3}\right). \quad (7)$$

Here spatial discretization about grid poit (i, j) is accomplished as follows: Firstly we fit the following quadratic to ζ across grid points (i + 1,j), (i, j), (i–1,j), (i, j+1) and (i, j–1),

$$\zeta = c_1 + c_2\zeta + c_3\zeta^2 + c_4\eta + c_5\eta^2 + c_6\zeta\eta \quad (8)$$

Then we obtain

$$c_1 = \zeta_{i,j},\ c_2 = \frac{1}{2\Delta x}(\zeta_{i+1,j} - \zeta_{i-1,j}),$$
$$c_3 = \frac{1}{2\Delta x^2}(\zeta_{i+1,j} - 2\zeta_{i,j} + \zeta_{i-1,j}), \quad (9)$$

$$c_4 = \frac{1}{2\Delta y}(\zeta_{i,j+1} - \zeta_{i,j-1}),$$

$$c_5 = \frac{1}{2\Delta y^2}(\zeta_{i,j+1} - 2\zeta_{i,j} + \zeta_{i,j-1}),\ c_6 = 0. \tag{10}$$

Next we integrate to obtain the average value of *zeta* within the (i, j)th mesh cell as follows:

$$\bar{\zeta} = \frac{1}{\Delta x \Delta y}\int_{-\Delta y/2}^{\Delta y/2}\int_{-\Delta x/2}^{\Delta x/2} \zeta d\xi d\eta = c_1 + \frac{c_3}{12}\Delta x^2 + \frac{c_5}{12}\Delta y^2$$
$$= \zeta_{i,j} + \frac{1}{24}(\zeta_{i+1,j} - 2\zeta_{i,j} + \zeta_{i-1,j}) + \frac{1}{24}(\zeta_{i,j+1} - 2\zeta_{i,j} + \zeta_{i,j-1}). \tag{11}$$

Hence this average value is determined at time levels n and $n+1$, yielding $\bar{\zeta}^n$ and $\bar{\zeta}^{n+1}$ as follws:

$$\bar{\zeta}^{n+1} - \bar{\zeta}^n = \zeta_{i,j}^{n+1} - \zeta_{i,j}^n + \frac{1}{24}(\zeta_{i+1,j}^{n+1} - 2\zeta_{i,j}^{n+1} + \zeta_{i-1,j}^{n+1})$$
$$+ \frac{1}{24}(\zeta_{i,j+1}^{n+1} - 2\zeta_{i,j}^{n+1} + \zeta_{i,j-1}^{n+1}) - \frac{1}{24}(\zeta_{i+1,j}^n - 2\zeta_{i,j}^n + \zeta_{i-1,j}^n)$$
$$- \frac{1}{24}(\zeta_{i,j+1}^n - 2\zeta_{i,j}^n + \zeta_{i,j-1}^n). \tag{12}$$

The last four terms in (1) can be interpreted as

$$\frac{1}{24}(\zeta_{i+1,j}^{n+1} - 2\zeta_{i,j}^{n+1} + \zeta_{i-1,j}^{n+1}) - \frac{1}{24}(\zeta_{i+1,j}^n - 2\zeta_{i,j}^n + \zeta_{i-1,j}^n)$$
$$= -\frac{\Delta t \Delta x^2}{24}\frac{\partial^2}{\partial x^2}\left(\frac{\partial \zeta_{i,j}}{\partial t}\right) = -\frac{u\Delta t\Delta x^2}{24}\frac{\partial^3 \zeta_{i,j}}{\partial x^3} + O(\Delta x^4), \tag{13}$$

$$\frac{1}{24}(\zeta_{i+1,j}^{n+1} - 2\zeta_{i,j}^{n+1} + \zeta_{i,j-1}^{n+1}) - \frac{1}{24}(\zeta_{i,j+1}^n - 2\zeta_{i,j}^n + \zeta_{i,j-1}^n)$$
$$= -\frac{\Delta t \Delta y^2}{24}\frac{\partial^2}{\partial y^2}\left(\frac{\partial \zeta_{i,j}}{\partial t}\right) = -\frac{u\Delta t\Delta y^2}{24}\frac{\partial^3 \zeta_{i,j}}{\partial y^3} + O(\Delta y^4), \tag{14}$$

Then we have the following relation from (7), (12), (13) and (14):

$$\zeta_{i,j}^{n+1} - \zeta_{i,j}^n - \frac{u\Delta t\Delta x^2}{24}\frac{\partial^3 \zeta_{i,j}^n}{\partial x^3} - \frac{u\Delta t\Delta y^2}{24}\frac{\partial^3 \zeta_{i,j}^n}{\partial y^3}$$
$$\cong \Delta t\left(-u\frac{\partial \zeta_{i,j}^n}{\partial x} - \upsilon\frac{\partial \zeta_{i,j}^n}{\partial y} + \Gamma\frac{\partial^2 \zeta_{i,j}^n}{\partial x^2} + \Gamma\frac{\partial^2 \zeta_{i,j}^n}{\partial y^2}\right)$$
$$+ \frac{\Delta t^2}{2}\left(u^2\frac{\partial^2 \zeta_{i,j}^n}{\partial x^2} + \upsilon^2\frac{\partial^2 \zeta_{i,j}^n}{\partial y^2} - 2u\Gamma\frac{\partial^3 \zeta_{i,j}^n}{\partial x^3} - 2\upsilon\Gamma\frac{\partial^3 \zeta_{i,j}^n}{\partial y^3}\right)$$
$$+ \frac{\Delta t^3}{6}\left(-u^3\frac{\partial^3 \zeta_{i,j}^n}{\partial x^3} - \upsilon^3\frac{\partial^3 \zeta_{i,j}^n}{\partial y^3}\right). \tag{15}$$

Therefore

$$\zeta_{i,j}^{n+1} \cong \zeta_{i,j}^n - u\Delta t\frac{\partial \zeta_{i,j}^n}{\partial x} - \upsilon\Delta t\frac{\partial \zeta_{i,j}^n}{\partial y}$$
$$+ \left(\Gamma\Delta t + \frac{u^2\Delta t^2}{2}\right)\frac{\partial^2 \zeta_{i,j}^n}{\partial x^2} + \left(\Gamma\Delta t + \frac{\upsilon^2\Delta t^2}{2}\right)\frac{\partial^2 \zeta_{i,j}^n}{\partial y^2}$$
$$+ \left(\frac{u\Delta t\Delta x^2}{24} - u\Gamma\Delta t^2 - \frac{u^3\Delta t^3}{6}\right)\frac{\partial^3 \zeta_{i,j}^n}{\partial x^3}$$
$$+ \left(\frac{\upsilon\Delta t\Delta y^2}{24} - \upsilon\Gamma\Delta t^2 - \frac{\upsilon^3\Delta t^3}{6}\right)\frac{\partial^3 \zeta_{i,j}^n}{\partial y^3}. \tag{16}$$

Here we discretize the spatial derivatives of the above equation as follows:

$$\frac{\partial \zeta_{i,j}}{\partial x} = \frac{1}{\Delta x}(\zeta_{i+1/2,j} - \zeta_{i-1/2,j})$$
$$= \frac{1}{2}(\zeta_{i+1,j} - \zeta_{i-1,j}) - \frac{1}{8}(\zeta_{i+1,j} - 3\zeta_{i,j} + 3\zeta_{i-1,j} - \zeta_{i-2,j}). \tag{17}$$

$$\frac{\partial \zeta_{i,j}}{\partial y} = \frac{1}{\Delta y}(\zeta_{i,j+1/2} - \zeta_{i,j-1/2})$$
$$= \frac{1}{2}(\zeta_{i,j+1} - \zeta_{i,j-1}) - \frac{1}{8}(\zeta_{i,j+1} - 3\zeta_{i,j} + 3\zeta_{i,j-1} - \zeta_{i,j-2}). \tag{18}$$

$$\frac{\partial^2 \zeta_{i,j}}{\partial x^2} = \frac{1}{\Delta x^2}(\zeta_{i+1,j} - 2\zeta_{i,j} + \zeta_{i-1,j}), \tag{19}$$

$$\frac{\partial^2 \zeta_{i,j}}{\partial y^2} = \frac{1}{\Delta y^2}(\zeta_{i,j+1} - 2\zeta_{i,j} + \zeta_{i,j-1}), \tag{20}$$

$$\frac{\partial^3 \zeta_{i,j}}{\partial x^3} = \frac{1}{\Delta x^3}(\zeta_{i+1,j} - 3\zeta_{i,j} + 3\zeta_{i-1,j} - \zeta_{i-2,j}), \tag{21}$$

$$\frac{\partial^3 \zeta_{i,j}}{\partial y^3} = \frac{1}{\Delta y^3}(\zeta_{i,j+1} - 3\zeta_{i,j} + 3\zeta_{i,j-1} - \zeta_{i,j-2}), \tag{22}$$

Therefore we finally obtain the two-dimensional Leith type third-order upwind finite difference schemes for equation (1) as follows:

$$\zeta_{i,j}^{n+1} = \zeta_{i,j}^n + WR + WL + WU + WD$$
$$+ \frac{1}{2}c_{i+1/2}^2(\zeta_{i+1,j} - \zeta_{i,j}) - \frac{1}{2}c_{i-1/2}^2(\zeta_{i,j} - \zeta_{i-1,j})$$
$$+ \frac{1}{2}c_{j+1/2}^2(\zeta_{i,j+1} - \zeta_{i,j}) - \frac{1}{2}c_{j-1/2}^2(\zeta_{i,j} - \zeta_{i,j-1})$$
$$+ \gamma_x(\zeta_{i+1,j} - 2\zeta_{i,j} + \zeta_{i-1,j})$$
$$+ \gamma_y(\zeta_{i,j+1} - 2\zeta_{i,j} + \zeta_{i,j-1}), \tag{23}$$

where

$$c_{i+1/2} = \frac{\Delta t u_{i+1/2,j}}{\Delta x},\ c_{i-1/2} = \frac{\Delta t u_{i-1/2,j}}{\Delta x},$$

$$c_{j+1/2} = \frac{\Delta t \upsilon_{i,j+1/2}}{\Delta y}, \tag{24}$$

$$c_{j-1/2} = \frac{\Delta t \upsilon_{i,j-1/2}}{\Delta y},\ \gamma_x = \frac{\Gamma \Delta t}{\Delta x^2},\ \gamma_y = \frac{\Gamma \Delta t}{\Delta y^2}, \tag{25}$$

$$u_{i+1/2,j} = \frac{1}{2}(u_{i+1,j} + u_{i,j}),$$

$$u_{i-1/2,j} = \frac{1}{2}(u_{i,j} + u_{i-1,j}), \tag{26}$$

$$\upsilon_{i,j+1/2} = \frac{1}{2}(\upsilon_{i,j+1} + \upsilon_{i,j}),$$

$$\upsilon_{i,j-1/2} = \frac{1}{2}(\upsilon_{i,j} + \upsilon_{i,j-1}), \tag{27}$$

if $u_{i+1/2,j} \geq 0$,

$$WR = -\frac{1}{2} c_{i+1/2}(\zeta_{i+1,j} + \zeta_{i,j}) + c_{i+1/2}\left(\frac{1}{6} - \gamma_x - \frac{1}{6}c^2_{i+1/2}\right) * (\zeta_{i+1,j} - 2\zeta_{i,j} + \zeta_{i-1,j}), \tag{28}$$

if $u_{i+1/2,j} \leq 0$,

$$WR = -\frac{1}{2} c_{i+1/2}(\zeta_{i+1,j} + \zeta_{i,j}) + c_{i+1/2}\left(\frac{1}{6} - \gamma_x - \frac{1}{6}c^2_{i+1/2}\right) * (\zeta_{i+2,j} - 2\zeta_{i+1,j} + \zeta_{i,j}), \tag{29}$$

