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A Method for the Calculation of Lifting Potential Flow Problems (1)

Theoretical Basis

Masao EBIHARA

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A Method for the Calculation of Lifting Potential Flow Problems*

I. Theoretical Basis

By

Masao EBIHARA**

SUMMARY

A formulation of lifting potential flow problems is worked out in terms of a doublet distribution over the body surface and the trailing vortex sheet.

In the course of analysis, it is shown that the velocity field due to a surface distribution of doublets is equivalent to that due to a surface distribution of vortices. This fact is utilized to derive a non-singular expression of surface derivatives of potential due to a doublet distribution.

In view of the significance of the Kutta's condition in controlling the lifting flow field, the behaviour of the potential and its derivatives is examined in the neighbourhood of the trailing-edge of a wing. Conditions on the strength of doublets are thus obtained with which the flow velocity remains finite at the trailing-edge. These conditions are incorporated in the final formulation of the lifting potential flow field.

1. INTRODUCTION

Calculation of the exact solutions of the potential flow field around lifting wing-fuselage configurations has been one of the primary targets of the aerodynamic research. Restricting the discussion to cases where the incompressible fluid is concerned, the determination of the flow field is mathematically expressed as a boundary value problem of a harmonic function representing the potential of the flow field.

The solution of this boundary value problem being intractable to analytical treatment, a variety of approximations has been imposed to simplify the problem thus rendering it amenable to analysis. The historical development of the lifting wing theory can be seen as a process of step-by-step removal of these approximations, beginning at Lanchester's single vortex line representation (ref. 1, Chap. VIII) and reaching to the sophistication of Küchemann, Weber et al. (ref. 2) On the other hand, numerical approaches to the problem incited by the work of Multhopp (ref. 3) have also been continuously improved in accuracy owing much to the continual advance in computing facilities. (e.g. refs. 4 & 5) These analytical and numerical approaches, however, suffer the essential limitation that they are seeking for the solution of the linearized, i.e. approximated, boundary value problem.

As will be expatiated upon in the main body of this paper, the original boundary value problem can be converted into a system of integral equations by means of superposition of the fundamental solutions of the Laplace equation. The advent of electronic digital computers has facilitated the solution of these integral equations if only within the scope of a numerical approximation to the exact solution of the original system. A review of such numerical methods is found elsewhere. (ref. 6) A typical example of them is demonstrated in the work of Hess & Smith (refs. 7 & 8) where the potential flow problem around non-lifting bodies is treated. In their method the bodies are represented by distributions of sources over their surfaces the strength of which is to be determined from the boundary conditions. Although the capability of this procedure is amply demonstrated by a host of numerical examples, a lethal limitation about the method is that it is unable to encompass the cases where a lifting wing is involved as the source distribution cannot represent a trailing vortex sheet which is inherent in such situations.

Let us expound this point a little more fully. A region of rotational flow field exists downstream of a lifting wing embedded in the surrounding irrotational field in which the streamwise vorticity is prevalent. (ref. 1, Chap. III) The trailing vortex sheet

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** The Second Aerodynamics Division.

is thought to be a mathematical idealization of this region. Since this region plays a decisive role in determining the character of the entire flow field, the trailing vortex sheet is indispensable in the formulation of lifting potential flow problem. The mathematical implication of the trailing voltex sheet is as follows. A lifting wing is inevitably accompanied by non-zero circulation around its sections. The circulation by itself makes the flow field multi-valued with respect to the velocity potential. The trailing vortex sheet provides a boundary across which a jump takes place in the potential to render it single-valued. In other words, the sheet, added to the wing, reduces the flow field to a simply-connected region. It is in this context that the surface distribution of sources cannot replace a trailing vortex sheet because the potential due to it is continuous across the surface and hence no jumps take place.

An attempt has been made by Rubbert & Saaris (ref. 9) to extend the method of Hess & Smith so as to cope with lifting problems by supplementing the source distribution with a system of doublets distributed over the trailing vortex sheet. For the sake of computational expediency they have to continue this doublet sheet across the trailing-edge into the interior of wing so that the doublet strength is continuous across the trailing-edge. There are no restrictions otherwise upon the distribution of doublets inside the wing and one can specify it arbitrarily so as to facilitate the numerical computation. This indeterminacy is one of the unsatisfactory features of this method, if only from the aesthetical point of view. Another unsatisfactory point is that the consideration on the Kutta's condition at the trailing-edge is insufficient, which bears some relation to the first point. The third of the undesirable features is that the shape of the trailing vortex sheet is specified prior to the calculation, which is in fact to be determined as a part of the solution.

Theoretical prediction of unsteady flow field around a lifting wing executing a time-dependent motion is an example of cases where it is of primary importance to account for the actual configuration of the trailing vortex sheet. Djojodihardjo & Widnall have proposed a scheme (ref. 10) in which the location of the trailing vortex sheet as well as the strength of singularities is to be calculated as a part of the solution of the entire problem. In this scheme the potential is given in terms of doublet distributions over the solid surfaces as well as over the trailing vortex sheet.

One difficulty inherent in a surface distribution of doublets is that the evaluation of the surface derivatives of the potential becomes much more complicated compared with the case of the source distribution owing to the fact that the kernels of the doublet integrals are by one degree higher in singularity than those of the source integrals. Djojodihardjo and Widnall do not seem to have succeeded in disposing of this difficulty whence the applications are confined to simplest cases. In their scheme again the Kutta's condition is treated only insufficiently, which has motivated the present work.

The three methods so far referred to are all aiming at the exact solution of the potential flow problem. The method developed by Woodward et al. (ref. 11 & 12) may be deemed to be situated somewhere between these 'exact' methods and the conventional lifting surface theories. In this method the effects of isolated fuselage and wing-thickness are first calculated employing the linearized-theory approach. The effects of wing camber and incidence, and of interference between the fuselage and the wing in combination are then accounted for by distributing 'constant pressure singularities' upon their surfaces. This 'pressure singularity' is nothing more than a plane vortex sheets which has been widely used in the linearized wing theories. The vortex distribution assumed in this method is of approximate nature in the sense that only a component of the vortex vector is taken into account which gives rise to a pressure difference across the surface. This approximation may cause serious errors when the method is applied to the cases where the wing aspect ratio is small and hence the spanwise variation of aerodynamic loading is appreciable. It is possible that to this approximation is connected a foible which haunts this method that the downwash control point at which the boundary condition is applied should be located at 95% of the local panel chord through the centroid of the panel instead of the plausible mid-chord point or the centroid itself as is chosen in the case of the source distribution. There is neither a physical nor a mathematical reason for the choice of this figure except that the calculated results have thus been best fitted in either experimentally or theoretically obtained ones. Obscurity of this figure is manifested through the fact that other figures of 75% and 85% were suggested elsewhere for this business. (ref. 13)

In view of several unsatisfactory features identified in the existing methods mentioned so far, an attempt is made in the present work to place the theory for the calculation of lifting potential flow field on a more rigorous foundation.

Following the approach taken by Djojodihardjo & Widnall, the flow field is assumed to be represented by a distribution of doublets over the body surfaces and the trailing vortex sheet. As was already remarked in the foregoing, the expressions of the derivatives of the potential (i. e. the flow velocity) are such that one cannot evaluate the surface values of the derivatives effectively if he leaves them as they are. Hence it is attempted in Section 2 to transform the expressions by means of the integration by parts into forms in which the order of singularity of the integrands is reduced by one degree compared with the original ones. It happens that the transformed expressions indicate a velocity field induced by a vortex distribution over the same body surfaces and trailing vortex sheet. Thus a correspondence is established between a surface distribution of doublets and that of vortices.

Attention is then turned to the Kutta's condition at the trailing-edge of a wing because of its significance as a decisive factor to control the lift distribution on the wing. In Section 3, therefore, the behaviour of the potential and its derivatives is examined in the neighbourhood of the trailing-edge. Several conditions are derived with respect to the doublet strength and its derivatives at the trailing-edge as a consequence of the requirement that the flow velocity remains finite there.

In the subsequent Section these conditions are incorporated in the formulation of the potential flow problem in terms of a surface distribution of doublets. By virtue of the correspondence rule between the doublets and the vortices, the formulation of the problem is also possible by the use of a vortex distribution over the body surfaces and the trailing vortex sheet. Realization of this possibility is considered in Section 5 and the corresponding formulation of the problem is accomplished thereby completing our analysis.

SYMBOLS

$\boldsymbol{A}$, $\overline{\boldsymbol{A}}$	see (4. 5) and (5. 16a)
$\boldsymbol{B}$, $\overline{\boldsymbol{B}}$	see (4. 6) and (5. 16b)
$\boldsymbol{D}\mu$	see (2. 16)
$D\mu_\xi$, $D\mu_\eta$, $D\mu_\zeta$	components of $\boldsymbol{D}\mu$
F, G, H	components of a metric tensor, see (A. I. 1)
l, m, n	components of the unit normal vector $\boldsymbol{n}$
n	see (3. 8)
$\boldsymbol{n}$	unit normal vector
$\boldsymbol{q}$	flow velocity
r	distance between P (x, y, z) and Q (ξ, η, ζ)
S	surface on which singularities are distributed
s	arc-length
T_0	point on the trailing-edge
t	see (3. 7)
U_∞	free-stream velocity
u	surface coordinate
v	surface coordinate
$\boldsymbol{v}$	velocity induced by a vortex distribution
x, y, z	Cartesian coordinates
α_1, α_2, α_3	unit vector along the α-axis, see (A. I. 4)
β_1, β_2, β_3	unit vector along the β-axis, see (A. I. 4)
α, β, γ	tangential unit vector to S
δ_T	tailing-edge angle, see (3. 15)
δ	sign convention, see the paragraph following (2. 3)
$\delta\phi_S$, $\delta\phi_D$	$\delta\phi_S=\phi_{SU}-\phi_{SL}$ etc., see (3. 33)
ΔS	part of S in the neighbourhood of T_0
$\Delta\phi_S$, $\Delta\phi_D$	part of ϕ_S or ϕ_D due to the integration over ΔS
ξ, η, ζ	Cartesian coordinates of a point on S
θ	angle measured from the x-y plane
λ_1, λ_2, λ_3	components of $\boldsymbol{\lambda}$
$\boldsymbol{\lambda}$	see (2. 17) or, in Section 5, the vortex vector
λ_n	see (5. 9)
μ	strength of doublets
ν	coordinate along $\boldsymbol{n}$
ρ	see (3. 12) or, in Section 4, the density of fluid
σ	strength of sources
ϕ	potential
φ	perturbation potential or see (3. 12)
ϕ_1, ϕ_2, ϕ_3	components of $\boldsymbol{\phi}$
$\boldsymbol{\phi}$	vector potential of a vortex distribution, see (5. 1)

Subscripts

$_B$	refers to the body surface
$_D$	refers to a doublet distribution
$_L$	refers to the lower surface of a wing
$_S$	refers to a source distribution
$_U$	refers to the upper surface of a wing
$_W$	refers to the trailing vortex sheet
$_\alpha$	refers to the α-axis
$_\beta$	refers to the β-axis
$_\nu$	refers to the ν-axis

2. POTENTIALS AND THEIR DERIVATIVES DUE TO SURFACE DISTRIBUTIONS OF SINGULARITIES

Our first objective is to establish a mathematical formulation of the flow field around a lifting body with trailing vortex sheets in terms of distributions of singularities over the body surface and the sheets. Before launching the formulation it is advantageous to work out the expression of derivatives of potentials in a form convenient for the later manipulation.

2.1 Potential Due to a Surface Distribution of Sources

First, consider the potential ϕ_S given by a distribution of sources σ over a surface S:

$$\phi_S=\frac{1}{4\pi}\int_S \frac{\sigma}{r}dS \tag{2.1}$$

where r is the distance between a point $P(x, y, z)$ in the flow field and a point $Q(\xi, \eta, \zeta)$ lying on S:

$$r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2. \tag{2.2}$$

The derivatives of ϕ_S are obtained by differentiation under the integral sign as long as P remains outside S and provided σ and S satisfy certain regularity conditions. Mathematical discussions of this kind are found elsewhere. (e.g. ref. 14, Chap. IV or ref. 15) For the moment we assume that σ and S satisfy all the requirements relevant to the subsequent development of formulation.

When P approaches a point Q_0 (ξ_0, η_0, ζ_0) on S along the normal to S at Q_0, they become unintegrable in the usual sense and recourse must be made to the notion of the Cauchy's principal value for the proper evaluation.

The results are:

$$\left(\frac{\partial\phi_S}{\partial x}\right)_{Q_0}=-\frac{l}{2}\sigma(Q_0)\cdot\delta+\frac{1}{4\pi}\oint_S \sigma\frac{\xi-\xi_0}{r^3}dS, \tag{2.3a}$$

$$\left(\frac{\partial\phi_S}{\partial y}\right)_{Q_0}=-\frac{m}{2}\sigma(Q_0)\cdot\delta+\frac{1}{4\pi}\oint_S \sigma\frac{\eta-\eta_0}{r^3}dS \tag{2.3b}$$

and

$$\left(\frac{\partial\phi_S}{\partial z}\right)_{Q_0}=-\frac{n}{2}\sigma(Q_0)\cdot\delta+\frac{1}{4\pi}\oint_S \sigma\frac{\zeta-\zeta_0}{r^3}dS \tag{2.3c}$$

where (l, m, n) is the direction cosines of the normal to S at Q_0, and $\delta=+1$ when Q_0 is approached from the side of S toward which the normal is directed and $\delta=-1$ otherwise, and where the sign $\oint$ indicates that the integral should be evaluated in the sense of the Cauchy's principal value. Mathematically, two conditions are sufficient to ensure the legitimacy of equations (2.3), i. e.,

(1) the density σ is Hölder continuous at Q_0, i. e. three positive constants K, α and ε exist such that

$$|\sigma(Q)-\sigma(Q_0)|<K(\overline{QQ_0})^\alpha$$

for any point Q on S which satisfies

$$\overline{QQ_0}<\varepsilon,$$

where $\overline{QQ_0}$ denotes the distance from Q_0 to Q, and

(2) the surface S has continuous curvature in the neighbourhood of Q_0.