if $u_{i-1/2,j} \geq 0$,

$$WL = \frac{1}{2} c_{i-1/2}(\zeta_{i,j} + \zeta_{i-1,j}) - c_{i-1/2}\left(\frac{1}{6} - \gamma_x - \frac{1}{6}c^2_{i-1/2}\right) * (\zeta_{i,j} - 2\zeta_{i-1,j} + \zeta_{i-2,j}), \tag{30}$$

if $u_{i-1/2,j} \leq 0$,

$$WL = \frac{1}{2} c_{i-1/2}(\zeta_{i,j} + \zeta_{i-1,j}) - c_{i-1/2}\left(\frac{1}{6} - \gamma_x - \frac{1}{6}c^2_{i-1/2}\right) * (\zeta_{i+1,j} - 2\zeta_{i,j} + \zeta_{i-1,j}), \tag{31}$$

if $\upsilon_{i,j+1/2} \geq 0$,

$$WU = -\frac{1}{2} c_{j+1/2}(\zeta_{i,j+1} + \zeta_{i,j}) + c_{j+1/2}\left(\frac{1}{6} - \gamma_y - \frac{1}{6}c^2_{j+1/2}\right) * (\zeta_{i,j+1} - 2\zeta_{i,j} + \zeta_{i,j-1}), \tag{32}$$

if $\upsilon_{i,j+1/2} \leq 0$,

$$WU = -\frac{1}{2} c_{j+1/2}(\zeta_{i,j+1} + \zeta_{i,j}) + c_{j+1/2}\left(\frac{1}{6} - \gamma_y - \frac{1}{6}c^2_{j+1/2}\right) * (\zeta_{i,j+2} - 2\zeta_{i,j+1} + \zeta_{i,j}), \tag{33}$$

if $\upsilon_{i,j-1/2} \geq 0$,

$$WD = \frac{1}{2} c_{j-1/2}(\zeta_{i,j} + \zeta_{i,j-1}) - c_{j-1/2}\left(\frac{1}{6} - \gamma_y - \frac{1}{6}c^2_{j-1/2}\right) * (\zeta_{i,j} - 2\zeta_{i,j-1} + \zeta_{i,j-2}), \tag{34}$$

if $\upsilon_{i,j-1/2} \leq 0$,

$$WD = \frac{1}{2} c_{j-1/2}(\zeta_{i,j} + \zeta_{i,j-1}) - c_{j-1/2}\left(\frac{1}{6} - \gamma_y - \frac{1}{6}c^2_{j-1/2}\right) * (\zeta_{i,j+1} - 2\zeta_{i,j} + \zeta_{i,j-1}), \tag{35}$$

Next the finite diference scheme for (2) are given by

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} = -\zeta_{i,j}. \tag{36}$$

Finally the finite diference scheme for (3) are given by

$$\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x} = u_{i,j},\ \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y} = -\upsilon_{i,j}. \tag{37}$$

4. INITIAL CONDITIONS, BOUNDARY CONDITIONS AND SHARP CORNERS

Figure 1 shows geometry definition of three test problems. In (A), B1, B2, B3 and B5 are the no-slip solid walls, B4 the inlet and B6 an open boundary. Coordinates of points 1 and 2 are (2JH, JH) and (IN, 2JH), respectly. In (B), B1, B2, B3, B4 and B6 are the no-slip solid walls, B5 the inlet and B7 an open boundary. Coordinates of points 1, 2 and 3 are (0, JH), (2JH, 2JH) and (IN, 3JH), respectly. In (C), B1, B2, B3, B4, B6 and B7 are the no-slip solid walls, B5 the inlet and B8 an open boundary. Coordinates of points 1, 2 and 3 are (8JH, 2JH), (9JH, 3JH) and (IN, 5JH), respectly. We note that truncation occurs at x = IN for unbounded flow. At the inlet, a uniform inlet u-velocity profile

$$u\,(y) = 1 \tag{38}$$

is chosen.

Initial conditions

Initial conditions for the backward-facing step problem are from (3) and (38) as follows:

$$\psi_{0,JH} = 0, \tag{39}$$
$$\psi_{0,JH+1} = 0.945h, \tag{40}$$
$$\psi_{0,j} = h + \psi_{0,j-1},\ JH + 2 \le j \le 2JH - 1, \tag{41}$$
$$\psi_{0,JH} = 0.945h + \psi_{0,2JH-1}, \tag{42}$$
$$\psi_{i,JH} = 0,\ 0 \le i \le 2JH, \tag{43}$$
$$\psi_{2JH,j} = 0,\ 0 \le j \le JH, \tag{44}$$
$$\psi_{i,0} = 0,\ 2JH \le i \le IN, \tag{45}$$
$$\psi_{i,2JH} = \psi_{0,2JH},\ 0 \le i \le IN, \tag{46}$$

where $h = 1/JH$, and the value of 0.945 decided by numerical experiments.

Initial conditions for the blunt based body problem are from (3) and (38) as follows:

$$\psi_{0,0} = 0, \tag{47}$$
$$\psi_{0,1} = 0.945h, \tag{48}$$
$$\psi_{0,j} = h + \psi_{0,j-1},\ 2 \le j \le JH - 1, \tag{49}$$
$$\psi_{0,JH} = 0.945h + \psi_{0,JH-1}, \tag{50}$$
$$\psi_{0,2JH} = \psi_{0,JH}, \tag{51}$$
$$\psi_{0,2JH+1} = 0.945h + \psi_{0,2JH}, \tag{52}$$
$$\psi_{0,j} = h + \psi_{0,j-1},\ 2JH + 2 \le j \le 3JH - 1, \tag{53}$$
$$\psi_{0,3JH} = 0.945h + \psi_{0,3JH-1}, \tag{54}$$
$$\psi_{i,0} = 0,\ 0 \le i \le IN, \tag{55}$$
$$\psi_{i,JH} = \psi_{0,JH},\ 0 \le i \le 2JH, \tag{56}$$
$$\psi_{i,2JH} = \psi_{0,JH},\ 0 \le i \le 2JH, \tag{57}$$
$$\psi_{2JH,j} = \psi_{0,JH},\ JH \le j \le 2JH, \tag{58}$$
$$\psi_{i,3JH} = \psi_{0,3JH},\ 0 \le i \le IN. \tag{59}$$

Initial conditions for the rectangular cylinder obstacle problem are from (3) and (38) as follows:

$$\psi_{0,0} = 0, \tag{60}$$
$$\psi_{0,1} = 0.945h, \tag{61}$$
$$\psi_{0,j} = h + \psi_{0,j-1},\ 0 \le j \le 5JH - 1, \tag{62}$$
$$\psi_{0,5JH} = 0.945h + \psi_{0,5JH-1}, \tag{63}$$
$$\psi_{i,0} = 0,\ 0 \le i \le IN, \tag{64}$$
$$\psi_{8JH,j} = \psi_{0,2.5JH},\ 2JH \le j \le 3JH, \tag{65}$$
$$\psi_{9JH,j} = \psi_{0,2.5JH},\ 2JH \le j \le 3JH, \tag{66}$$
$$\psi_{i,2JH} = \psi_{0,2.5JH},\ 8JH \le i \le 9JH, \tag{67}$$
$$\psi_{i,3JH} = \psi_{0,2.5JH},\ 8JH \le i \le 9JH, \tag{68}$$
$$\psi_{i,5JH} = \psi_{0,5JH},\ 0 \le i \le IH. \tag{69}$$

Boundary conditions

Since all the wall boundaries (WB) are no-slip, the following equations are obtained:

$$\zeta^{n+1}_{i,WB} = -\frac{2}{h^2}\,(\psi^{n}_{i,WB+1} - \psi^{n}_{i,WB}), \tag{70}$$

or

$$\zeta^{n+1}_{i,WB} = -\frac{2}{h^2}\,(\psi^{n}_{i,WB-1} - \psi^{n}_{i,WB}), \tag{71}$$

and

$$\zeta^{n+1}_{WB,j} = -\frac{2}{h^2}\,(\psi^{n}_{WB+1,j} - \psi^{n}_{WB,j}), \tag{72}$$

or

$$\zeta^{n+1}_{WB,j} = -\frac{2}{h^2}\,(\psi^{n}_{WB-1,j} - \psi^{n}_{WB,j}), \tag{73}$$

Also one phoney grid point becomes necessarily outside the boundaries (B). We extrapolate the values of these points by the method similar to equation (8) to obtain

$$\zeta^{n+1}_{i,\,B-1} = 3\zeta^{n}_{i,B} - 3\zeta^{n}_{i,B+1} + \zeta^{n}_{i,B+2}, \tag{74}$$

$$\zeta^{n+1}_{i,\,B+1} = 3\zeta^{n}_{i,B} - 3\zeta^{n}_{i,B-1} + \zeta^{n}_{i,B-2}, \tag{75}$$

and

$$\zeta_{B-1,j}^{n+1} = 3\zeta_{B,j}^{n} - 3\zeta_{B+1,j}^{n} - \zeta_{B+2,j}^{n}, \qquad (76)$$

or

$$\zeta_{B+1,j}^{n+1} = 3\zeta_{B,j}^{n} - 3\zeta_{B-1,j}^{n} - \zeta_{B-2,j}^{n}, \qquad (77)$$

Also at the inlet boundary the following equation is necessary:

$$\zeta_{0,j}^{n+1} = -\frac{1}{h^2}(\psi_{2,j}^{n} + \psi_{0,j}^{n} - 2\psi_{1,j}^{n}). \qquad (78)$$

Sharp corners

Sharp corners (SC) is the point (2JH, JH) in (A) of Figure 1, the two points (2JH, JH) and (2JH, 2JH) in (B), and the four points (8JH, 2JH), (8JH, 3JH), (9JH, 2JH) and (9JH, 3JH) in (C). We assume the values of ζ at the sharp corners to take two-values. Then we obtain the following equations:

For the upstream walls

$$\zeta_{SC,SC}^{n+1} = \frac{2}{h^2}\psi_{SC,SC+1}^{n}, \qquad (79)$$

or

$$\zeta_{SC,SC}^{n+1} = \frac{2}{h^2}\psi_{SC,SC-1}^{n}, \qquad (80)$$

and for the downstream walls

$$\zeta_{SC,SC}^{n+1} = \frac{2}{h^2}\psi_{SC+1,SC}^{n}, \qquad (81)$$

or

$$\zeta_{SC,SC}^{n+1} = \frac{2}{h^2}\psi_{SC-1,SC}^{n}, \qquad (82)$$

5. OPEN BOUNDARY CONDITIONS FOR UNBOUNDED FLOWS

OBC: no. 1

The following open boundary condition was firstly used by Thoman and Szewczyk (1966) [3]:

$$\frac{\partial \upsilon}{\partial x^2}\Big|_{OB} = -\frac{\partial^2 \psi}{\partial x}\Big|_{OB} = 0, \frac{\partial \zeta}{\partial x}\Big|_{OB} = 0. \qquad (83)$$

Hence

$$\psi_{OB,j}^{n+1} = 2\psi_{OB-1,j}^{n} - \psi_{OB-2,j}^{n}, \qquad (84)$$

$$\zeta_{OB,j}^{n+1} = \zeta_{OB-1,j}^{n}. \qquad (85)$$

OBC: no. 2

The following open boundary condition was proposed by Mehta and Lavan (1975) [4]:

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial (u\zeta)}{\partial x} - \frac{\partial (\upsilon\zeta)}{\partial y},$$
$$\frac{\partial \psi}{\partial x} = -\upsilon, \frac{\partial \upsilon}{\partial t} = -u\frac{\partial \upsilon}{\partial x} - \upsilon\frac{\partial \upsilon}{\partial y}$$
$$= -\left(u\zeta + \frac{1}{2}\frac{\partial (u^2 + \upsilon^2)}{\partial y} \right)$$
$$\text{at } OB, \qquad (86)$$

in case that at the open boundary the inertia terms are dominant.

Hence

$$\zeta_{OB,j}^{n+1} = \zeta_{OB,j}^{n} - \frac{\Delta t}{\Delta x}((u\zeta)_{OB,j}^{n} - (u\zeta)_{OB-1,j}^{n})$$
$$- \frac{\Delta t}{2\Delta y}((u\zeta)_{OB,j+1}^{n} - (u\zeta)_{OB,j-1}^{n}),$$
$$\text{for } u_{OB,j}^{n} \geq 0, \qquad (87)$$

$$\zeta_{OB,j}^{n+1} = \zeta_{OB,j}^{n} - \frac{\Delta t}{\Delta x}((u\zeta)_{OB+1,j}^{n} - (u\zeta)_{OB,j}^{n})$$
$$- \frac{\Delta t}{2\Delta y}((u\zeta)_{OB,j+1}^{n} - (u\zeta)_{OB,j-1}^{n}),$$
$$\text{for } u_{OB,j}^{n} \leq 0, \qquad (88)$$

$$\upsilon_{OB,j}^{n+1} = \upsilon_{OB,j}^{n} - \Delta t u_{OB,j}^{n}\zeta_{OB,j}^{n}$$
$$- \frac{\Delta t}{4\Delta x}((u_{OB,j+1}^{n})^2 - (u_{OB,j-1}^{n})^2 - (\upsilon_{OB,j-1}^{n})^2), \qquad (89)$$

$$\psi_{OB,j}^{n+1} = \psi_{OB-1,j}^{n} - \Delta x \upsilon_{OB,j}^{n+1}. \qquad (90)$$

OBC: no.3

The following open boundary condition is the Sommerfeld radiation condition firstly used by Orlanski (1976) [5]:

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \text{ at } OB, \qquad (91)$$

where ϕ is any variable, and c is the phase velocity of the waves. Orlanski proposed the following method which numerically evaluates the phase speed at the closet interior points every time: Using a leapfrog finite-difference representation, we have

$$\frac{\phi_{OB}^{n+1} - \phi_{OB}^{n-1}}{2\Delta t} = -\frac{c}{2\Delta x}\left(\phi_{OB}^{n+1} + \phi_{OB}^{n-1} - 2\phi_{OB-1}^{n}\right). \tag{92}$$

Hence the phase speed is numerically evaluated at the closet interior points from the above equation as follows:

$$c = -\frac{\Delta x}{\Delta t}\frac{\phi_{OB-1}^{n} - \phi_{OB-1}^{n-2}}{\phi_{OB-1}^{n} + \phi_{OB-1}^{n-2} - 2\phi_{OB-2}^{n-1}}. \tag{93}$$

From the above two equations, we can also obtain the boundary conditions $\{\phi_{OB}^{n+1}\}$ as follows:

$$\phi_{OB}^{n+1} = \frac{1 - c\Delta t/\Delta x}{1 + c\Delta t/\Delta x}\phi_{OB}^{n-1} + \frac{2c\Delta t/\Delta x}{1 + c\Delta t/\Delta x}\phi_{OB-1}^{n}. \tag{94}$$

OBC: no. 4

The following open boundary condition is the Sommerfeld radiation condition used by Bottaro (1990) and Kobayashi, Pereira and Sousa (1993) [6]:

$$\frac{\partial\phi}{\partial t} + c\frac{\partial\phi}{\partial x} = 0 \text{ at } OB, \tag{95}$$

where ϕ is any variable, and c is the phase velocity of the waves. Bottaro took the average streamwise speed in the channel as c, and Kobayashi et al. the mean channel velocity as c. The author proposes to take the uniform inlet velocity as c. Therefore, c = 1.

Hence

$$\phi_{OB,j}^{n+1} = \phi_{OB,j}^{n} - \frac{\Delta t}{\Delta x}\left(\phi_{OB,j}^{n} - \phi_{OB\text{-}1,j}^{n}\right). \tag{96}$$

Under the above preparation, we computed the numerical solution of flows to the three test problems for unbounded flows to obtain the results shown in the following Sections. As seen from the results, the Leith type third-order upwind scheme is stable and accurate. Also the results showed that there is severe difference among the above four open boundary conditions in flows in the domain near the open boundary, when the problem becomes more complicate. By the way all the computations conducted under the condition of JH = 40, uniform grid and hence $\Delta x = \Delta y = h = 1/JH$.

6. RESULTS ON THE BACKWARD-FACING STEP FLOWS

In this problem, we show results for Re = 1,000 and IN = 800. Figure 2 shows aspects of flows according to advance of every simulation time t = 5 for OBC: no. 1, Figure 3 for OBC: no. 2, Figure 4 for OBC: no. 3, and Figure 5 for OBC: no 4. (In Figure 2, we use symbols of DTHP and DTHN. DTHP expresses the pitch drawn on streamlines for $\psi \leq 0$, and DTHN expresses the pitch drawn on streamlines for $\psi \leq 0$. Also ω in this Figure is equal to ζ. Hence we use symbols in similar mean for this ω.) Also Figure 6 compares aspects of flows at t = 30, 35 and 40 among the four OBCs. In (A) of Figure 6, we can see that variation of flow does not yet arrive at the open boundary at t = 30. Meanwhile in (B), we can find that variation of flow already arrives at the open boundary at t = 35, and hence it is the same as well as at t = 40, as shown in (C). As seen from (B) and (C), there is severe difference among the four OBCs in flows in the domain near the open boundary. Hence we cannot at all conclude which of four OBCs gives the most excellent solution, only from these Figures. (We here note that computation of flow by OBC: no. 2 was broken off due to overflow in computation of ζ at $t > 35.1$.) Hence it is necessary to investigate in detail which of the four OBCs is the most excellent, but this is not the purpose in this paper. Detailed investigation will be given in the other paper.

7. RESULTS ON THE BLUNT BASED BODY FLOWS

This problem is more complicated than the previous one, and hence OBC: no. 2 could not bear practically for this problem due to this complexity. We discuss about OBCs for this problem similarly to the previous problem. We show here results for Re = 1,000 and IN = 800. Figure 7 shows aspects of flows according to advance of every simulation time t = 5 for OBC: no. 1, Figure 8 for OBC: no. 3, and Figure 9 for OBC: no. 4. Also Figure 10 compares aspects of flows at t = 30, 35 and 40 among three OBCs. In (A) of Figure 10, we can see that variation of flow does not yet arrive at the open boundary at t = 30. Meanwhile in (B), we can find that variation of flow already arrives at the open boundary at t = 35, and hence it is the same as well at t = 40, as shown in (C). As seen from (B) and (C), there is severe difference among three OBCs in flows in the domain near the open boundary. Hence we cannot at all conclude which of the three OBCs gives the most excellent solution, only from these Figures. Hence it is necessary to investigate in detail which of the three OBCs is the most excellent, but this is not the purpose in this paper. Detailed investigation will be given in the other paper.

8. RESULTS ON THE RECTANGULAR CYLINDER OBSTACLE FLOWS

This problem is the most complicated among the three problem, and hence even OBC: no. 1 could not bear practically for this problem due to this complexity. We discuss about OBCs for this problem similarly to the previous problem. We show here results for Re = 1,000 and IN = 1400. Figure 11 shows aspects of flows according to advance of every simulation time t = 5 for OBC: no. 3, and Figure 12 for OBC: no. 4. Also Figure 13 compares aspects of flows at t = 45, 55 and 65 between two OBCs. In (A) of Figure 13, we can see that variation of flow does not yet arrive at the open boundary at t = 45. Meanwhile in (B), we can find that variation of flow already arrives at the open boundary at t = 55, and hence it is the same as well at t = 65, as shown in (C). As seen from (B) and (C), there is severe difference between the two OBCs in flows in the domain near the open boundary. Hence we cannot at all conclude which of the two OBCs gives the more excellent solution, only from these Figures. Hence it is necessary to investigate in detail which of the two OBCs is the more excellent, but this is not the purpose in this paper. Detailed investigation will be given in the other paper. By the way we note that initial aspects of flows in this problem are like shown in Figure 14.

9. CONCLUSION

In this paper we gave detailed description for the Leith type third-order upwind finite difference schemes. Development of such a stable and accurate schemes is indispensable to compute numerical solutions of incompressible unbounded flows for Re ≥ 1,000. To test effectiveness of this scheme, we defined three problems, gave detailed description of finite difference approximations of initial conditions, boundary conditions and sharp corners for each problem, and gave detailed description of finite difference approximations for the four investigated open boundary conditions.

The results showed that this scheme is stable and accurate as was expected. Also the results showed that there is severe difference among the four open boundary conditions in flows in the domain near the open boundary, when the problem becomes more complicated. Hence it is necessary to investigate in detail which of the four open boundary conditions is the most excellent, but this is not the purpose in this paper. Detailed investigation will be given in the other paper.

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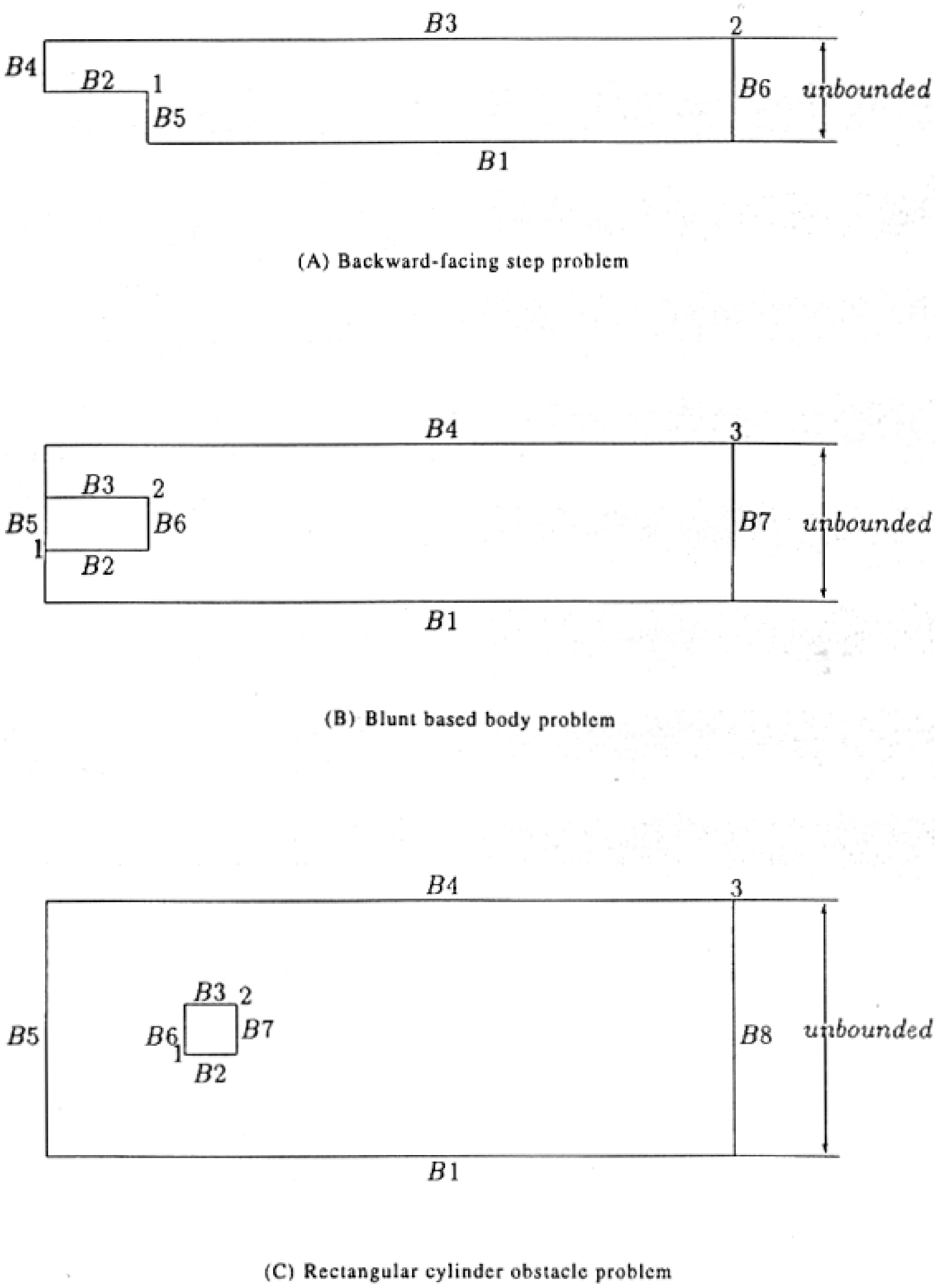