It is expected that in most physical situations likely to be encountered in aerodynamical applications these conditions are invariably satisfied.

Defining the differential operators $\partial/\partial n$ and $\partial/\partial s$ as

$$\left(\frac{\partial}{\partial n}\right)_{Q_0}=\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)_{P=Q_0} \tag{2.4}$$

and

$$\left(\frac{\partial}{\partial s}\right)_{Q_0}=\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+\gamma\frac{\partial}{\partial z}\right)_{P=Q_0}, \tag{2.5}$$

where (α, β, γ) is a unit vector lying in the plane tangent to S at Q_0, we obtain

$$\left(\frac{\partial\phi_S}{\partial n}\right)_{Q_0}=-\frac{1}{2}\sigma(Q_0)\cdot\delta+\frac{1}{4\pi}\oint\sigma\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \tag{2.6}$$

and

$$\left(\frac{\partial\phi_S}{\partial s}\right)_{Q_0}=\frac{1}{4\pi}\oint\sigma\frac{\partial}{\partial s}\left(\frac{1}{r}\right)dS. \tag{2.7}$$

Although only the formula (2.6) is written out explicitly in the literature (e. g. ref. 15) the expressions (2.3), and hence (2.7), are easily obstained by following the procedure similar to the one adopted there.

2.2 Potential Due to a Surface Distribution of Doublets

Next the potential of a surface distribution of doublets is discussed: it is given by

$$\phi_D(x, y, z)=\frac{1}{4\pi}\int_S \mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS \tag{2.8}$$

where

$$\frac{\partial}{\partial\nu}=l\frac{\partial}{\partial\xi}+m\frac{\partial}{\partial\eta}+n\frac{\partial}{\partial\zeta}, \tag{2.9a}$$

(l, m, n) being the same as in the case of ϕ_S.

As is easily seen, the kernel of ϕ_D, $\partial(1/r)/\partial\nu$, is by one order more singular on S than the kernel $1/r$ of ϕ_S.

Because of this fact the evaluation of surface derivatives, i. e. the values which the derivatives would assume as the point $P(x, y, z)$ approaches a point Q_0 on S, becomes much more difficult with ϕ_D than with ϕ_S, and to the auther's knowledge there have been no detailed account in the literature on the explicit form of the surface derivatives of ϕ_D.

However, assuming that the strength μ of doublets is adequately regular on S (for instance the conditions are sufficient that μ is differentiable and that the derivatives are Hölder-continuous), the derivatives of ϕ_D on S can be obtained in the following way. Since

$$\frac{\partial}{\partial\xi}\left(\frac{1}{r}\right)=-\frac{\partial}{\partial x}\left(\frac{1}{r}\right) \quad \text{etc.,}$$

we have

$$\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)=-\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right). \tag{2.9b}$$

Then for a point P outside S,

$$\begin{aligned}\left(\frac{\partial\phi_D}{\partial x}\right)_P&=\frac{1}{4\pi}\int_S\frac{\partial}{\partial x}\left[\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)\right]dS\\&=\frac{1}{4\pi}\int_S\left\{-\mu\left(l\frac{\partial^2}{\partial x^2}+m\frac{\partial^2}{\partial x\partial y}+n\frac{\partial^2}{\partial x\partial z}\right)\times\left(\frac{1}{r}\right)\right\}dS\\&=\frac{1}{4\pi}\int_S\mu\left\{l\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)-m\frac{\partial^2}{\partial x\partial y}-n\frac{\partial^2}{\partial x\partial z}\right\}\left(\frac{1}{r}\right)dS\\&=\frac{1}{4\pi}\int_S\mu\left\{\frac{\partial}{\partial y}\left(l\frac{\partial}{\partial y}-m\frac{\partial}{\partial x}\right)-\frac{\partial}{\partial z}\left(n\frac{\partial}{\partial x}-l\frac{\partial}{\partial z}\right)\right\}\left(\frac{1}{r}\right)dS\\&=\frac{1}{4\pi}\left\{\frac{\partial}{\partial y}\int_S\mu\left(m\frac{\partial}{\partial\xi}-l\frac{\partial}{\partial\eta}\right)\left(\frac{1}{r}\right)dS-\frac{\partial}{\partial z}\int_S\mu\left(l\frac{\partial}{\partial\zeta}-n\frac{\partial}{\partial\xi}\right)\left(\frac{1}{r}\right)dS\right\}.\end{aligned}\tag{2.10a}$$

In passing from the second line to the third in the above equation use has been made of the fact that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\right)=0.$$

Similarly,

$$\begin{aligned}\left(\frac{\partial\phi_D}{\partial y}\right)_P=\frac{1}{4\pi}\left\{\frac{\partial}{\partial z}\int_S\mu\left(n\frac{\partial}{\partial\eta}-m\frac{\partial}{\partial\zeta}\right)\left(\frac{1}{r}\right)dS-\frac{\partial}{\partial x}\int_S\mu\left(m\frac{\partial}{\partial\xi}-l\frac{\partial}{\partial\eta}\right)\left(\frac{1}{r}\right)dS\right\}\end{aligned}\tag{2.10b}$$

and

$$\begin{aligned}\left(\frac{\partial\phi_D}{\partial z}\right)_P=\frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\int_S\mu\left(l\frac{\partial}{\partial\zeta}-n\frac{\partial}{\partial\xi}\right)\left(\frac{1}{r}\right)dS-\frac{\partial}{\partial y}\int_S\mu\left(n\frac{\partial}{\partial\eta}-m\frac{\partial}{\partial\zeta}\right)\left(\frac{1}{r}\right)dS\right\}.\end{aligned}\tag{2.10c}$$

These expressions of the derivatives of ϕ_D are written down concisely using the notations of vector analysis as

$$\operatorname{grad}\phi_D=\operatorname{rot}\phi\tag{2.11}$$

where $\phi=(\phi_1,\phi_2,\phi_3)$ is given by

$$\phi_1(x,y,z)=\frac{1}{4\pi}\int_S\mu\left(n\frac{\partial}{\partial\eta}-m\frac{\partial}{\partial\zeta}\right)\left(\frac{1}{r}\right)dS,\tag{2.12a}$$

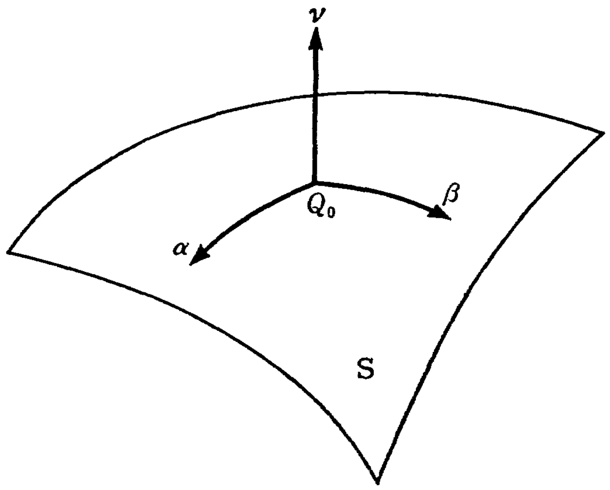
$$\phi_2(x,y,z)=\frac{1}{4\pi}\int_S\mu\left(l\frac{\partial}{\partial\zeta}-n\frac{\partial}{\partial\xi}\right)\left(\frac{1}{r}\right)dS\tag{2.12b}$$

and

$$\phi_3(x,y,z)=\frac{1}{4\pi}\int_S\mu\left(m\frac{\partial}{\partial\xi}-l\frac{\partial}{\partial\eta}\right)\left(\frac{1}{r}\right)dS.\tag{2.12c}$$

We now attempt to transform the right-hand sides of (2.12) so that the kernels of the integrals emerge with singularities of order of $1/r$ on S instead of $1/r^3$.

Let us define an orthogonal curvilinear coordinate system (α,β) confined within the surface S. To



Sketch 1

this system is added the third coordinate ν, which is the distance along the normal to S, to form a three-dimensional coordinate system (α,β,ν).

Let ds_α and ds_β be the element arc-lengths along the α- and β-axes respectively. Further, let $(\alpha_1,\alpha_2,\alpha_3)$ and $(\beta_1,\beta_2,\beta_3)$ be the expressions in the Cartesian system (x,y,z) of the unit vectors in the directions of the α-axis and the β-axis respectively. The differential operators appearing under the integral signs of equations (2.12) are shown to be inner operators on the surface S and are expressed as

$$n\frac{\partial}{\partial\eta}-m\frac{\partial}{\partial\zeta}=\alpha_1\frac{\partial}{\partial s_\beta}-\beta_1\frac{\partial}{\partial s_\alpha},\tag{2.13a}$$

$$l\frac{\partial}{\partial\zeta}-n\frac{\partial}{\partial\xi}=\alpha_2\frac{\partial}{\partial s_\beta}-\beta_2\frac{\partial}{\partial s_\alpha}\tag{2.13b}$$

and

$$m\frac{\partial}{\partial\xi}-l\frac{\partial}{\partial\eta}=\alpha_3\frac{\partial}{\partial s_\beta}-\beta_3\frac{\partial}{\partial s_\alpha}.\tag{2.13c}$$

A detailed account of how these and subsequent results are obtained is given in Appendix I.

Since the surface element dS is given by

$$dS=ds_\alpha\,ds_\beta,$$

the right-hand sides of (2.12) thus become amenable to the integration by parts resulting in

$$\phi_1=\frac{1}{4\pi}\left\{\int_S\frac{\lambda_1}{r}dS-\oint_{\partial S}\frac{\mu}{r}d\xi\right\},\tag{2.14a}$$

$$\phi_2=\frac{1}{4\pi}\left\{\int_S\frac{\lambda_2}{r}dS-\oint_{\partial S}\frac{\mu}{r}d\eta\right\}\tag{2.14b}$$

and

$$\phi_3=\frac{1}{4\pi}\left\{\int_S\frac{\lambda_3}{r}dS-\oint_{\partial S}\frac{\mu}{r}d\zeta\right\}\tag{2.14c}$$

where the symbol $\oint$ indicates the line integral along the boundary ∂S of S in the sense that the side of S from which the ν-axis is directed away is seen on the left-hand side as the boundary is travelled.

It goes without saying that the line integrals in (2.14) do not emerge at all when S is a closed surface.

The function λ_1, λ_2 and λ_3 are given by

$$\lambda_1=\beta_1\frac{\partial\mu}{\partial s_\alpha}-\alpha_1\frac{\partial\mu}{\partial s_\beta},\tag{2.15a}$$

$$\lambda_2=\beta_2\frac{\partial\mu}{\partial s_\alpha}-\alpha_2\frac{\partial\mu}{\partial s_\beta} \tag{2.15b}$$

and

$$\lambda_3=\beta_3\frac{\partial\mu}{\partial s_\alpha}-\alpha_3\frac{\partial\mu}{\partial s_\beta}. \tag{2.15c}$$

To put the expressions (2.15) into a concise form we introduce a vector $\boldsymbol{D}\mu$, which is the gradient of μ within S, defined in the (α, β, ν)-system as

$$\boldsymbol{D}\mu=\left(\frac{\partial\mu}{\partial s_\alpha}, \frac{\partial\mu}{\partial s_\beta}, 0\right). \tag{2.16}$$

Now consider a vector $\boldsymbol{\lambda}$ given by

$$\boldsymbol{\lambda}=\boldsymbol{\nu}\times\boldsymbol{D}\mu \tag{2.17}$$

where $\boldsymbol{\nu}$ is the unit vector along the ν-axis which is expressed as (0, 0, 1) in the (α, β, ν)-system.

Then it is easily shown that λ_1, λ_2 and λ_3 of (2.15) are the three components of the vector $\boldsymbol{\lambda}$ viewed in Cartesian (x, y, z)-system:

$$(\lambda_1, \lambda_2, \lambda_3)=(\boldsymbol{\nu}\times\boldsymbol{D}\mu)_{x-y-z\ \text{system}} \tag{2.18}$$

As is seen from the definition of $\boldsymbol{\lambda}$, it is tangent to S, and is orthogonal to the gradient $\boldsymbol{D}\mu$.

The expressions (2.14) are the desired ones. According to them $\boldsymbol{\phi}$ is given in terms of integrals of which the kernels are of order of $1/r$ as a point on S is approached, and is in this respect similar to the potential due to a surface distribution of sources. Therefore a process similar to the one taken in deriving the surface derivatives (2.3) from the potential ϕ_S (2.1) can be employed to obtain surface derivatives of ϕ_D using the relations (2.11) and (2.14).