Fig. 1 Geometry definition of three test problems

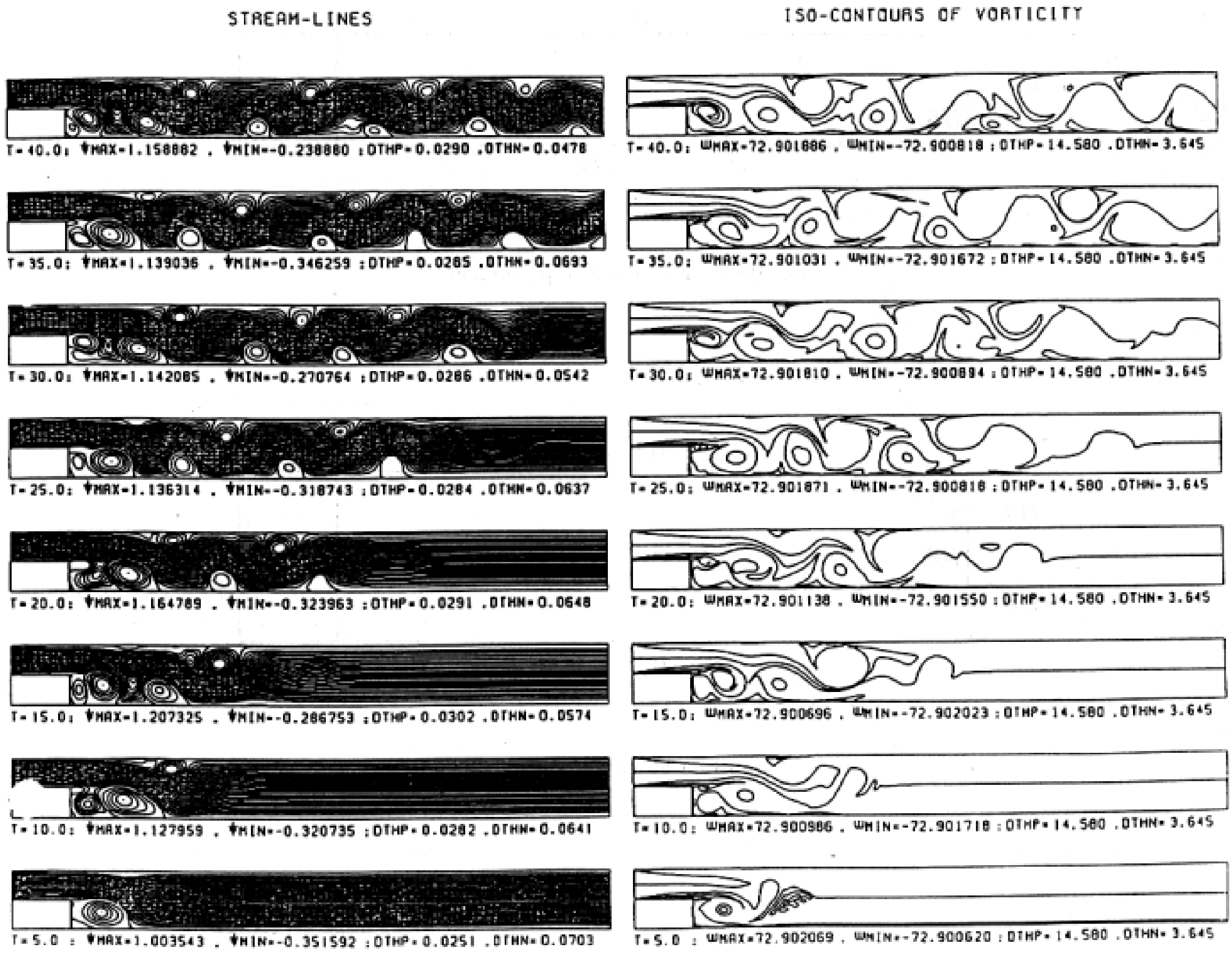


Fig. 2 Difference of flows for open boundary: IN = 800 (Re = 1,000, OBC: no. 1)

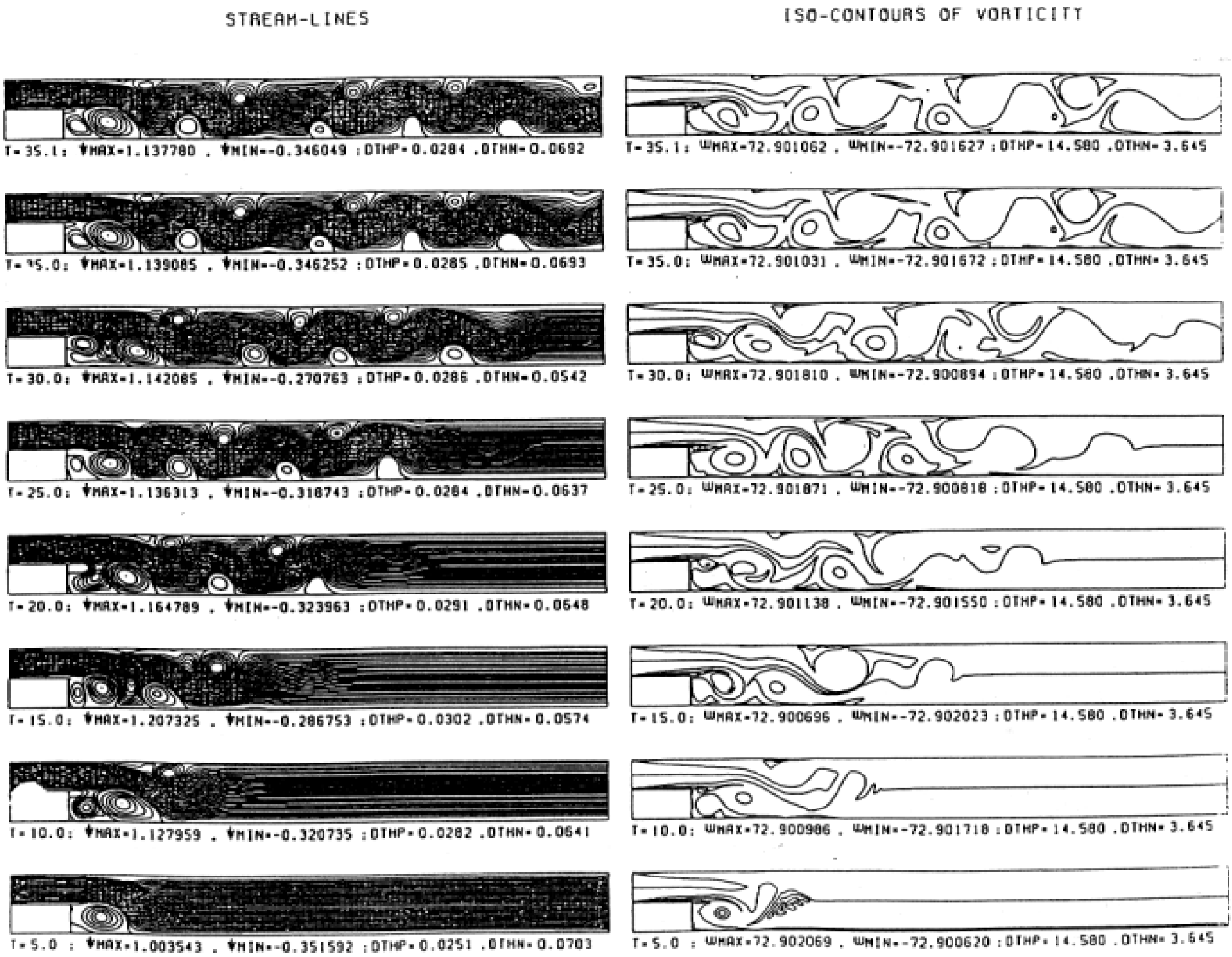


Fig. 3 Difference of flows for open boundary: IN = 800 (Re = 1,000, OBC: no. 2)

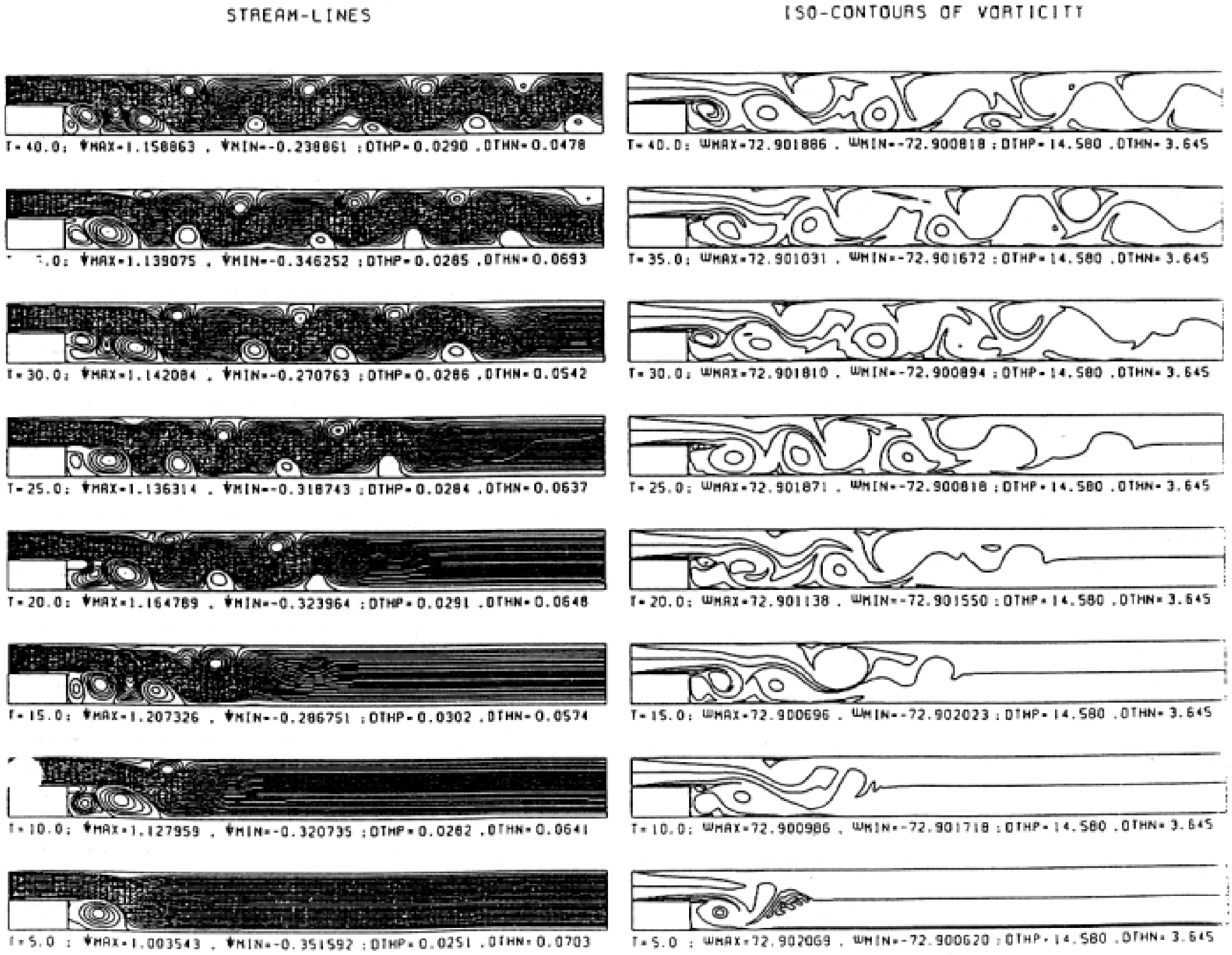


Fig. 4 Difference of flows for open boundary: IN = 800 (Re = 1,000, OBC: no. 3)