The final results then are as follows:

Let $(D\mu_\xi, D\mu_\eta, D\mu_\zeta)$ denote $\boldsymbol{D}\mu$ viewed in the (x, y, z)-system:

$$D\mu_\xi=\alpha_1\frac{\partial\mu}{\partial s_\alpha}+\beta_1\frac{\partial\mu}{\partial s_\beta}, \tag{2.19a}$$

$$D\mu_\eta=\alpha_2\frac{\partial\mu}{\partial s_\alpha}+\beta_2\frac{\partial\mu}{\partial s_\beta} \tag{2.19b}$$

and

$$D\mu_\zeta=\alpha_3\frac{\partial\mu}{\partial s_\alpha}+\beta_3\frac{\partial\mu}{\partial s_\beta}. \tag{2.19c}$$

Using $\boldsymbol{D}\mu$ in the sense of $(D\mu_\xi, D\mu_\eta, D\mu_\zeta)$ and understanding $\boldsymbol{\nu}$ to be (l, m, n), we have

$$\left(\frac{\partial\phi_D}{\partial x}\right)_{Q_0}=\frac{1}{2}D\mu_\xi(Q_0)\cdot\delta+\frac{1}{4\pi}\oint\Big[(l\boldsymbol{D}\mu-D\mu_\xi\boldsymbol{\nu})\cdot\text{grad}\left(\frac{1}{r}\right)\Big]_{Q_0}dS+\frac{1}{4\pi}\left\{\frac{\partial}{\partial z}\oint\frac{\mu}{r}\times d\eta-\frac{\partial}{\partial y}\oint\frac{\mu}{r}d\zeta\right\}_{Q_0}, \tag{2.20a}$$

$$\left(\frac{\partial\phi_D}{\partial y}\right)_{Q_0}=\frac{1}{2}D\mu_\eta(Q_0)\cdot\delta+\frac{1}{4\pi}\oint\Big[(m\boldsymbol{D}\mu-D\mu_\eta\boldsymbol{\nu})\cdot\text{grad}\left(\frac{1}{r}\right)\Big]_{Q_0}dS+\frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\oint\frac{\mu}{r}\times d\zeta-\frac{\partial}{\partial z}\oint\frac{\mu}{r}d\xi\right\}_{Q_0} \tag{2.20b}$$

and

$$\left(\frac{\partial\phi_D}{\partial z}\right)_{Q_0}=\frac{1}{2}D\mu_\zeta(Q_0)\cdot\delta+\frac{1}{4\pi}\oint\Big[(n\boldsymbol{D}\mu-D\mu_\zeta\boldsymbol{\nu})\cdot\text{grad}\left(\frac{1}{r}\right)\Big]_{Q_0}dS+\frac{1}{4\pi}\left\{\frac{\partial}{\partial y}\oint\frac{\mu}{r}\times d\xi-\frac{\partial}{\partial x}\oint\frac{\mu}{r}d\eta\right\}_{Q_0} \tag{2.20c}$$

where

$$\text{grad}\left(\frac{1}{r}\right)=\left(\frac{\xi-x}{r^3}, \frac{\zeta-y}{r^3}, \frac{\zeta-z}{r^3}\right) \tag{2.21}$$

and $\delta=+1$ or $\delta=-1$ according as the point Q_0 on S is approached along the positive or negative side of the ν-axis.

Corresponding to the expressions (2.6) and (2.7) for the case of ϕ_S, we have the following expressions of the normal and tangential derivatives of ϕ_D:

$$\left(\frac{\partial\phi_D}{\partial n}\right)_{Q_0}=\left(l\frac{\partial\phi_D}{\partial x}+m\frac{\partial\phi_D}{\partial y}+n\frac{\partial\phi_D}{\partial z}\right)_{Q_0}=l_0V_1+m_0V_2+n_0V_3, \tag{2.22a}$$

$$\left(\frac{\partial\phi_D}{\partial s_\alpha}\right)_{Q_0}=\left(\alpha_1\frac{\partial\phi_D}{\partial x}+\alpha_2\frac{\partial\phi_D}{\partial y}+\alpha_3\frac{\partial\phi_D}{\partial z}\right)_{Q_0}=\frac{1}{2}\left(\frac{\partial\mu}{\partial s_\alpha}\right)_{Q_0}\cdot\delta+\alpha_1V_1+\alpha_2V_2+\alpha_3V_3 \tag{2.22b}$$

and

$$\left(\frac{\partial\phi_D}{\partial s_\beta}\right)_{Q_0}=\left(\beta_1\frac{\partial\phi_D}{\partial x}+\beta_2\frac{\partial\phi_D}{\partial y}+\beta_3\frac{\partial\phi_D}{\partial z}\right)_{Q_0}=\frac{1}{2}\left(\frac{\partial\mu}{\partial s_\beta}\right)_{Q_0}\cdot\delta+\beta_1V_1+\beta_2V_2+\beta_3V_3 \tag{2.22c}$$

where (l_0, m_0, n_0) stands for (l, m, n) at Q_0 and

$$V_1=\frac{1}{4\pi}\left\{\oint(l\boldsymbol{D}\mu-D\mu_\xi\boldsymbol{\nu})\cdot\text{grad}\left(\frac{1}{r}\right)dS+\frac{\partial}{\partial z}\oint\frac{\mu}{r}d\eta-\frac{\partial}{\partial y}\oint\frac{\mu}{r}d\zeta\right\}_{Q_0}, \tag{2.23a}$$

$$V_2=\frac{1}{4\pi}\left\{\oint(m\boldsymbol{D}\mu-D\mu_\eta\boldsymbol{\nu})\cdot\text{grad}\left(\frac{1}{r}\right)dS+\frac{\partial}{\partial x}\oint\frac{\mu}{r}d\zeta-\frac{\partial}{\partial z}\oint\frac{\mu}{r}d\xi\right\}_{Q_0}, \tag{2.23b}$$

$$V_3=\frac{1}{4\pi}\left\{\oint(n\boldsymbol{D}\mu-D\mu_\zeta\boldsymbol{\nu})\cdot\text{grad}\left(\frac{1}{r}\right)dS+\frac{\partial}{\partial y}\oint\frac{\mu}{r}d\xi-\frac{\partial}{\partial x}\oint\frac{\mu}{r}d\eta\right\}_{Q_0}. \tag{2.23c}$$

The relations (2.22) and (2.23) indicate that in contrast to the case for the source distribution, the normal derivative of the doublet distribution $\partial\phi_D/\partial n$ is continuous across S while the tangential derivatives $\partial\phi_D/\partial s_\alpha$ and $\partial\phi_D/\partial s_\beta$ suffer jumps of $\partial\mu/\partial s_\alpha$ and $\partial\mu/\partial s_\beta$ respectively in passing through the surface.

The principal results in this section are the equations (2.11) and (2.14).

The equation (2.11) implies that the velocity field can be given in terms of a vector potential $\boldsymbol{\phi}$ instead of in terms of a scalar potential ϕ_D. On the other hand we know that a velocity field induced by a vorticity distribution is conveniently described in terms of a vector potential $\boldsymbol{A}$ three components of which are given as integrals related to that vorticity (e.g. ref. 16, Sec. 148). Our expression of $\boldsymbol{\phi}$, (2.14), is equivalent to $\boldsymbol{A}$ if $\boldsymbol{\lambda}$ in (2.14) is interpreted as the vorticity itself occupying the place of the surface S while μ appearing in the line integrals interpreted as the strength of the vorticity concentrated on a line which takes the position of the boundary of S. Thus the equations (2.11) and (2.14) indicate that a flow field due to a doublet distribution over a surface is equivalent to that due to a vorticity distribution over the same surface, provided equation (2.15) is satisfied between them.

Though the term 'vorticity' concerns with an attribute of fluid in motion rather than implies what gives rise to a particular motion in fluid as the terms 'source' and 'doublet' do, the term 'vortex' is perverted in this paper to refer to the singularity demonstrated in equation (2.14). A 'vortex' distribution over a solid surface is a fiction in the sense that it is capable of sustaining a pressure difference and adheres to the surface without being convected with the fluid. (ref. 1, p. 296)

Unlike source and doublet, vortex is a vector just as the vorticity is, of which strength is either $\boldsymbol{\lambda}(\lambda_1, \lambda_2, \lambda_3)$ or $(\mu\cdot d\xi/ds,\ \mu\cdot d\eta/ds,\ \mu\cdot d\zeta/ds)$ according as the vortex is distributed on a surface or along a line. More will be discussed about the vortex distribution in Section 5. The equivalence between a doublet distribution and corresponding vortex distribution is, although in a rather cursory manner, already suggested elsewhere. (ref. 16, Secs. 150 & 151)

Although the finding of this equivalence law is in a sense a by-product obtained in an effort to achieve a convenient expression of surface derivatives of the potential due to a doublet distribution, this finding will be exploited fully to facilitate the formulation of problems in the later stage of numerical computation.

3. BEHAVIOUR OF POTENTIALS DUE TO SINGULARITY DISTRIBUTIONS AT THE TRAILING-EDGE OF A WING

A variety of techniques has been employed in the existing literature of potential flow calculations in order to meet the Kutta's condition at the trailing-edge of a wing. None of them, however, are deemed satisfactory because even the finiteness of flow velocity has not been guaranteed when a point on the trailing-edge is approached. Therefore a more detailed study of the condition is desired and this is accomplished to an extent by examining the behaviour of potentials and their derivatives due to singularity distributions such as sources and doublets in the neighbourhood of the trailing-edge of a wing.

3.1 Behaviour of ϕ_S

We investigate ϕ_S, the potential due to a source distribution first, because the knowledge acquired with this case is directly applicable to the case of ϕ_D, the potential due to a doublet distribution.

Let us fix a Cartesian coordinate system (x, y, z) as follows: the origin is placed at a point T_0 on the trailing-edge. The y-axis is taken in the direction of the tangent to the trailing-edge at T_0 and the x-axis lies in the plane which is determined by the y-axis and the tangent to the streamline emanating from T_0. The z-axis is fixed so that the system is right-handed.

Let ϕ_S be divided into three parts:

$$\begin{aligned}\phi_S(x, y, z) &= \frac{1}{4\pi}\int_S \frac{\sigma}{r} dS \\ &= \frac{1}{4\pi}\int_{S-\Delta S} \frac{\sigma}{r} dS \\ &\quad + \frac{1}{4\pi}\int_{\Delta S-\Delta S_0} \frac{\sigma}{r} dS \\ &\quad + \frac{1}{4\pi}\sigma(T_0)\int_{\Delta S_0} \frac{dS}{r} \end{aligned} \tag{3.1}$$

where ΔS is a sufficiently small neighbourhood of T_0 on S and ΔS_0 is part of the plane folded along the y-axis (see Sketch 2 (b)) with the angles θ_U for the upper sheet and θ_L for the lower one, θ_U and θ_L being the tangent angles at T_0 of the upper and the lower branches of the section of the wing intercepted by the x-z plane (Sketch 2 (c)).

The first integral of the right-hand side of (3.1) obviously does not contribute to any singularities in ϕ_S and in its derivatives which are likely to occur when the point P (x, y, z) tends to T_0.

As for the second integral, which is the abbreviation of

$$\int_{\Delta S} \frac{\sigma}{r} dS - \sigma(T_0)\int_{\Delta S_0} \frac{dS}{r}, \tag{3.2}$$

it is shown that this integral nor gives rise to any singularities in them provided σ and ΔS satisfy certain conditions. A detailed account of this assertion is given in Appendix II.

Thus the singular behaviour of ϕ_S and its derivatives will be known by examining the behaviour of the third integral as P approaches T_0.

We designate the contribution of the third integral to ϕ_S by affixing Δ to it:

$$\Delta\phi_S = \frac{1}{4\pi}\sigma(T_0)\int_{\Delta S_0} \frac{dS}{r}. \tag{3.3}$$

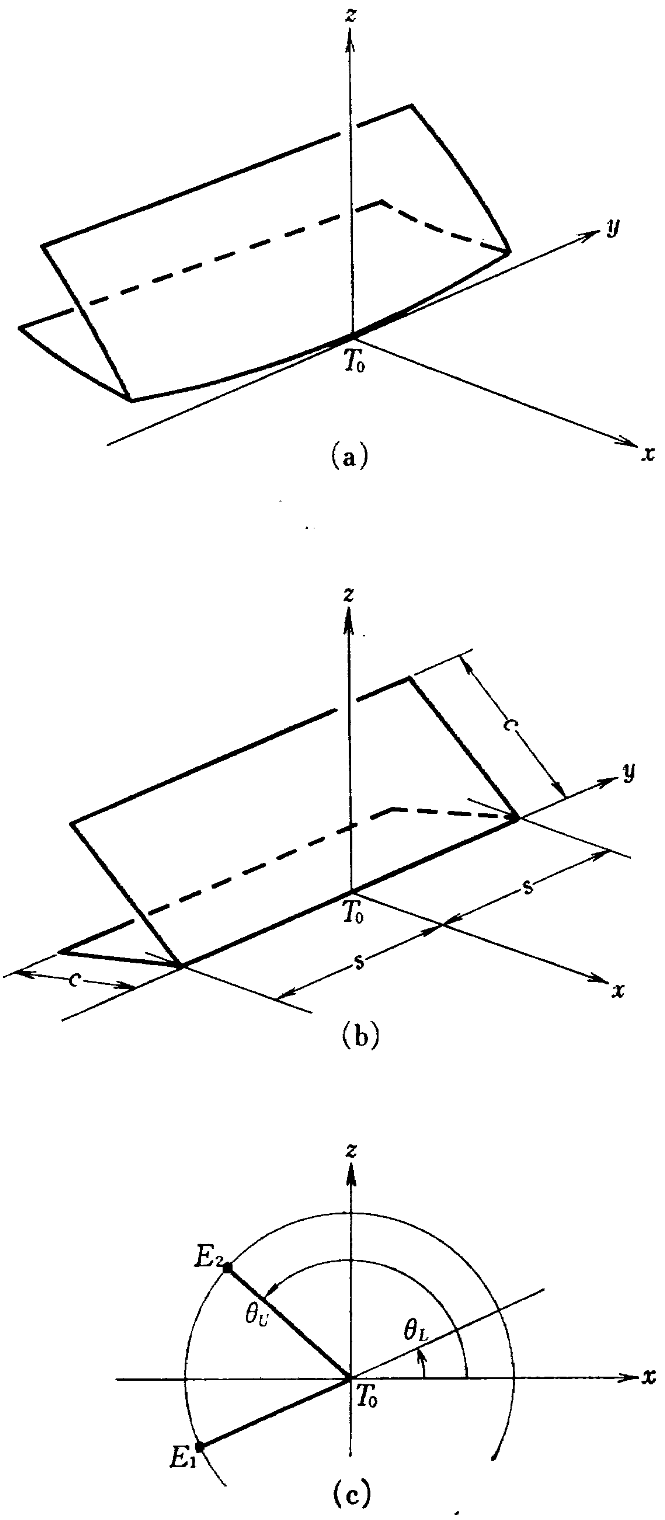
Likewise we define

$$\Delta\frac{\partial\phi_S}{\partial x} \equiv \frac{\partial}{\partial x}\Delta\phi_S = \frac{1}{4\pi}\sigma(T_0)\frac{\partial}{\partial x}\int_{\Delta S_0} \frac{dS}{r}. \tag{3.4}$$

Let a point Q (ξ, η, ζ) on ΔS_0 be given in terms of a serface coordinate system (u, v) as

$$\left.\begin{aligned}\xi &= u\cos\theta \\ \eta &= v \\ \zeta &= u\sin\theta\end{aligned}\right\} \tag{3.5}$$

where



Sketch 2

$$\theta=\begin{cases}\theta_U \text{ for } 0<u<c,\\ \theta_L \text{ for } -c<u<0.\end{cases} \tag{3.6}$$

u is the coordinate along the section $E_1\ T_0\ E_2$ of ΔS_0 cut by a plane normal to the y-axis and v is the coordinate along a line parallel to the y-axis.