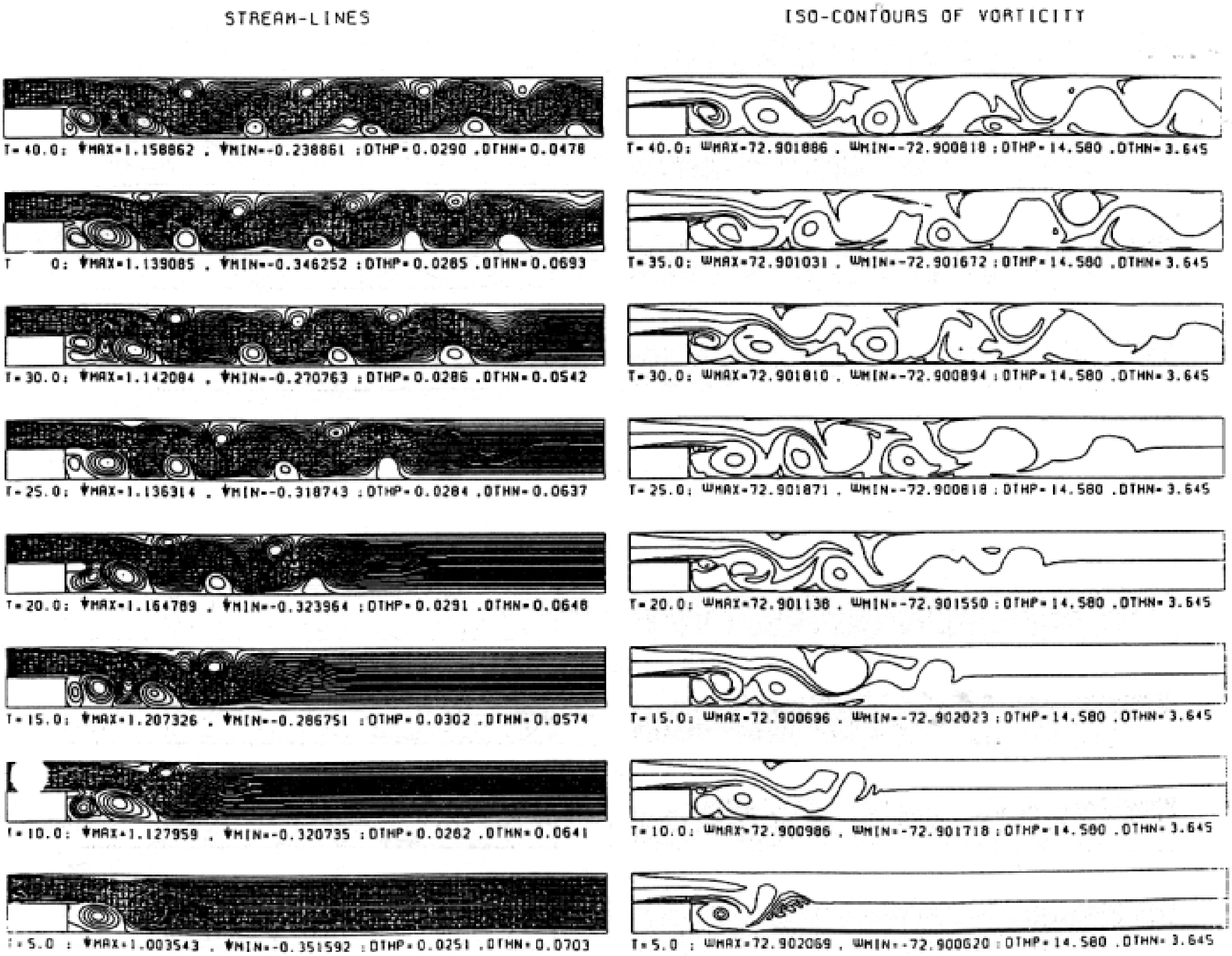
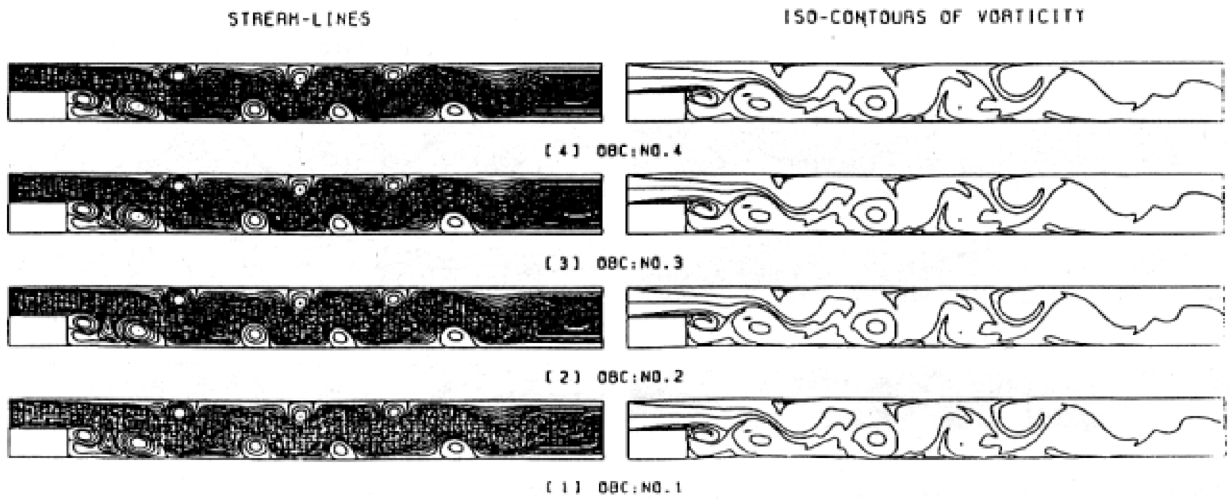
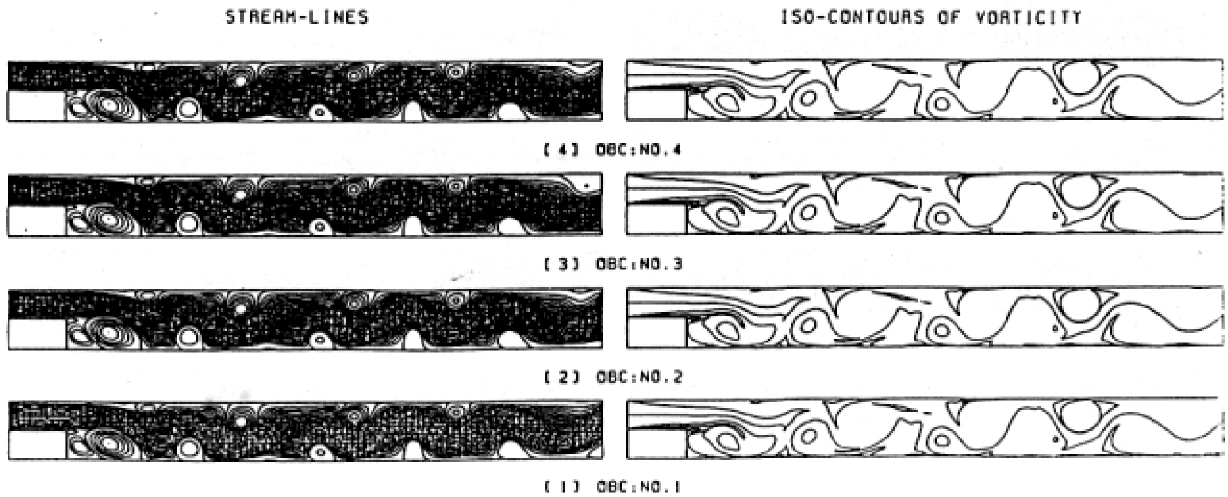


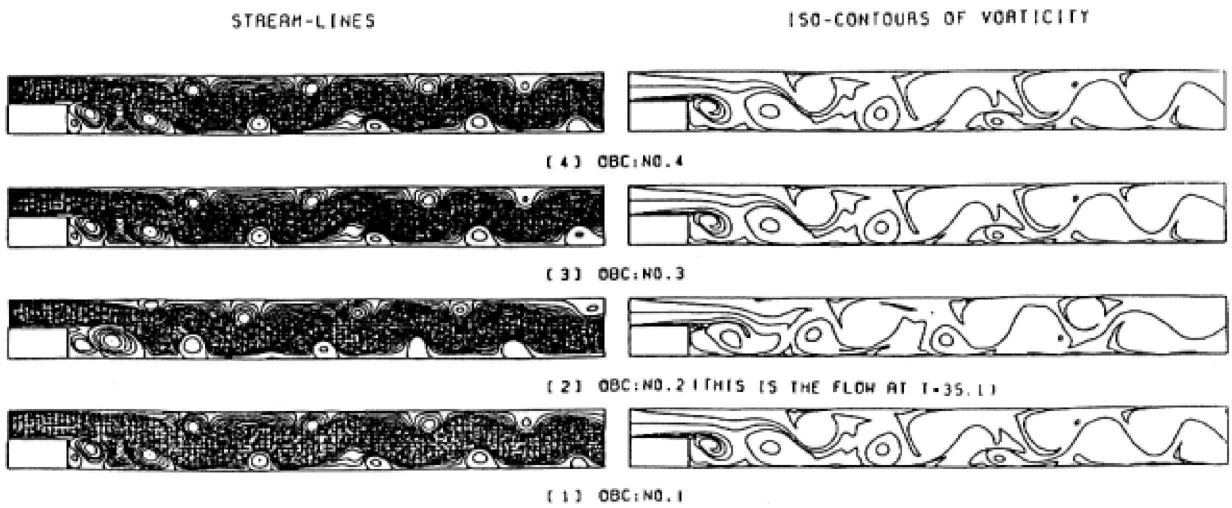
Fig. 5 Difference of flows for open boundary: IN = 800 (Re = 1,000, OBC: no. 4)