Let, further, t and n be given by

$$t=x\cos\theta+z\sin\theta \tag{3.7}$$

and

$$n=x\sin\theta-z\cos\theta, \tag{3.8}$$

and let the subscripts $_U$ and $_L$ stand for the values relevant to the upper and the lower sheets respectively. For instance

$$t_U=x\cos\theta_U+z\sin\theta_U.$$

To allow for a possible discontinuity in the density σ across the trailing-edge, we discriminate σ_U and σ_L defined as

$$\sigma_U=\lim_{P\to T_0}\sigma(P): \text{ for } P\in \text{ the upper sheet of } \Delta S$$

and

$$\sigma_L=\lim_{P\to T_0}\sigma(P): \text{ for } P\in \text{ the lower sheet of } \Delta S.$$

Now we have

$$\begin{aligned} r^2&=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\\ &=(u-t)^2+n^2+(v-y)^2 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} dS&=\left|\left(\frac{\partial\xi}{\partial u},\ \frac{\partial\eta}{\partial u},\ \frac{\partial\zeta}{\partial u}\right)\times\left(\frac{\partial\xi}{\partial v},\ \frac{\partial\eta}{\partial v},\ \frac{\partial\zeta}{\partial v}\right)\right|dudv\\ &=dudv \end{aligned}$$

Then the integration is performed yielding the following:

$$\begin{aligned}
4\pi\Delta\phi_S&=\sigma(T_0)\int_{\Delta S_0}\frac{dS}{r}\\
&=\sigma_U\int_0^c du\int_{-s}^{s}\frac{dv}{\sqrt{(u-t)^2+n^2+(v-y)^2}}\\
&\quad+\sigma_L\int_{-c}^{0} du\int_{-s}^{s}\frac{dv}{\sqrt{(u-t)^2+n^2+(v-y)^2}}\\
&=\sigma_U\Big\{(c-t_U)\log\frac{\sqrt{(c-t_U)^2+n_U^2+(s-y)^2}+s-y}{\sqrt{(c-t_U)^2+n_U^2+(s+y)^2}-s-y}\\
&\quad+t_U\log\frac{\sqrt{t_U^2+n_U^2+(s-y)^2}+s-y}{\sqrt{t_U^2+n_U^2+(s+y)^2}-s-y}\\
&\quad+(s-y)\log\frac{\sqrt{(c-t_U)^2+n_U^2+(s-y)^2}+c-t_U}{\sqrt{t_U^2+n_U^2+(s-y)^2}-t_U}\\
&\quad+(s+y)\log\frac{\sqrt{(c-t_U)^2+n_U^2+(s+y)^2}+c-t_U}{\sqrt{t_U^2+n_U^2+(s+y)^2}-t_U}\\
&\quad-n_U\Big[\mathrm{Tan}^{-1}\frac{(s-y)(c-t_U)}{n_U\sqrt{(c-t_U)^2+n_U^2+(s-y)^2}}\\
&\quad+\mathrm{Tan}^{-1}\frac{(s-y)t_U}{n_U\sqrt{t_U^2+n_U^2+(s-y)^2}}\\
&\quad+\mathrm{Tan}^{-1}\frac{(s+y)(c-t_U)}{n_U\sqrt{(c-t_U)^2+n_U^2+(s+y)^2}}\\
&\quad+\mathrm{Tan}^{-1}\frac{(s+y)t_U}{n_U\sqrt{t_U^2+n_U^2+(s+y)^2}}\Big]\Big\}\\
&\quad+\sigma_L\Big\{(c+t_L)\log\frac{\sqrt{(c-t_L)^2+n_L^2+(s-y)^2}+s-y}{\sqrt{(c-t_L)^2+n_L^2+(s+y)^2}-s-y}\\
&\quad-t_L\log\frac{\sqrt{t_L^2+n_L^2+(s-y)^2}+s-y}{\sqrt{t_L^2+n_L^2+(s+y)^2}-s-y}\\
&\quad-(s-y)\log\frac{\sqrt{(c-t_L)^2+n_L^2+(s-y)^2}-c-t_L}{\sqrt{t_L^2+n_L^2+(s-y)^2}-t_L}\\
&\quad-(s+y)\log\frac{\sqrt{(c-t_L)^2+n_L^2+(s+y)^2}-c-t_L}{\sqrt{t_L^2+n_L^2+(s+y)^2}-t_L}\\
&\quad-n_L\Big[\mathrm{Tan}^{-1}\frac{(s-y)(c+t_L)}{n_L\sqrt{c+t_L)^2+n_L^2+(s-y)^2}}\\
&\quad-\mathrm{Tan}^{-1}\frac{(s-y)t_L}{n_L\sqrt{t_L^2+n_L^2+(s-y)^2}}
\end{aligned}$$

$$+\mathrm{Tan}^{-1}\frac{(s+y)(c+t_L)}{n_L\sqrt{(c+t_L)^2+n_L^2+(s+y)^2}}$$
$$-\mathrm{Tan}^{-1}\frac{(s+y)t_L}{n_L\sqrt{t_L^2+n_L^2+(s+y)^2}}\Bigg]\Bigg\}, \quad (3.10)$$

$$4\pi\Delta\left(\frac{\partial\phi_S}{\partial x}\right)=\sigma(T_0)\int_{\Delta S_0}\frac{\xi-x}{r^3}dS$$
$$=\sigma_U\left\{\frac{\cos\theta_U}{2}\left[\log\frac{\sqrt{(c-t_U)^2+n_U^2+(s-y)^2}-s+y}{\sqrt{(c-t_U)^2+n_U^2+(s-y)^2}+s-y}\right.\right.$$
$$-\log\frac{\sqrt{t_U^2+n_U^2+(s-y)^2}-s+y}{\sqrt{t_U^2+n_U^2+(s-y)^2}+s-y}$$
$$+\log\frac{\sqrt{(c-t_U)^2+n_U^2+(s+y)^2}-s-y}{\sqrt{(c-t_U)^2+n_U^2+(s+y)^2}+s+y}$$
$$\left.-\log\frac{\sqrt{t_U^2+n_U^2+(s+y)^2}-s-y}{\sqrt{t_U^2+n_U^2+(s+y)^2}+s+y}\right]$$
$$-\sin\theta_U\left[\mathrm{Tan}^{-1}\frac{(s-y)(c-t_U)}{n_U\sqrt{(c-t_U)^2+n_U^2+(s-y)^2}}\right.$$
$$+\mathrm{Tan}^{-1}\frac{(s-y)t_U}{n_U\sqrt{t_U^2+n_U^2+(s-y)^2}}$$
$$+\mathrm{Tan}^{-1}\frac{(s+y)(c-t_U)}{n_U\sqrt{(c-t_U)^2+n_U^2+(s+y)^2}}$$
$$\left.\left.+\mathrm{Tan}^{-1}\frac{(s+y)t_U}{n_U\sqrt{t_U^2+n_U^2+(s+y)^2}}\right]\right\}$$
$$+\sigma_L\left\{\frac{\cos\theta_L}{2}\left[\log\frac{\sqrt{t_L^2+n_L^2+(s-y)^2}-s+y}{\sqrt{t_L^2+n_L^2+(s-y)^2}+s-y}\right.\right.$$
$$-\log\frac{\sqrt{(c+t_L)^2+n_L^2+(s+y)^2}-s+y}{\sqrt{(c+t_L)^2+n_L^2+(s+y)^2}+s-y}$$
$$+\log\frac{\sqrt{t_L^2+n_L^2+(s+y)^2}-s-y}{\sqrt{t_L^2+n_L^2+(s+y)^2}+s+y}$$
$$\left.-\log\frac{\sqrt{(c+t_L)^2+n_L^2+(s+y)^2}-s-y}{\sqrt{(c+t_L)^2+n_L^2+(s+y)^2}+s+y}\right]$$
$$-\sin\theta_L\left[\mathrm{Tan}^{-1}\frac{(s-y)(c+t_L)}{n_L\sqrt{(c+t_L)^2+n_L^2+(s-y)^2}}\right.$$
$$-\mathrm{Tan}^{-1}\frac{(s-y)t_L}{n_L\sqrt{t_L^2+n_L^2+(s-y)^2}}$$
$$+\mathrm{Tan}^{-1}\frac{(s+y)(c+t_L)}{n_L\sqrt{(c+t_L)^2+n_L^2+(s+y)^2}}$$
$$\left.\left.-\mathrm{Tan}^{-1}\frac{(s+y)t_L}{n_L\sqrt{t_L^2+n_L^2+(s+y)^2}}\right]\right\}, \quad (3.11a)$$

$$4\pi\Delta\left(\frac{\partial\phi_S}{\partial y}\right)=\sigma(T_0)\int_{\Delta S_0}\frac{\eta-y}{r^3}dS$$
$$=\sigma_U\left\{\log\frac{\sqrt{(c-t_U)^2+n_U^2+(s+y)^2}+c-t_U}{\sqrt{(c-t_U)^2+n_U^2+(s-y)^2}+c-t_U}\right.$$
$$\left.-\log\frac{\sqrt{t_U^2+n_U^2+(s+y)^2}-t_U}{\sqrt{t_U^2+n_U^2+(s-y)^2}-t_U}\right\}$$
$$-\sigma_L\left\{\log\frac{\sqrt{(c+t_L)^2+n_L^2+(s+y)^2}-c-t_L}{\sqrt{(c+t_L)^2+n_L^2+(s-y)^2}-c-t_L}\right.$$
$$\left.-\log\frac{\sqrt{t_L^2+n_L^2+(s+y)^2}-t_L}{\sqrt{t_L^2+n_L^2+(s+y)^2}-t_L}\right\} \quad (3.11b)$$

and $\Delta(\partial\phi_S/\partial z)$ being obtained by substituting $\theta_U-\pi/2$ and $\theta_L-\pi/2$ for θ_U and θ_L respectively in the right-hand side of $\Delta(\partial\phi_S/\partial x)$ given above.

Now suppose that $P(x, y, z)$ tends to T_0 along a path which does not touch the y-axis at T_0. Let the symbol $R(A)$ denote the singular part of a quantity A resulting from this process.

Putting
$$\left.\begin{aligned} x&=\rho\cos\varphi\\ y&=0\\ z&=\rho\sin\varphi\end{aligned}\right\} \quad (3.12)$$
and letting ρ tend to zero in the right-hand sides of (3.10) and (3.11), we obtain the following results:

$$R(\phi_S)=0,$$
$$R\left(\frac{\partial\phi_S}{\partial x}\right)=\frac{1}{4\pi}(\sigma_L\cos\theta_L-\sigma_U\cos\theta_U)\log\rho^2, \quad (3.13a)$$
$$R\left(\frac{\partial\phi_S}{\partial y}\right)=0, \quad (3.13b)$$
and
$$R\left(\frac{\partial\phi_S}{\partial z}\right)=\frac{1}{4\pi}(\sigma_L\sin\theta_L-\sigma_U\sin\theta_U)\log\rho^2. \quad (3.13c)$$

These results are independent of what path is taken actually provided the path does not touch the y-axis at T_0. They show that the velocity becomes logarithmically infinite as P approaches the trailing-edge unless the following condition is satisfied:

$$(\sigma_U\cos\theta_U-\sigma_L\cos\theta_L)^2+(\sigma_U\sin\theta_U-\sigma_L\sin\theta_L)^2=0. \quad (3.14)$$

Using the trailing-edge angle δ_T defined as
$$\delta_T=\pi-(\theta_U-\theta_L), \quad (3.15)$$
this condition reduces to
$$\sigma_U^2+\sigma_L^2+2\sigma_U\sigma_L\cos\delta_T=0. \quad (3.16)$$

Restricting ourselves to physically plausible cases where
$$0\leq\delta_T<\pi,$$
we see that this condition implies that
$$\sigma_U+\sigma_L=0 \quad \text{if} \quad \delta_T=0 \quad (3.17)$$
and
$$\sigma_U=\sigma_L=0 \quad \text{otherwise.} \quad (3.18)$$

That is, the source strength must vanish at the trailing-edge in order that the velocity remains finite there as dictated by the Kutta's condition unless the trailing-edge is cusped, in wich case the sum of the source strengths on both sides of the wing surface at the edge should vanish.

The condition (3. 16) renders the flow velocity not only finite but also continuous at the trailing-edge. If a source distribution exists which satisfiies the boudary condition of vanishing normal velocity on the body surface, then it is inferred that the condition (3.18) at T_0 implies the vanishing of the velocity component normal to the trailing-edge at T_0 since otherwise the velocity would become discontinuous.

3. 2 Behaviour of ϕ_D

Now we proceed to the case of ϕ_D, the potential due to a doublet distribution.

ϕ_D is used to describe a flow field around a lifting wing. Inherent in such a flow field is the existence of the trailing vortex sheet which emanates from the trailing-edge of the wing riding, as it were, on a stream surface. ϕ_D then includes terms due to a doublet distribution over the trailing vortex sheet.

Incidentally, it is possible to try to represent a lifting potential flow field by a combination of a source distribution on the wing surface and a doublet distribution over the trailing vortex sheet. In fact, as will be shown later, this flow model does not work well unless one extends the doublet sheet across the trailing-edge into the interior of the wing just as was done by Rubbert & Saaris, whose approach we discarded however for several reasons. (cf. Introduction.)

As has been done with ϕ_S, ϕ_D is split as follows:

$$\begin{aligned}\phi_D(x, y, z) &= \frac{1}{4\pi}\int_S \mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS \\ &= \frac{1}{4\pi}\int_{S-\Delta S}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS \\ &\quad+\frac{1}{4\pi}\int_{\Delta S-\Delta S_0}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS \\ &\quad+\frac{1}{4\pi}\int_{\Delta S_0}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS \qquad (3.19)\end{aligned}$$

where ΔS is a neighbourhood on S of a point T_0 on the trailing-edge which includes a part of the trailing vortex sheet as depicted in Sketch 3(a).

As before the integral on ΔS is replaced by that on ΔS_0, the difference, the second integral in the right-hand side of (3.19), between them giving rise to no singularities in ϕ_D and in its derivatives as $P(x, y, z)$ approaches T_0. ΔS consists of three parts: ΔS_U lying on the upper surface of the wing, ΔS_L on the lower surface and ΔS_W on the trailing vortex sheet. Correspondingly ΔS_0 consists of ΔS_{0U}, ΔS_{0L} and ΔS_{0W} as is illustrated in Sketch 3(b).

Let us define μ_U, μ_L and μ_W as

$$\mu_U = \lim_{P\to T_0}\mu(P) \quad \text{for} \quad P\in\Delta S_U,$$

$$\mu_L = \lim_{P\to T_0}\mu(P) \quad \text{for} \quad P\in\Delta S_L$$

and

$$\mu_W = \lim_{P\to T_0}\mu(P) \quad \text{for} \quad P\in\Delta S_W.$$

Since the direction ν of the normal is given as

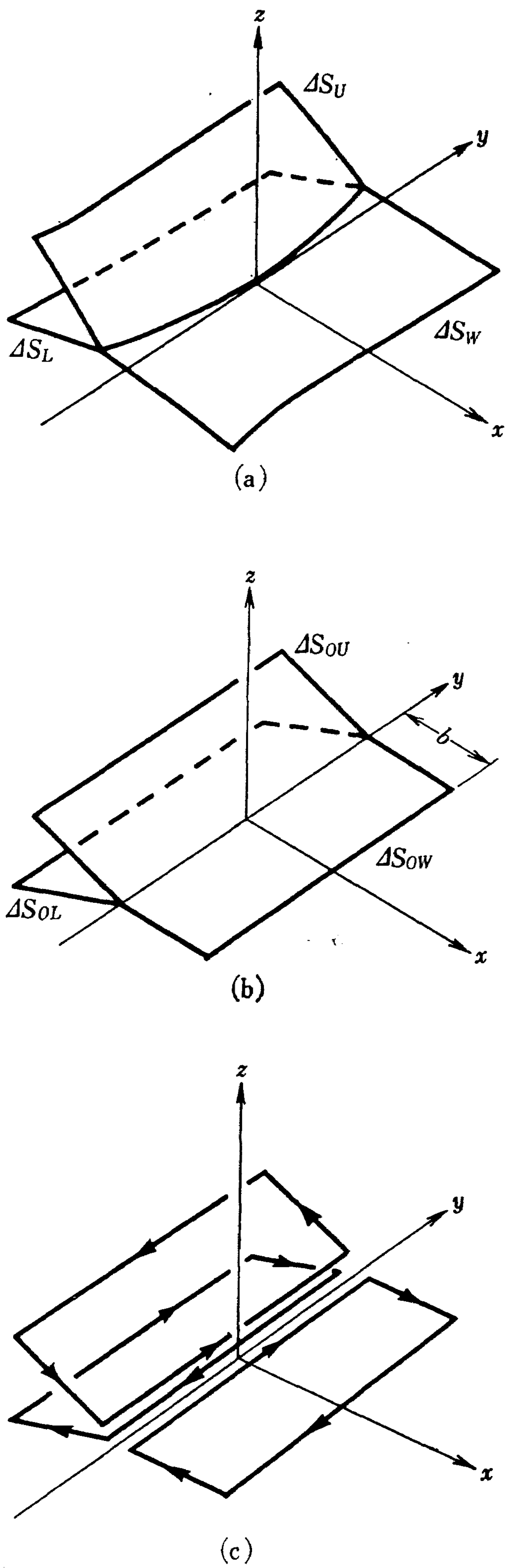
$$(\sin\theta,\ 0,\ -\cos\theta),$$

were θ is as defined in Sketch 2(c), we have

$$\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)=\frac{n}{r^3}. \qquad (3.20)$$

Then

$$\begin{aligned}4\pi\Delta\phi_D &= \int_{\Delta S_0}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS \\ &= \left\{\mu_U\int_0^c du\int_{-s}^{s}dv+\mu_L\int_{-c}^{0}du\int_{-s}^{s}dv+\mu_W\int_0^b du\int_{-s}^{s}dv\right\} \\ &\quad\times\frac{n}{\sqrt{(u-t)^2+n^2+(v-y)^2}^{\,3}} \\ &= \mu_U\left\{\mathrm{Tan}^{-1}\frac{(s-y)(c-t_U)}{n_U\sqrt{(c-t_U)^2+n_U^2+(s-y)^2}}\right.\end{aligned}$$



Sketch 3

$$+\mathrm{Tan}^{-1}\frac{(s-y)t_U}{n_U\sqrt{t_U^2+n_U^2+(s-y)^2}}$$
$$+\mathrm{Tan}^{-1}\frac{(s+y)(c-t_U)}{n_U\sqrt{(c-t_U)^2+n_U^2+(s+y)^2}}$$
$$+\mathrm{Tan}^{-1}\frac{(s+y)t_U}{n_U\sqrt{t_U^2+n_U^2+(s+y)^2}}\Bigg\}$$
$$+\mu_L\Bigg\{\mathrm{Tan}^{-1}\frac{(s-y)(c+t_L)}{n_L\sqrt{(c+t_L)^2+n_L^2+(s-y)^2}}$$
$$-\mathrm{Tan}^{-1}\frac{(s-y)t_L}{n_L\sqrt{t_L^2+n_L^2+(s-y)^2}}$$
$$+\mathrm{Tan}^{-1}\frac{(s+y)(c+t_L)}{n_L\sqrt{(c+t_L)^2+n_L^2+(s+y)^2}}$$
$$-\mathrm{Tan}^{-1}\frac{(s+y)t_L}{n_L\sqrt{t_L^2+n_L^2+(s+y)^2}}\Bigg\}$$
$$-\mu_W\Bigg\{\mathrm{Tan}^{-1}\frac{(s-y)(b-x)}{z\sqrt{(b-x)^2+(s-y)^2+z^2}}$$
$$+\mathrm{Tan}^{-1}\frac{(s-y)x}{z\sqrt{x^2+(s-y)^2+z^2}}$$
$$+\mathrm{Tan}^{-1}\frac{(s+y)(b-x)}{z\sqrt{(b-x)^2+(s+y)^2+z^2}}$$
$$+\mathrm{Tan}^{-1}\frac{(s+y)x}{z\sqrt{x^2+(s+y)^2+z^2}}\Bigg\}. \qquad (3.21)$$

Here it should be noted that the normal to ΔS_{0W} is taken in the direction of the negative z-axis. In other words, the axis of doublets on this sheet is positive downward.

It is seen from (3.21) that $\phi_D(x, y, z)$ remains finite as the point $P(x, y, z)$ tends to T_0. The limit value of ϕ_D in this process, however, depends on the path along which P approaches T_0. For instance, in case where the path is such that it lies in the $x-z$ plane and the tangent to it at T_0 makes an angle φ with the x-axis, the limit value is given by

$$\lim_{P\to T_0}\Delta\phi_D$$
$$=\frac{1}{2\pi}\{\mu_U(\pi-\theta_U+\varphi)+\mu_L(\theta_L-\varphi)-\mu_W(\pi-\varphi)\} \qquad (3.22)$$

provided that

$$0<\theta_U-\varphi<2\pi,\quad -\pi<\theta_L-\varphi<\pi \text{ and } 0<\varphi<2\pi.$$

In fact the right-hand side of (3.22) does not exhibit an explicit dependence on φ because of the relation

$$\mu_W=\mu_L-\mu_U$$

which is a consequence of the Kutta's condition as will be shown in the subsequent discussion of $\Delta(\partial\phi_D/\partial x)$, etc.

However the first term $\mu_U(\pi-\theta_U+\varphi)$, and the second term $\mu_L(\theta_L-\varphi)$, in the right-hand side of (3.22) should be replaced respectively by $\mu_U(-\pi-\theta_U+\varphi)$ when φ is such that

$$-2\pi<\theta_U-\varphi<0,$$

and by $\mu_L(2\pi+\theta_L-\varphi)$ when

$$-3\pi<\theta_L-\varphi<-\pi,$$

which indicates that the limit value actually depends on the approaching path.

Next we turn to the evaluation of $\Delta(\partial\phi_D/\partial x)$, etc.

The curvature of S is in general not continuous along the trailing-edge. The strength μ of doublets together with its derivatives either may not be continuous across the edge. These peculiarities preclude inadvertent application of equation (2.14) in obtaining $\Delta(\partial\phi_D/\partial x)$, etc.

The expression (2.14) as it is fails to be valid along the trailing-edge, and the interchange of the differentiation and the integration, as is required in view of equation (2.11), may also cease to be valid in a limiting process of approaching the edge.

However these difficulties are obviated by applying a cut to S along the trailing-edge thus rendering it a union of three open surfaces with the trailing-edge as a part of their boundaries, on each of which the curvature is continuous and the strength μ is regular.

ΔS_0 is then made up of three separate sheets as depicted in Sketch 3(c) on each of which the direction of line integrals are defined as shown in the Sketch according to the convention stated in connection with equation (2.14).

Now we have

$$\Delta\left(\frac{\partial\phi_D}{\partial x}\right)=\frac{1}{4\pi}\left\{\frac{\partial}{\partial y}\left(\int\frac{\lambda_3}{r}dS-\oint\frac{\mu}{r}d\zeta\right)-\frac{\partial}{\partial z}\left(\int\frac{\lambda_2}{r}dS-\oint\frac{\mu}{r}d\eta\right)\right\}, \qquad (3.23a)$$

$$\Delta\left(\frac{\partial\phi_D}{\partial y}\right)=\frac{1}{4\pi}\left\{\frac{\partial}{\partial z}\left(\int\frac{\lambda_1}{r}dS-\oint\frac{\mu}{r}d\xi\right)-\frac{\partial}{\partial x}\left(\int\frac{\lambda_3}{r}dS-\oint\frac{\mu}{r}d\zeta\right)\right\} \qquad (3.23b)$$

and

$$\Delta\left(\frac{\partial\phi_D}{\partial z}\right)=\frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\left(\int\frac{\lambda_2}{r}dS-\oint\frac{\mu}{r}d\eta\right)-\frac{\partial}{\partial y}\left(\int\frac{\lambda_1}{r}dS-\oint\frac{\mu}{r}d\xi\right)\right\} \qquad (3.23c)$$

where the surface integration ranges over ΔS_0 while the line integration goes along $\partial\Delta S_0$, the periphery of each of ΔS_{0U}, ΔS_{0L} and ΔS_{0W}.

Let us first consider the line integrals. The integrals $\oint(\mu/r)d\xi$ and $\oint(\mu/r)d\zeta$ remains regular as the point P approaches T_0 because the path of integration is at finite distances from T_0.

The remaining one is

$$\oint\frac{\mu}{r}d\eta=\int_{-s}^{s}\frac{\mu_U dv}{\sqrt{t_U^2+n_U^2+(v-y)^2}}+\int_{s}^{-s}\frac{\mu_U dv}{\sqrt{(c-t_U)^2+n_U^2+(v-y)^2}}$$
$$+\int_{-s}^{s}\frac{\mu_L dv}{\sqrt{(c+t_L)^2+n_L^2+(v-y)^2}}+\int_{s}^{-s}\frac{\mu_L dv}{\sqrt{t_L^2+n_L^2+(v-y)^2}}$$
$$+\int_{-s}^{s}\frac{\mu_W d\eta}{\sqrt{x^2+(y-\eta)^2+z^2}}+\int_{s}^{-s}\frac{\mu_W d\eta}{\sqrt{(x-b)^2+(y-\eta)^2+z^2}}. \qquad (3.24)$$

The second, third and sixth integrals in the right-hand side of (3.24) are regular for the same reason as for $\oint (\mu/r)d\xi$ and $\oint (\mu/r)d\zeta$.