(A) The aspect of flow at t = 30



(B) The aspect of flow at t = 35



(C) The aspect of flow at t = 40

Fig. 6 Difference of flows among four OBCs (Re = 1,000, IN = 800)

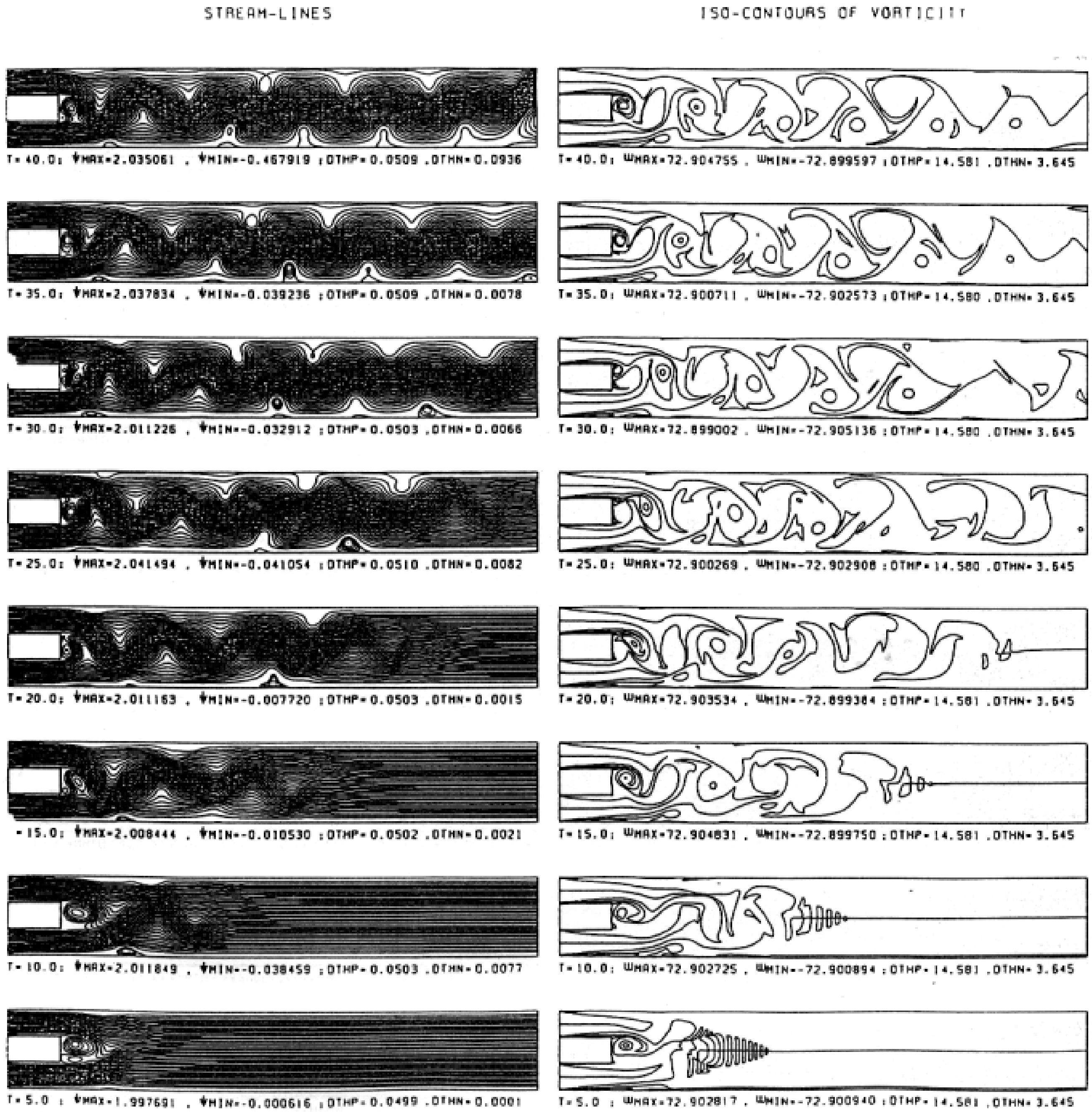


Fig. 7 Difference of flows for open boundary: IN = 800 (Re = 1,000, OBC: no. 1)

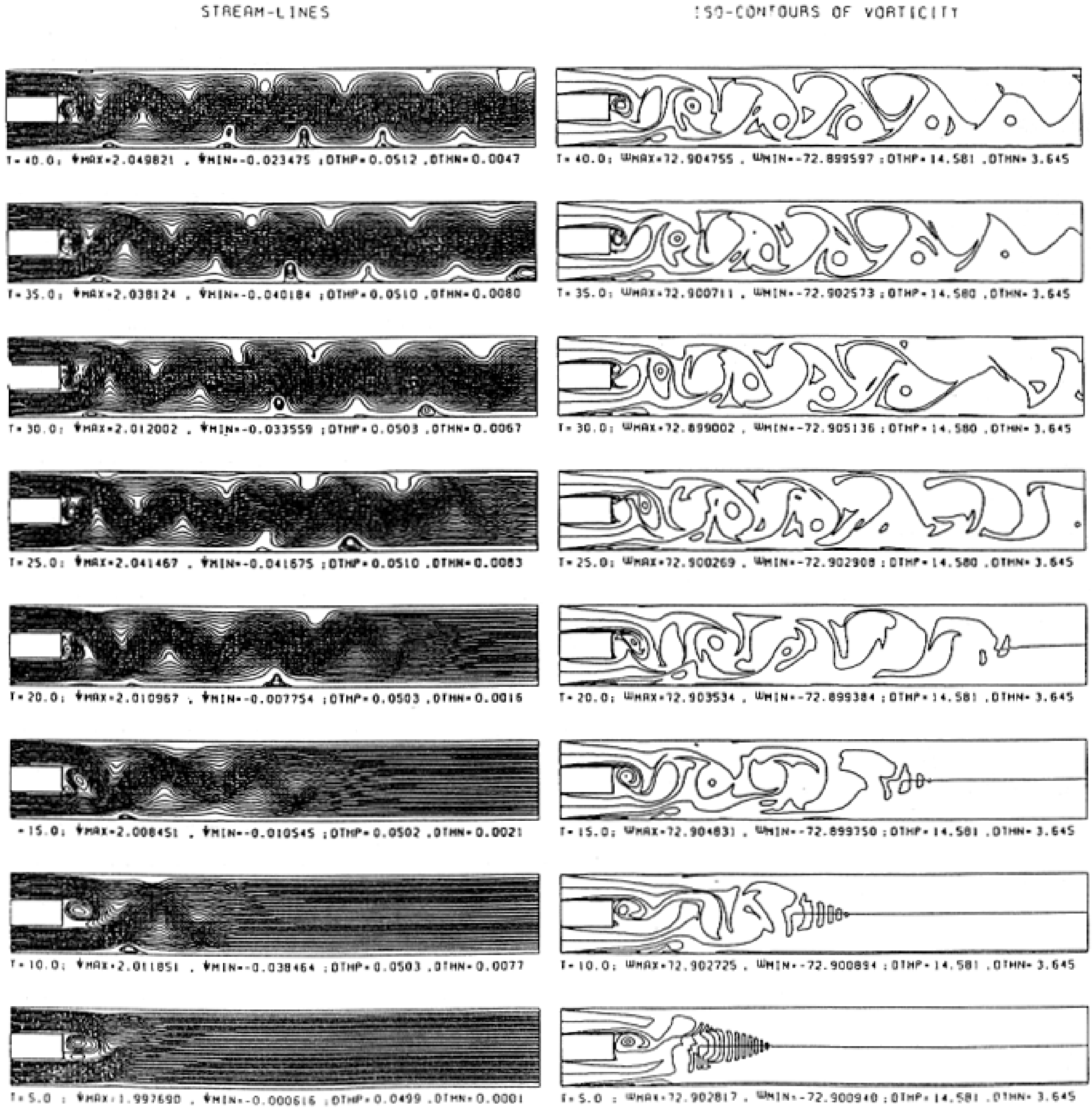


Fig. 8 Difference of flows for open boundary: IN = 800 (Re = 1,000, OBC: no. 3)

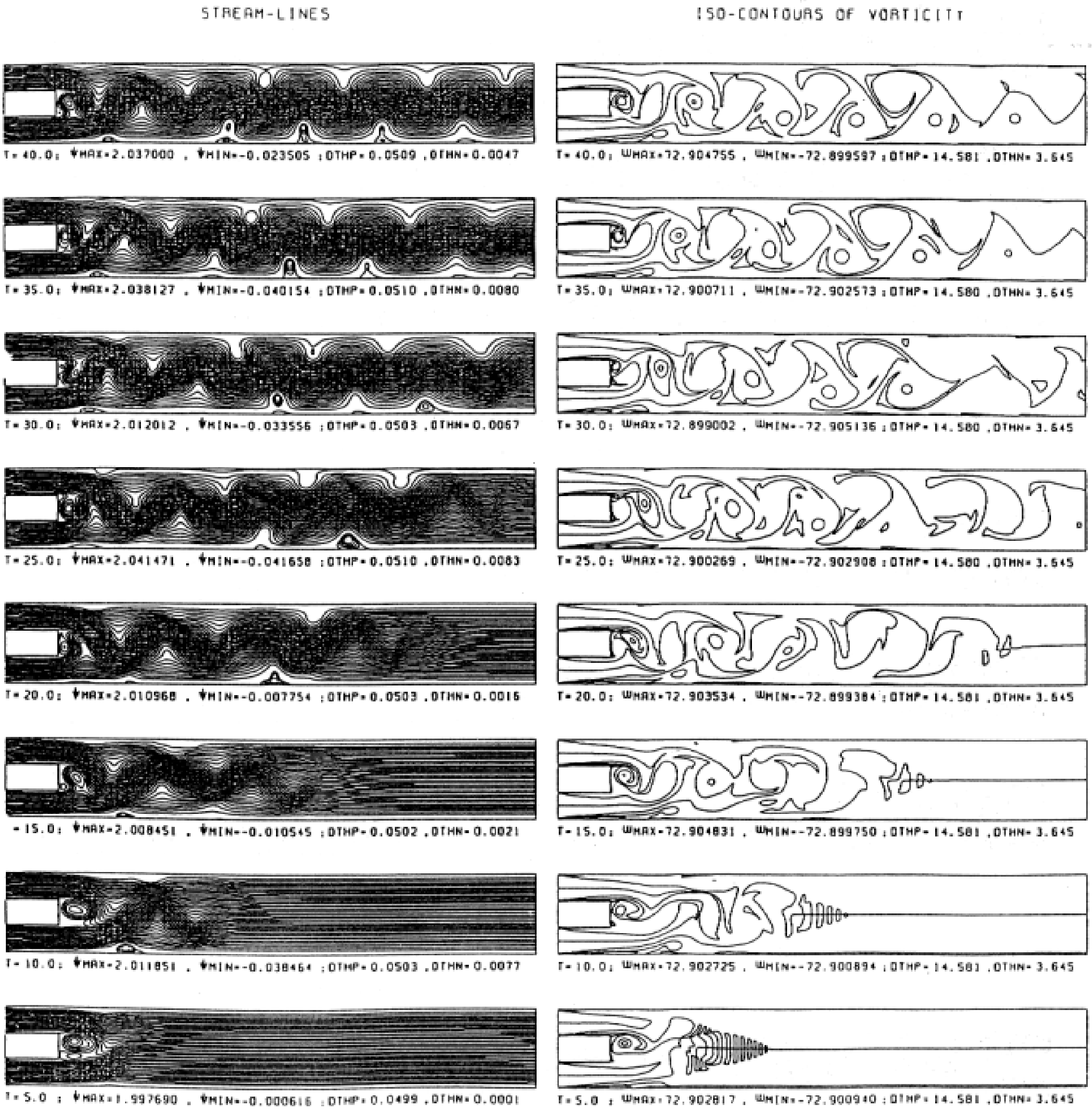
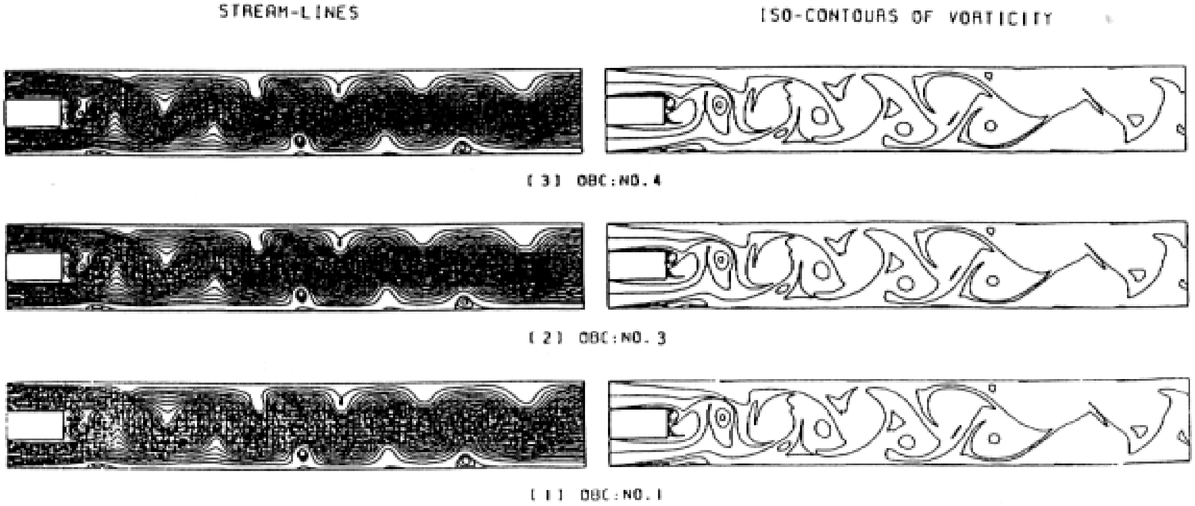
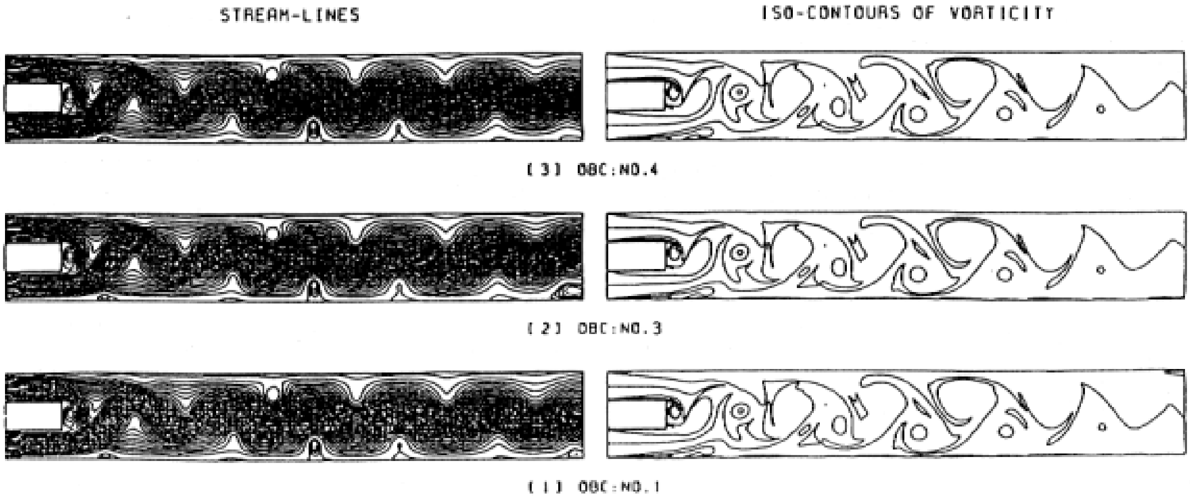


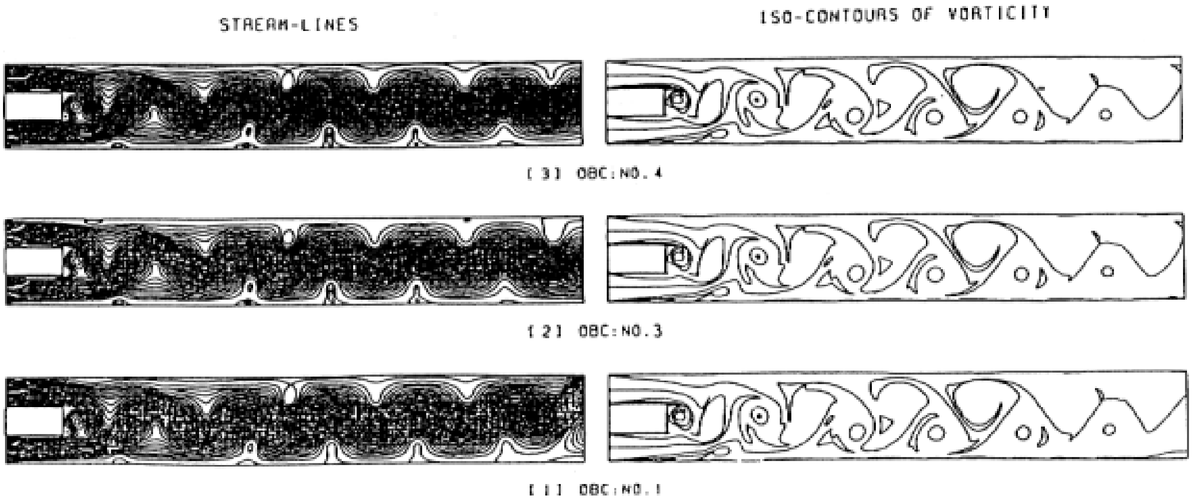
Fig. 9 Difference of flows for open boundary: IN = 800 (Re = 1,000, OBC: no. 4)



(A) The aspect of flow at t = 30



(B) The aspect of flow at t = 35



(C) The aspect of flow at t = 40

Fig. 10 Difference of flows among three OBCs (Re = 1,000, IN = 800)

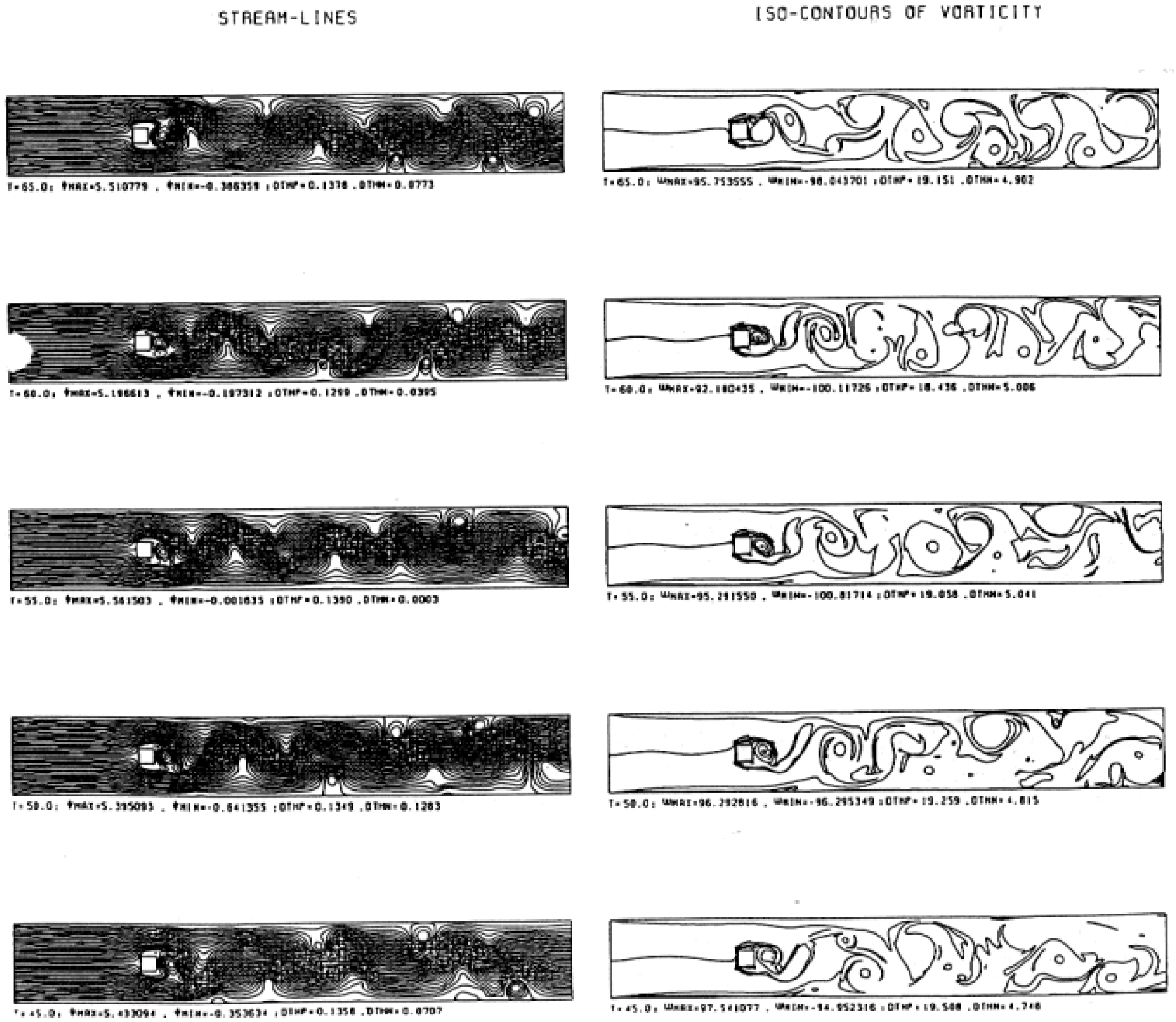


Fig. 11 Difference of flows for open boundary: IN = 1400 (Re = 1,000, OBC: no. 3)