Then the other terms are evaluated resulting in

$$\oint \frac{\mu}{r} d\eta = (\mu_U - \mu_L + \mu_W) \times \log\left[\frac{\sqrt{x^2+(s-y)^2+z^2}+s-y}{\sqrt{x^2+(s+y)^2+z^2}-s-y}\right] + (\text{terms regular at } x^2+z^2=0). \quad (3.25)$$

Hence the integral $\oint (\mu/r)d\eta$ gives rise to a singularity of the type of $(x^2+z^2)^{-1/2}$ in $\Delta(\text{grad } \phi_D)$.

In deriving this result it is assumed that the strength μ takes constant values. If μ satisfies the Lipschitz condition:

$$|\mu(x)-\mu(a)| < K|x-a|$$

along the trailing-edge then allowing for the variation of μ as a function of v (or η) does not affect the above result.

Because:

for instance,

$$\left|\frac{\partial}{\partial x}\int_{-s}^{s} \frac{\mu(v)dv}{\sqrt{x^2+(v-y)^2+z^2}} - \frac{\partial}{\partial x}\int_{-s}^{s} \frac{\mu(y)dv}{\sqrt{x^2+(v-y)^2+z^2}}\right| < K\int_{-s}^{s} \frac{|x(v-y)|dv}{\sqrt{x^2+(v-y)^2+z^2}^3}$$
$$= K\left\{\frac{2|x|}{\sqrt{x^2+z^2}} - \frac{|x|}{\sqrt{x^2+(s-y)^2+z^2}} - \frac{|x|}{\sqrt{x^2+(s+y)^2+z^2}}\right\} \quad (3.26)$$

which verifies the assertion.

Next the contribution from the surface integrals is considered. As is inferred from the calculation of $\Delta \text{ grad } \phi_S$, an integral $\int (\alpha/r) dS$ gives rise to the singularities of $\alpha \cos\theta \cdot \log \rho^2$ and $\alpha \sin\theta \cdot \log\rho^2$ in $(\partial/\partial x)\int (\alpha/r)dS$ and $(\partial/\partial z)\int(\alpha/r)dS$ respectively where $\rho^2 = x^2+z^2$ and θ is the angle which the intersection of S and the x-z plane makes with the x-axis towards T_0.

Hence the term, for instance,

$$\frac{\partial}{\partial z}\int_{\Delta S_0} \frac{\lambda_1}{r} dS - \frac{\partial}{\partial x}\int_{\Delta S_0} \frac{\lambda_3}{r} dS$$

is singular by

$$\{(-\lambda_{1U}\sin\theta_U + \lambda_{1L}\sin\theta_L) - (-\lambda_{3U}\cos\theta_U + \lambda_{3L}\cos\theta_L - \lambda_{3W})\}\log\rho^2$$

at T_0.

Here we have defined, as before, the following:

$$\left.\begin{aligned} \lambda_{kU} &= \lim_{P\to T_0} \lambda_k \quad \text{for} \quad P \in \Delta S_U \\ \lambda_{kL} &= \lim_{P\to T_0} \lambda_k \quad \text{for} \quad P \in \Delta S_L \\ \lambda_{kW} &= \lim_{P\to T_0} \lambda_k \quad \text{for} \quad P \in \Delta S_W \end{aligned}\right\} k=1,2,3.$$

Collecting all the singular terms we finally obtain the following result:

$$R\left(\frac{\partial \phi_D}{\partial x}\right) = -\frac{1}{2\pi}\left\{(\mu_U-\mu_L+\mu_W)\frac{z}{\rho^2} + (-\lambda_{2U}\sin\theta_U + \lambda_{2L}\sin\theta_L)\log\rho\right\}, \quad (3.27\text{a})$$

$$R\left(\frac{\partial \phi_D}{\partial y}\right) = \frac{1}{2\pi}\left\{(-\lambda_{1U}\sin\theta_U+\lambda_{1L}\sin\theta_L) - (-\lambda_{3U}\cos\theta_U + \lambda_{3L}\cos\theta_L - \lambda_{3W})\right\}\log\rho \quad (3.27\text{b})$$

and

$$R\left(\frac{\partial \phi_D}{\partial z}\right) = -\frac{1}{2\pi}\left\{(\mu_U-\mu_L+\mu_W)\frac{x}{\rho^2} + (-\lambda_{2U}\cos\theta_U + \lambda_{2L}\cos\theta_L - \lambda_{2W})\log\rho\right\}. \quad (3.27\text{c})$$

Now the Kutta's condition is called into play.

First, the singularities of order of $1/\rho$ should vanish:

$$\mu_U - \mu_L + \mu_W = 0 \quad (3.28)$$

which is a kind of conservation law of the doublet strength at the trailing-edge. This condition has already been aquired by Djojodihardjo & Widnall (ref. 10) as a consequence of the continuity in the potential jump across the trailing-edge.

Next, the singularities of order of $\log\rho$ are to be suppressed:

$$-\lambda_{2U}\sin\theta_U + \lambda_{2L}\sin\theta_L = 0, \quad (3.29\text{a})$$

$$(-\lambda_{1U}\sin\theta_U + \lambda_{1L}\sin\theta_L) - (-\lambda_{3U}\cos\theta_U + \lambda_{3L}\cos\theta_L - \lambda_{3W}) = 0 \quad (3.29\text{b})$$

and

$$-\lambda_{2U}\cos\theta_U + \lambda_{2L}\cos\theta_L - \lambda_{2W} = 0. \quad (3.29\text{c})$$

In view of equation (2.18),

$\boldsymbol{\lambda}(\lambda_1, \lambda_2, \lambda_3)$ is given by

$$\lambda_1 = D\mu_\eta \cos\theta, \quad (3.30\text{a})$$

$$\lambda_2 = -(D\mu_\xi\cos\theta + D\mu_\zeta \sin\theta) \quad (3.30\text{b})$$

and

$$\lambda_3 = D\mu_\eta \sin\theta \quad (3.30\text{c})$$

where $\theta=\theta_U$ for ΔS_{0U}, $\theta=\theta_L$ for ΔS_{0L} and $\theta=\pi$ for ΔS_{0W}. (Note that the normal to ΔS_{0W} is in the direction of the negative z-axis.)

By substituting (3.30) into (3.29b) we see that the singularity in $\partial\phi_D/\partial y$ vanishes automatically.

Equations (3.29a) and (3.29c) imply that

$$\lambda_{2U} = k\sin\theta_L,$$
$$\lambda_{2L} = k\sin\theta_U$$

and

$$\lambda_{2W} = k\sin(\theta_U - \theta_L) = k\sin\delta_T$$

where δ_T is the trailing-edge angle (eq. (3.15)) and k is an arbitrary constant.

Let alone k, λ_{2U} depends on θ_L while λ_{2L} on θ_U. This is rather a strange situation and we are tempted to put $k=0$ resulting in

$$\lambda_{2U} = \lambda_{2L} = \lambda_{2W} = 0 \quad (3.31)$$

except for the case of a cusped trailing-edge, for which instead we have

$$\left.\begin{array}{l}\lambda_{2U}+\lambda_{2L}=0\\ \lambda_{2W}=0\end{array}\right\}. \tag{3.32}$$

Summing up, the Kutta's condition demands that for a doublet distribution representing a wing surface and accompanying trailing vortex sheet, the strength μ must satisfy the following conditions at the trailing-edge:

$$\mu_U-\mu_L+\mu_W=0,$$
$$\lambda_{2W}=0$$

and

either $\lambda_{2U}=\lambda_{2L}=0$

for a wedge-type trailing-edge,

or $\lambda_{2U}+\lambda_{2L}=0$

for a cusped trailing-edge.

Looking in terms of vortex distribution, λ_2 is the component of vortex vector $\boldsymbol{\lambda}$ in the direction tangent to the trailing-edge. The condition $\lambda_2=0$ at the trailing-edge is convincing since λ_2 is one of the agents through which the pressure difference is generated across the surface.

3. 3 Potential jumps at the trailing-edge

So far the behaviour of potentials and their derivatives are examined when the point $P(x, y, z)$ tends to T_0 from the inside of the flow field.

Now what will happen if T_0 is approached along the wing surface?

As for the Kutta's condition, the situation is not altered in view of equation (2.20) provided $D\mu$ remains finite at T_0. The question is of interest, however, since the circulation around the wing section through T_0 is equivalent to the jump in potential which may arise when T_0 is approached along the upper and the lower surfaces.

Let us first consider the case of ϕ_S.

We wish to evaluate $\delta\phi_S$ defined as

$$\delta\phi_S=\phi_{SU}-\phi_{SL} \tag{3.33}$$

where ϕ_{SU} is the limit of $\phi_S(x, y, z)$ as $P(x, y, z)$ tends to T_0 lying on the upper surface of the wing while ϕ_{SL} is that of ϕ_S as P is kept within the lower surface.

Suppose that ϕ_S is partitioned as given in expression (3.1). In the first integral there we can put simply $P=T_0$ because T_0 locates outside the range of integration. In Appendix II it is shown that the second integral is continuous at T_0. The third one is written as

$$\int_{\Delta S_0}\frac{\sigma}{r}dS=\int_{-c}^{c}\sigma du\int_{-s}^{s}\frac{dv}{\sqrt{(u-t)^2+n^2+(v-y)^2}}$$
$$=\sigma_U\int_0^c du\log\frac{\sqrt{(u-t_U)^2+n_U{}^2+(s-y)^2}+s-y}{\sqrt{(u-t_U)^2+n_U{}^2+(s+y)^2}-(s+y)}$$
$$+\sigma_L\int_{-c}^{0} du\log\frac{\sqrt{(u-t_L)^2+n_L{}^2+(s-y)^2}+s-y}{\sqrt{(u-t_L)^2+n_L{}^2+(s+y)^2}-(s+y)}$$

where either

$$\left.\begin{array}{ll}t_U=u_0, & n_U=0,\\ t_L=u_0\cos\Delta\theta, & n_L=-u_0\sin\Delta\theta_1\end{array}\right\} \tag{3.35a}$$

for $P(x, y, z)$ lying on the upper surface:

$$x=u_0\cos\theta_U,\quad y=v_0,\quad z=u_0\sin\theta_U,\quad u_0>0,$$

or

$$\left.\begin{array}{ll}t_U=u_0\cos\Delta\theta, & n_U=u_0\sin\Delta\theta,\\ t_L=u_0, & n_L=0\end{array}\right\} \tag{3.35b}$$

for $P(x, y, z)$ located on the lower surface:

$$x=u_0\cos\theta_L,\quad y=v_0,\quad z=u_0\sin\theta_L,\quad u_0<0$$

with

$$\Delta\theta=\theta_U-\theta_L.$$

By letting P tend to T_0, i.e. by letting $u_0\to 0$ we observe that the third integral assumes the same value in ϕ_{SU} and ϕ_{SL}.

Hence we conclude that

$$\delta\phi_S=0. \tag{3.36}$$

That is, in a flow field represented by a source distribution over a wing surface, the circulation around any wing section is identically zero. Therefore a source distribution is incapable of representing a flow field around a wing with non-zero circulation.

Now let us suppose that the source distribution on the wing surface is supplemented by a doublet distribution over the trailing vortex sheet so that it becomes capable of representing a lifting potential flow field.

As a matter of fact this flow model is disproved by the following two observations.

First, let us evaluate the potential jump $\delta\phi$ as before, which may arise when a point T_0 on the trailing-edge is approached along the upper and the lower wing surfaces. According to (3.36), the contribution of the source distribution to $\delta\phi$ is null. The contribution from the doublet distribution is obtained by taking the difference of the right-hand side of (3.22) for $\varphi=\theta_U$ and for $\varphi=\pi+\theta_L$.

Thus we have

$$\delta\phi(T_0)=-\mu_W\frac{\delta_T}{2\pi}$$

where δ_T is the trailing-edge angle, see (3.15). (Note that in the present flow model, $\mu_U=\mu_L=0$ since the doublet distribution is confined on the trailing vortex sheet.)

On the other hand, the potential jump across the trailing vortex sheet evaluated at a point Q_0 on the sheet is

$$\delta\phi(Q_0)=-\mu(Q_0)$$

(cf. eqation (3.37) to follow and the passage preceding it.)

Letting Q_0 tend to T_0 we obtain

$$\delta\phi(T_0)=-\mu_W$$

which goes against the foregoing result.

This discontinuity in the potential jump is serious because $\delta\phi(T_0)$ is equal to the circulation around the wing section through T_0. The contradiction has arisen because of the assumption that the doublet distribution begins at the trailing-edge. It can be dissolved by extending the doublet sheet across the trailing-edge into the interior of the wing, as was done by Rubbert & Saaris. They say (ref. 9) that they did this because of computational expediency but it is seen that this extension has a vital concern to the consistency of their model.

As the second evidence to disprove the flow model under consideration, we apply the Kutta's condition to it. The singular parts in the velocity at the trailing-edge are given by the sum of the right-hand sides of (3.13) and (3.27). As a consequence of enforcing the Kutta's condition we have

$$\mu_W=0$$

which implies that the circulation around any wing section is zero. This in turn indicates that the present model can handle only the symmetrical non-lifting case and hence the addition of the doublet sheet is utterly futile.

Obviously this situation can be salvaged by extending the doublet sheet into the interior of the wing, since then the requirement is only the connuity of the strength of doublets across the trailing-edge.

Next the case of ϕ_D is examined.

In paralled to ϕ_S we define $\delta\phi_D$ as

$$\delta\phi_D=\phi_{DU}-\phi_{DL}.$$

Comparing equation (2.9b) with the definition (2.4) we see that $\phi_D(x,y,z)$ takes a form corresponding to the right-hand side of equation (2.6) as $P(x,y,z)$ approaches a point $Q_0(\xi_0,\eta_0,\zeta_0)$ on S:

$$\phi_D(\xi_0,\eta_0,\zeta_0)=\frac{1}{2}\mu(Q_0)\cdot\delta+\frac{1}{4\pi}\oint_S\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS. \tag{3.37}$$

We may put $\delta=1$ with the understanding that the normal is directed into the flow field from which P approached Q_0.