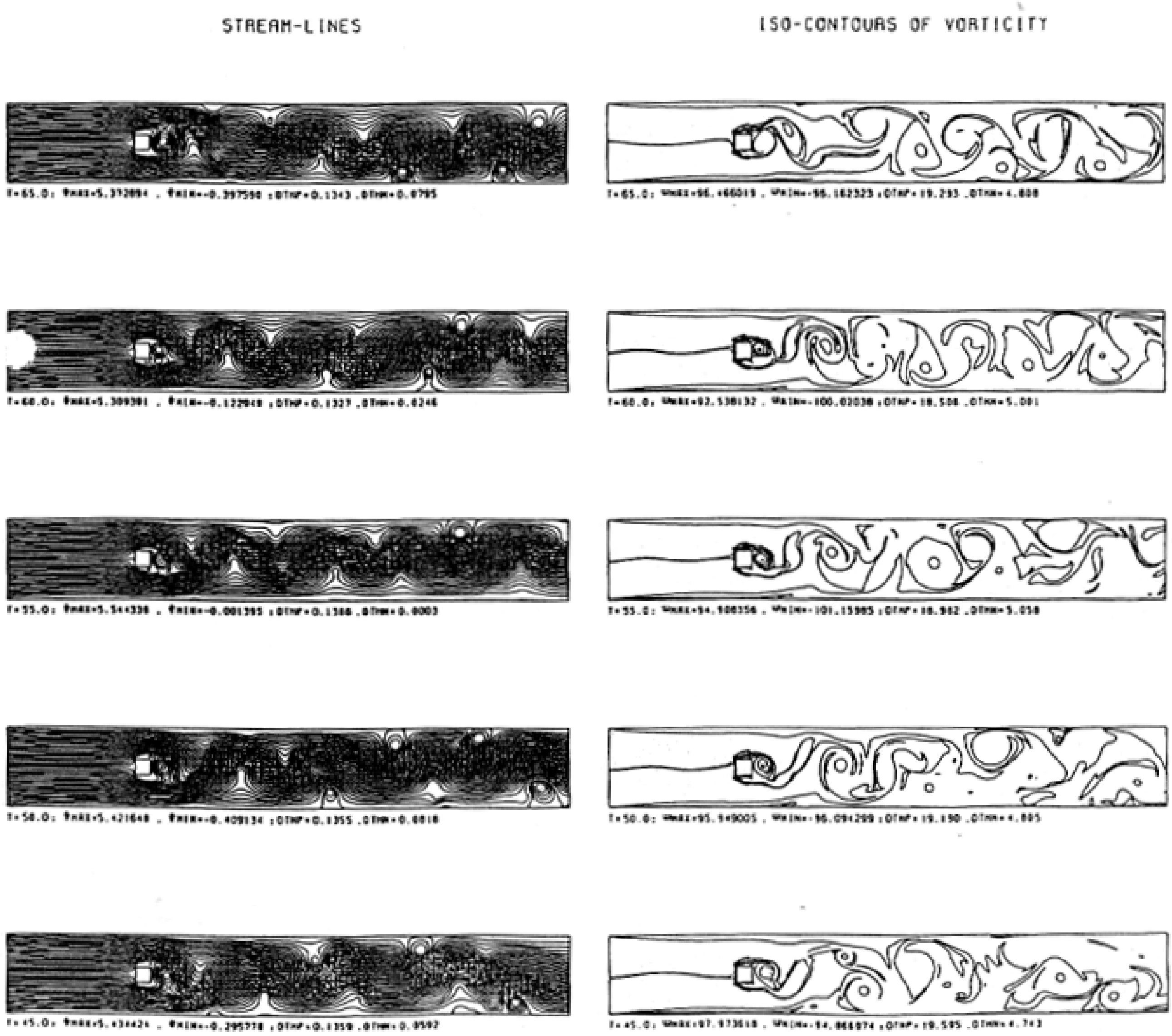
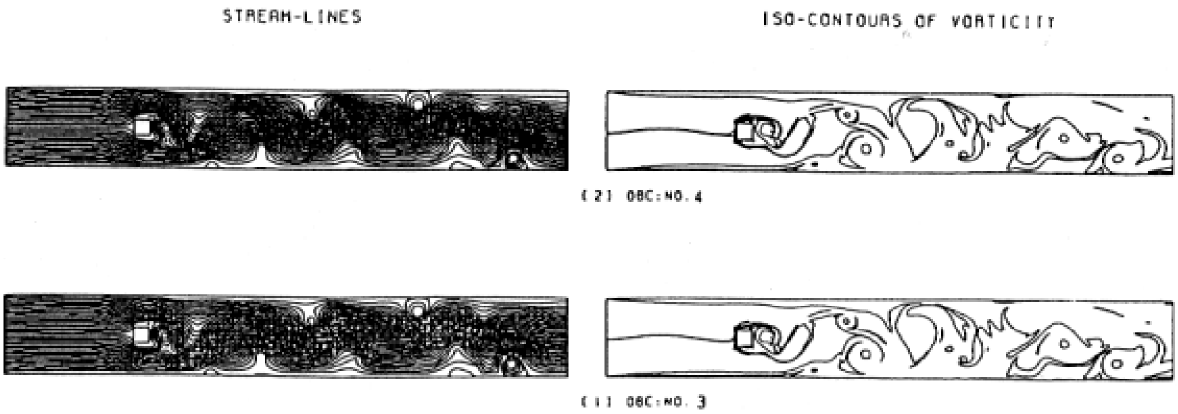
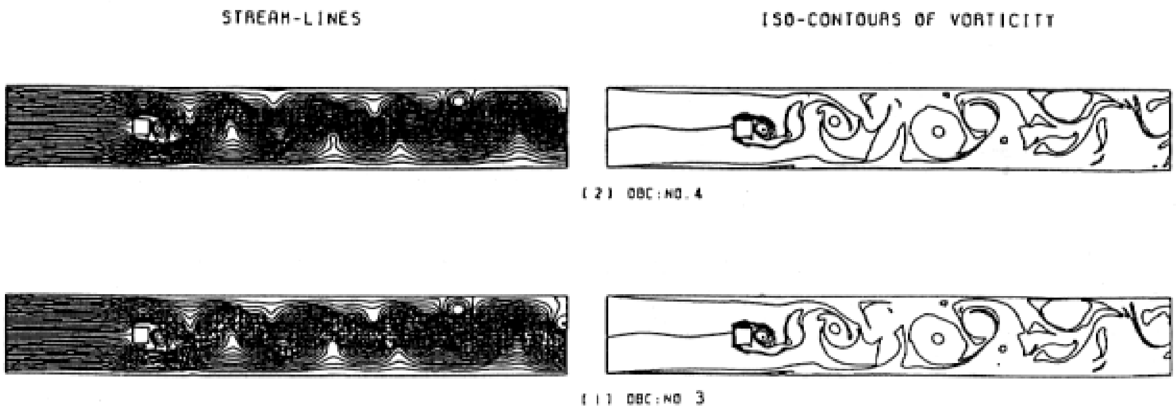


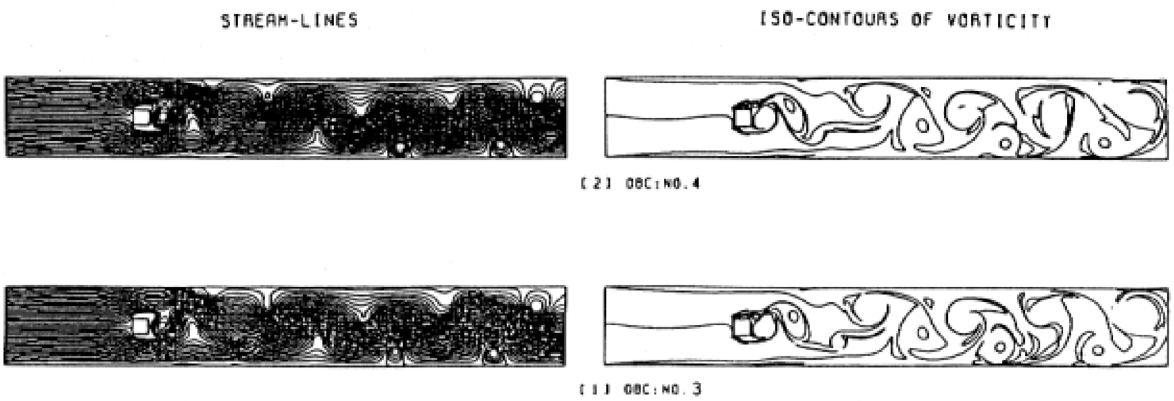
Fig. 12 Difference of flows for open boundary: IN = 1400 (Re = 1,000, OBC: no. 4)



(A) Aspects of flow at t = 45



(B) Aspects of flow at t = 55



(C) Aspects of flow at t = 65

Fig. 13 Difference of flows between two OBCs (Re = 1,000, IN = 1400)

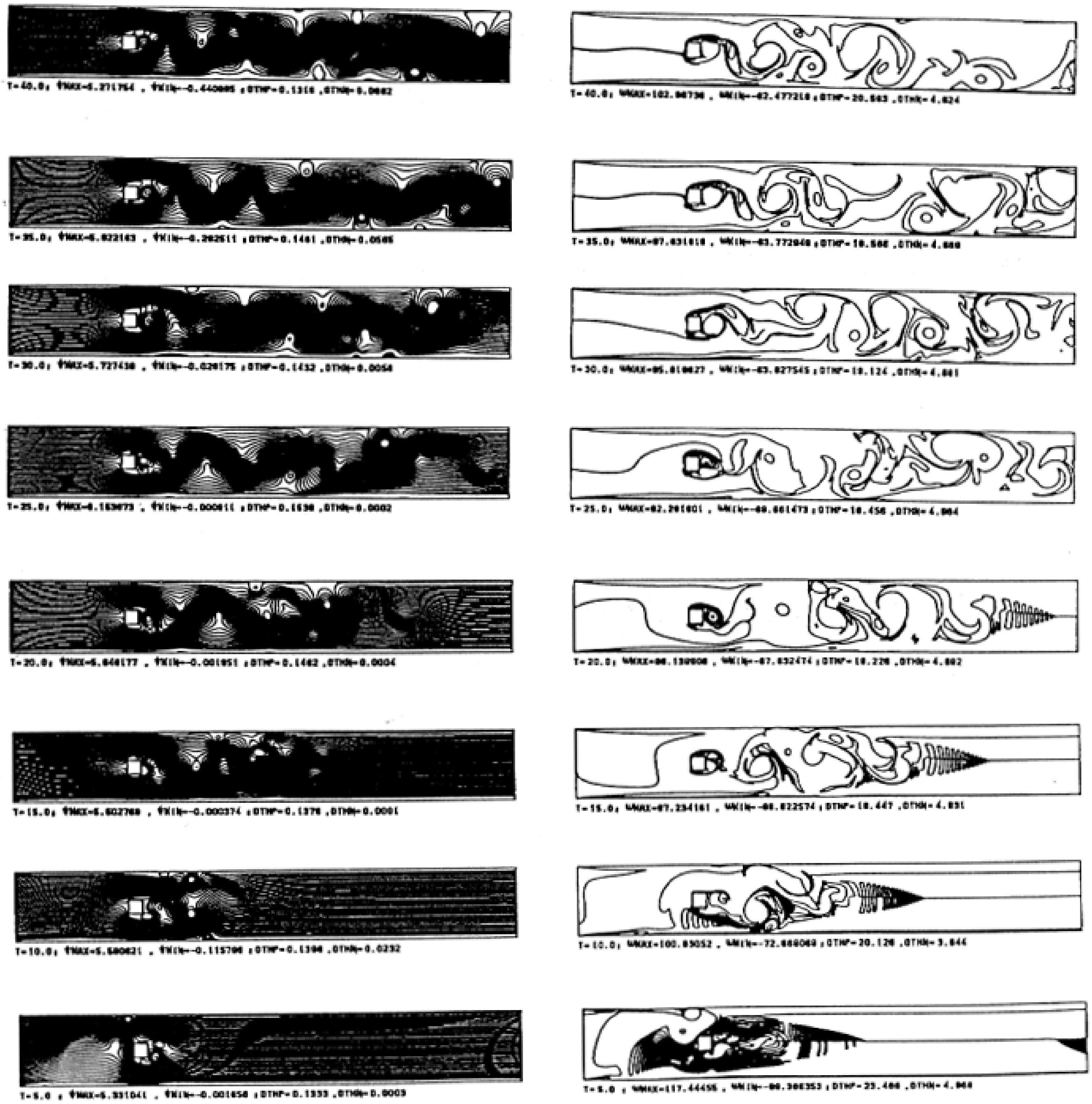


Fig. 14 Initial aspects of flows for open boundary: IN = 1400 (Re = 1,000, OBC: no. 4)