Then the first term in (3.37) contributes to $\delta\phi_D$ by

$$\frac{1}{2}(\mu_U-\mu_L)$$

In evaluating the contribution from the second term in (3.37) we again split this integral into three as was done in equation (3.19). In parallel to the case of ϕ_S, the first and the second integrals there are continuous at T_0 thus having no effects upon $\delta\phi_D$. The third one, the integral over ΔS_0, should be evaluated in the sense of Cauchy's principal value. The result, however, is identical with equation (3.21) provided that

(1) t_U, n_U, t_L and n_L there are to be given either as equations (3.35a) in evaluating ϕ_{DU}, i.e. when $P(x,y,z)$ is lying on the upper surface:

$$(x,y,z)=(u_0\cos\theta_U,\ v_0,\ u_0\sin\theta_U),\quad u_0>0,$$

or as equations (3.35b) in evaluating ϕ_{DL}, i.e. when $P(x,y,z)$ is on the lower surface:

$$(x,y,z)=(u_0\cos\theta_L,\ v_0,\ u_0\sin\theta_L),\ u_0<0,$$

and

(2) in evaluating ϕ_{DU} the term proportional to μ_U in (3.21) is eliminated because $n_U=0$ for P lying on the upper surface, and in evaluating ϕ_{DL} the term proportional to μ_L is to be deleted since $n_L=0$ for P lying on the lower surface.

Then the contributions of the third integral to ϕ_{DU} and ϕ_{DL} are obtained from the expression within the curly bracket in equation (3.22) by putting $\varphi=\theta_U-0$ and $\varphi=\pi+\theta_L+0$ respectively. Here the symbols -0 and $+0$ denote that φ tends to θ_U or $\pi+\theta_L$ always taking smaller or greater values respectively than the limits.

Thus this integral contributes to $\delta\phi_D$ the following:

$$\frac{1}{2\pi}\{[\mu_L(\theta_L-\theta_U)+\mu_W(\theta_U-\pi)]-[\mu_U(\theta_L-\theta_U)+\mu_W\theta_L]\}$$
$$=\frac{1}{2\pi}\{(\mu_U-\mu_L+\mu_W)(\theta_U-\theta_L)-\pi\mu_W\}.$$

Adding the contribution from the first term in (3.37), we obtain

$$\delta\phi_D=\left(\frac{1}{2}+\frac{\theta_U-\theta_L}{2\pi}\right)(\mu_U-\mu_L+\mu_W)-\mu_W \tag{3.38}$$

which, by virtue of equation (3.28), reduces to

$$\delta\phi_D=-\mu_W=\mu_U-\mu_L. \tag{3.39}$$

That is, the Kutta's condition ensures that the jump in ϕ_D which emerges when the trailing-edge is approached via two distinct paths, one being on the upper and the other on the lower surfaces of the wing, is continuously transferred to the jump in ϕ_D across the trailing vortex sheet. In other words, the circulation around a wing section passing through a point T_0 on the trailing-edge is given either by $\mu_U-\mu_L$ at T_0 or by $-\mu_W$ there, a fact which facilitates the calculation of the sectional lift forces acting on the wing. Incidentally, the relation (3.38) indicates that we cannot dispense with the trailing vortex sheet unless we restrict ourselves to the calculation of non-lifting potential flow fields.

Because:

assume that the trailing vortex sheet is absent. Then $\mu_W\equiv0$ and we have the condition

$$\mu_U-\mu_L=0$$

instead of (3.28) at the trailing-edge. As is seen from (3.38) this makes $\delta\phi_D$ vanish identically and hence no circulation around any wing section.

3.4 Singularities due to the line integrals along the boundary

We have seen, by way of equation (3.25), that the line integrals in equation (2.14) give rise to singularities of order of $1/\rho$ in grad ϕ_D as one approaches a point on the path of integration, where ρ is the distance from this point. The line integrals being along the boundaries ∂S it follows that the strength μ of doublets should vanish along ∂S if the velocity of fluid is to remain finite there. When dealing with potential flow problems around lifting wing-fuselage configurations, the boundaries ∂S are likely to consist of the following:

(1) the trailing-edge.
This is the boundary artificially introduced by applying a cut to render the formulae (2.11) and (2.14) applicable to the neighbourhood of the edge.

(2) Lines on the surface of the body along which discontinuity in curvature exists: possible examples of such lines are the wing-fuselage and -wing junctures. Cuts must be applied to these lines, too, for the same reason as with the trailing-edge.

(3) The spanwise ends of the trailing vortex sheet.

It is evident that the velocity of fluid is finite along the side edges of the trailing vortex sheet. Therefore $\mu=0$ along the side edges, and the third member of ∂S listed above has no contributions to the line integrals in (2.14).

For the first member the Kutta's condition entails the continuity of the strength μ, equation (3.28), to render the line integrals along the edges null.

For the lines of discontinuity on the body surface we do not know exactly what is happening there. If, at least, the continuity of μ across those lines is established, the line integrals along them vanish and there would be no singularities of order of $1/\rho$. In the practical situation, the fairings are invariably applied to such discontinuities, let alone the masking effects of the boundary layers. Hence we have some justifications in neglecting the contributions to the line integrals from sources of this sort.

All in all we can enunciate the following in conclusion: the line integrals in equation (2.14) are identically null when potential flow problems around lifting wing-fuselage configurations are considered.

4. DERIVATION OF THE BASIC EQUATIONS OF POTENTIAL FLOW FIELD IN TERMS OF A DOUBLET DISTRIBUTION

Suppose that the flow field around a lifting wing is described in terms of a doublet distribution over the wing surface S_B and the trailing vortex sheet S_W. That is, suppose that the perturbation potential φ of the flow field is written as

$$\varphi(x, y, z)=\frac{1}{4\pi}\int_S \mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS \qquad (4.1)$$

where $S=S_B+S_W$.

Let gradφ denote the derivatives of φ:

$$\text{grad}\,\varphi=\left(\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial z}\right). \qquad (4.2)$$

When the point $P\,(x, y, z)$ approaches a point $Q_0\,(\xi_0, \eta_0, \zeta_0)$ on S, gradφ tend to the limit given by equation (2.20).

In a summarized form it can be written as

$$(\text{grad}\,\varphi)_{Q_0}=\frac{1}{2}\boldsymbol{D}\mu(Q_0)\cdot\delta + \frac{1}{4\pi}\oint_S(\boldsymbol{\nu}\boldsymbol{D}u-\boldsymbol{D}\mu\boldsymbol{\nu})\cdot\text{grad}\left(\frac{1}{r}\right)dS \qquad (4.3)$$

where grad$(1/r)$ is given by (2.21).

It is noted that the line integrals contained in equation (2.20) are suppressed here according to the conclusion given at the end of the last Section.

For the sake of brevity we write the right-hand side of (4.3) as

$$(\text{grad}\,\varphi)_{Q_0}=\boldsymbol{A}\cdot\delta+\boldsymbol{B} \qquad (4.4)$$

where

$$\boldsymbol{A}=\frac{1}{2}\boldsymbol{D}\mu \qquad (4.5)$$

and

$$\boldsymbol{B}=\frac{1}{4\pi}\oint_S(\boldsymbol{\nu}\boldsymbol{D}\mu-\boldsymbol{D}\mu\boldsymbol{\nu})\cdot\text{grad}\left(\frac{1}{r}\right)dS. \qquad (4.6)$$

Let $\boldsymbol{U}_\infty$ be the free-stream velocity.

The velocity $\boldsymbol{q}$ at any point in the flow field is given as a sum of $\boldsymbol{U}_\infty$ and gradφ.

Let $\boldsymbol{q}_0$ denote the value of $\boldsymbol{q}$ at Q_0:

$$\boldsymbol{q}_0=\boldsymbol{U}_\infty+\boldsymbol{A}\cdot\delta+\boldsymbol{B}. \qquad (4.7)$$

The component of $\boldsymbol{q}_0$ normal to S is then given as

$$q_n=\boldsymbol{q}_0\cdot\boldsymbol{n}=(\boldsymbol{U}_\infty+\boldsymbol{B})\cdot\boldsymbol{n} \qquad (4.8)$$

since

$$\boldsymbol{A}\cdot\boldsymbol{n}=0 \qquad (4.9)$$

in view of the definition of $\boldsymbol{D}\mu$ (equation (2.16)) where $\boldsymbol{n}$ stands for the normal to S.

Hence the boundary condition on the wing surface S_B is readily established:

$$(\boldsymbol{U}_\infty+\boldsymbol{B})\cdot\boldsymbol{n}=F \qquad (4.10)$$

where F is a function prescribed on S_B, taken to be null for a solid surface in steady flow.

Next let us examine conditions to be satisfied on the trailing vortex sheet S_W.

We discriminate between the front and the reverse sides of S_W by writing P_U or P_L according as a point P is supposed to be lying on the front or the reverse side.

Since $\delta=+1$ at Q_{0U} and $\delta=-1$ at Q_{0L} in equation (4.3), a jump takes place in $\boldsymbol{q}$ as one shifts between Q_{0U} and Q_{0L}.

Hence Q_{0U} and Q_{0L} cannot be positioned on a single streamline. It follows that a streamline cannot cross the trailing vortex sheet. That is, the trailing vortex sheet is a surface of discontinuity which separates two stream surfaces an infinitesimal distance apart from each other.

Let $\boldsymbol{q}_{0U}$ and $\boldsymbol{q}_{0L}$ be defined as

$$q_{0U}=U_\infty+A+B \tag{4.11a}$$

and

$$q_{0L}=U_\infty-A+B. \tag{4.11b}$$

Then the above statement is expressed as

$$q_{0U}\cdot n=q_{0L}\cdot n=0$$

which, by virtue of (4.9), reduces to

$$(U_\infty+B)\cdot n=0 \tag{4.12}$$

on the trailing vortex sheet.

Equation (4.12) implies that $U_\infty+B$ is lying on the sheet. Incidentally we have, from (4.11),

$$\frac{1}{2}(q_{0U}+q_{0L})\equiv q_{0m}=U_\infty+B \tag{4.13}$$

which means that the velocity component $U_\infty+B$ on the trailing vortex sheet is given by an arithmetical mean of the velocities on the front side and the reverse side.

Since the trailing vortex sheet is a hypothetical surface embedded within the fluid, a dynamical condition accrues that there is no pressure difference across the sheet.

By virtue of the Bernoulli's equation, the pressure difference Δp across the sheet is given by

$$\Delta p=\frac{\rho}{2}(q_{0L}^2-q_{0U}^2)$$

which, in view of equations (4.11), reduces to

$$\frac{\Delta p}{\rho}=-2A\cdot(U_\infty+B)=-D\mu\cdot(U_\infty+B)$$

Hence the dynamical condition on the trailing vortex sheet amounts to the following condition:

$$D\mu\cdot(U_\infty+B)=0. \tag{4.14}$$

That is, the two components $(1/2)D\mu$ and $U_\infty+B$ of q_0 are orthogonal with each other, both being confined within S_W. It follows that the strength μ of doublets is constant along such a curve C on S_W that at every point on it the tangent is in the direction of $U_\infty+B$, i.e. in the direction of the mean velocity q_{0m}.

Let us consider the circulation Γ around a loop which crosses S_W at a point Q_0 and never does so at any other points on $S=S_B+S_W$. Γ is equivalent to the difference in potential at points Q_{0U} and Q_{0L}, which in turn is equivalent to the strength of doublets at Q_0:

$$\Gamma=\varphi(Q_{0U})-\varphi(Q_{0L})=-\mu(Q_0). \tag{4.15}$$

The latter relation in the above is easily established by observing that the second term in the right-hand side of equation (3.37) is continuous across Q_0.

From the preceding arguments, then, we see that the circulation Γ is kept constant as Q_0 moves along any curve C which is tangent to $U_\infty+B$ at every point on it. This implies that the curve C is in fact the path of the vorticity which is shed downstream from a point T on the trailing-edge. In other words, the path of shed vorticity is determined by tracing a curve which starts at T and is always tangent to the direction of the mean velocity q_{0m}.

In short, the trailing vortex sheet is generated by a family of curves which is constructed by tracing the direction of the mean velocity q_{0m} starting from each point on the trailing-edge. On each curve thus formed, the strength μ of doublets is constant, and there is another family of curves orthogonal to the former which consists of curves that lie on the sheet and are always in the direction of the gradient $D\mu$.

Now the basic equations of the flow field are put together:

$$(U_\infty+B)\cdot n=0 \tag{4.16}$$

on the solid part of the surface S,

$$(U_\infty+B)\cdot n=0 \tag{4.17a}$$

and

$$(U_\infty+B)\cdot A=0 \tag{4.17b}$$

for the trailing vortex sheet, and

$$\mu_U-\mu_L+\mu_W=0 \tag{4.18}$$

and

$$\lambda_{2W}=0.$$

Further, either

$$\left.\begin{array}{l}\lambda_{2U}=\lambda_{2L}=0 \text{ for a wedge-type edge}\\ \text{or}\\ \lambda_{2U}+\lambda_{2L}=0 \text{ for a cusped edge}\end{array}\right\} \tag{4.19}$$

at the trailing-edge of the wing,

where n is the normal to the surface, A is given by (4.5) and B by (4.6).

In the above system of equations the unknown quantities are the strength μ of doublets over $S=S_B+S_W$ and the location of the trailing vortex sheet S_W. There is at present no way of knowing whether the system is in general solvable for these unknowns.

At least, however, it can be said that the system seems to be mathematically consistent and worth taking as a mathematical model of potential flow fields around lifting wings or wing-fuselage combinations.

5. FORMULATION IN TERMS OF A VORTEX DISTRIBUTION

In Sections 2 and 3 it was shown that the velocity field induced by a surface distribution of doublets could be interpreted as that due to a surface distribution of vortices. Based on this fact an attempt is made in this Section to formulate the basic equations in terms of a vortex distribution.

Suppose that the body and the trailing vortex sheet are replaced by a distribution of vortex $\lambda(\lambda_1, \lambda_2, \lambda_3)$ over their surfaces S_B and S_W.

Let ϕ be defined as

$$\phi(x,y,z)=\frac{1}{4\pi}\int_S\frac{\lambda}{r}\,dS \tag{5.1}$$

where $S=S_B+S_W$ and

$$r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2,$$

$Q(\xi,\eta,\zeta)$ being a point on S.

The velocity v induced by this vortex distribution is then given by

$$v=\text{rot}\,\phi. \tag{5.2}$$

5. 1 Compatibility Conditions

Since by definition $\nabla^2\boldsymbol{\phi}=0$ in the domain outside S we have

$$\begin{aligned}\text{rot } \boldsymbol{v} &= \text{rot}(\text{rot } \boldsymbol{\phi}) \\ &= \text{grad}(\text{div } \boldsymbol{\phi}) - \nabla^2\boldsymbol{\phi} \\ &= \text{grad}(\text{div } \boldsymbol{\phi}). \end{aligned} \tag{5.3}$$

Hence the condition

$$\text{grad}(\text{div } \boldsymbol{\phi}) = 0 \tag{5.4}$$

must hold in order that the induced velocity field is irrotational.

To investigate this condition we employ the orthogonal curvilinear coordinate system (α, β, ν) already introduced in Section 2. Then we see that

$$\begin{aligned}\text{div } \boldsymbol{\phi} &= \frac{1}{4\pi}\int_S \frac{\lambda_1(\xi-x)+\lambda_2(\eta-y)+\lambda_3(\zeta-z)}{r^3}dS \\ &= -\frac{1}{4\pi}\int_S\left(\lambda_1\frac{\partial}{\partial\xi}+\lambda_2\frac{\partial}{\partial\eta}+\lambda_3\frac{\partial}{\partial\zeta}\right)\left(\frac{1}{r}\right)dS \\ &= -\frac{1}{4\pi}\int_S\left(\frac{\lambda_\alpha}{F}\frac{\partial}{\partial\alpha}+\frac{\lambda_\beta}{G}\frac{\partial}{\partial\beta}+\frac{\lambda_\nu}{H}\frac{\partial}{\partial\nu}\right) \\ &\quad \times\left(\frac{1}{r}\right)F\cdot Gd\alpha d\beta \end{aligned} \tag{5.5}$$

where λ_α, λ_β and λ_ν are the components of $\boldsymbol{\lambda}$ in the directions of the α-, β- and ν-axes respectively given by

$$\lambda_\alpha = \lambda_1\frac{\xi_\alpha}{F}+\lambda_2\frac{\eta_\alpha}{F}+\lambda_3\frac{\zeta_\alpha}{F}, \tag{5.6a}$$

$$\lambda_\beta = \lambda_1\frac{\xi_\beta}{G}+\lambda_2\frac{\eta_\beta}{G}+\lambda_3\frac{\zeta_\beta}{G}, \tag{5.6b}$$

and

$$\lambda_\nu = \lambda_1\frac{\xi_\nu}{H}+\lambda_2\frac{\eta_\nu}{H}+\lambda_3\frac{\zeta_\nu}{H}. \tag{5.6c}$$

Here ξ_α etc., F, G and H are defined as before. (see Appendix I)

In view of equation (2.17) the vectex vector $\boldsymbol{\lambda}$ should lie within the surface S. In other words the component λ_ν is to vanish identically:

$$\lambda_\nu = 0. \tag{5.7}$$

Then integration by parts reduces the right-hand side of equation (5.5) to

$$\begin{aligned}&-\frac{1}{4\pi}\int_S\left(\frac{\lambda_\alpha}{F}\frac{\partial}{\partial\alpha}+\frac{\lambda_\beta}{G}\frac{\partial}{\partial\beta}\right)\left(\frac{1}{r}\right)F\cdot Gd\alpha d\beta \\ &= \frac{1}{4\pi}\left\{\int_S\frac{1}{r}\left[\frac{\partial}{\partial\alpha}(\lambda_\alpha G)+\frac{\partial}{\partial\beta}(\lambda_\beta F)\right]d\alpha d\beta \right. \\ &\quad \left. -\oint_{\partial S}\frac{\lambda_n}{r}dS\right\}\end{aligned} \tag{5.8}$$

where

$$\lambda_n = \lambda_\alpha\frac{ds_\beta}{ds}-\lambda_\beta\frac{ds_\alpha}{ds} \tag{5.9}$$

is the component of $\boldsymbol{\lambda}$ normal to the boudary ∂S.

Hence the condition (5.4) requires that

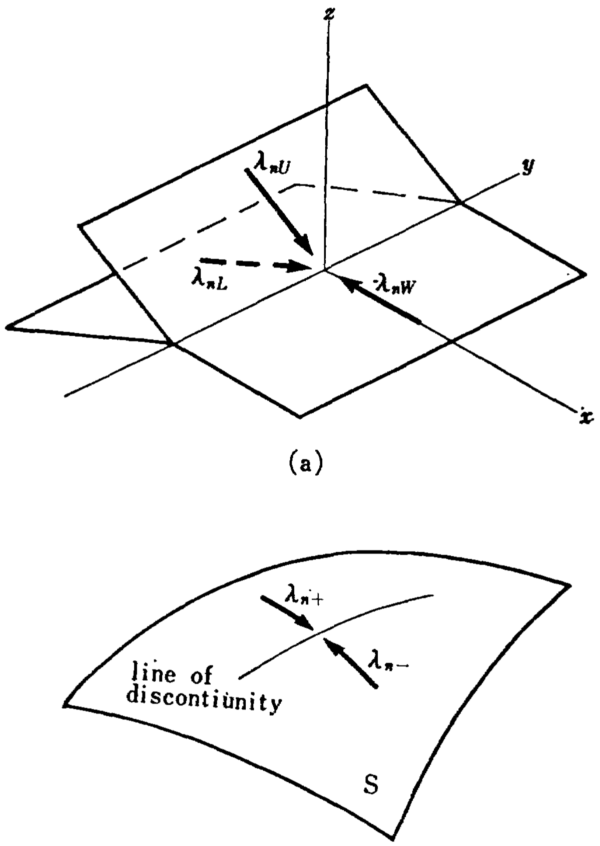
$$\frac{\partial}{\partial\alpha}(\lambda_\alpha G)+\frac{\partial}{\partial\beta}(\lambda_\beta F) = 0 \tag{5.10}$$

on S, and

$$\lambda_n = 0 \tag{5.11}$$

along ∂S.

Let us term the conditions (5.10) and (5.11) the compatibility conditions for a surface distribution of vortex $\boldsymbol{\lambda}$. As was discussed towards the end of Section 3, the boundary ∂S is likely to consist of (1) the cut along the trailing-edge, (2) the cuts along the lines of discontinuity on S, and (3) the spanwise ends of the trailing vortex sheet. Applying the condition (5.11) to these constituents of ∂S we have



(a)

(b)

Sketch 4

$$\lambda_{nU}+\lambda_{nL}+\lambda_{nW} = 0 \tag{5.12a}$$

along the trailing-edge,

$$\lambda_{n+}+\lambda_{n-} = 0 \tag{5.12b}$$

along the lines of discontinuity, and

$$\lambda_n = 0 \tag{5.12c}$$

along the side ends of the trailing vortex sheet.

The sense of λ_{nU} etc. is as depicted in Sketch 4.

5. 2 Boundary Conditions

The procedure similar to that taken in deriving (2.20) leads to the following expression of the value of $\boldsymbol{v}$ at a point $Q_0(\xi_0, \eta_0, \zeta_0)$ on S. (cf. equation (4.3))

$$\boldsymbol{v}(Q_0) = \frac{1}{2}(\boldsymbol{\lambda}\times\boldsymbol{n})_{Q_0}\cdot\delta - \frac{1}{4\pi}\oint_S\boldsymbol{\lambda}\times\text{grad}\left(\frac{1}{r}\right)dS \tag{5.13}$$

where grad $(1/r)$ is as given by (2.21), $\boldsymbol{n}$ is the normal to S and δ is as was defined in connection with equation (2.3). Taking $\boldsymbol{v}$ as the perturbation velocity of the flow field, we obtain the following boundary conditions after the manner in which equation (4.16) and (4.17) were derived:

$$(\boldsymbol{U}_\infty+\overline{\boldsymbol{B}})\cdot\boldsymbol{n}=0 \tag{5.14}$$

on the solid surface, and

$$(\boldsymbol{U}_\infty+\overline{\boldsymbol{B}})\cdot\overline{\boldsymbol{A}}=0 \tag{5.15 a}$$

together with

$$(\boldsymbol{U}_\infty+\overline{\boldsymbol{B}})\cdot\boldsymbol{n}=0 \tag{5.15 b}$$

on the trailing vortex sheet, where

$$\overline{\boldsymbol{A}}=\frac{1}{2}(\boldsymbol{\lambda}\times\boldsymbol{n}) \tag{5.16 a}$$

and

$$\overline{\boldsymbol{B}}=-\frac{1}{4\pi}\oint_S \boldsymbol{\lambda}\times\mathrm{grad}\left(\frac{1}{r}\right)dS. \tag{5.16 b}$$

The physical significance of equation (5.15) is in the same tenor as was traced in the discussion given in the paragraph subsequent to equation (4.14). Reiterating it, $\boldsymbol{\lambda}$ on the trailing vortex sheet comes to its own as the *vorticity* vector shed from the trailing-edge. The conservation law (5.10) ensures that along a curve tangent to $\boldsymbol{\lambda}$ the potential difference, and hence the circulation, is held constant thus reproducing the well-known Kelvin's theorem. Truly, the equation (5.10) implies that there exists a scalar function μ defined on S such that

$$\lambda_\alpha G=-\frac{\partial\mu}{d\beta} \quad\text{and}\quad \lambda_\beta F=\frac{\partial\mu}{\partial\alpha}$$

i. e.,

$$\lambda_\alpha=-\frac{\partial\mu}{\partial s_\beta} \quad\text{and}\quad \lambda_\beta=\frac{\partial\mu}{\partial s_\alpha} \tag{5.17}$$

which, together with the condition (5.7), is exactly the relation already given as equation (2.17).

Thus the process in which a vortex distribution is deduced from a doublet distribution is shown to be reversible, suggesting a procedure by which a potential of the flow field due to a vortex distribution can be calculated. Incidentally, another one, (5.11), of the compatibility conditions reduces to

$$\mu=\text{const. along } \partial S \tag{5.18}$$

in view of equations (5.9) and (5.17). The conditions then lead to

$$\mu_U-\mu_L+\mu_W=\text{const.}$$

along the trailing-edge,

$$\mu_+-\mu_-=\text{const.}$$

along the lines of discontinuity and

$$\mu=\text{const.}$$

along the spanwise ends of the trailing vortex sheet.

That these constants should be zero was already argued towards the end of Section 3.

5. 3 Kutta's Condition

Let us now examine the Kutta's condition in terms of $\boldsymbol{\lambda}$. Comparing equations (3.23) and (3.27) we see from (5.1) and (5.2) that the Kutta's condition entails the condition (3.29).

The relation (3.30) is not availabe in the present case. Instead the condition (5.7) ensures the condition (3.29 b) because

$$\lambda_{1U}\sin\theta_U-\lambda_{3U}\cos\theta_U=\lambda_{\nu U},$$
$$\lambda_{1L}\sin\theta_L-\lambda_{3L}\cos\theta_L=\lambda_{\nu L}$$

and

$$\lambda_{3W}=\lambda_{\nu W}.$$

The remaining two of (3.29) lead to, as before, the following:

$$\lambda_{2U}=\lambda_{2L}=\lambda_{2W}=0$$

for the wedge-type trailing-edge, and

$$\lambda_{2U}+\lambda_{2L}=\lambda_{2W}=0$$

for the cusped trailing-edge.

5. 4 Basic equations

We are now in a position to enunciate the basic equations stated in terms of the vortex vector $\boldsymbol{\lambda}$.

They are:

Condition I & II (compatibility conditions)

$$\text{I.}\quad \lambda_\nu=\boldsymbol{\lambda}\cdot\boldsymbol{n}=0 \quad\text{on } S,$$

and

$$\text{II.}\quad \frac{\partial}{\partial\alpha}(\lambda_\alpha G)+\frac{\partial}{\partial\beta}(\lambda_\beta F)=0 \quad\text{on } S.$$

Condition III (boundary conditions)

$$(\boldsymbol{U}_\infty+\overline{\boldsymbol{B}})\cdot\boldsymbol{n}=0$$

on the solid part S_B of S, and

$$(\boldsymbol{U}_\infty+\overline{\boldsymbol{B}})\cdot\boldsymbol{n}=0,$$

and

$$(\boldsymbol{U}_\infty+\overline{\boldsymbol{B}})\cdot\overline{\boldsymbol{A}}=0$$

on the trailing vortex sheet.

Condition IV (Kutta's condition)

$$\lambda_{nU}+\lambda_{nL}+\lambda_{nW}=0,$$

and

$$\lambda_{tU}=\lambda_{tL}=\lambda_{tW}=0$$

along the trailing-edge (for the wedge-type case), where λ_n is the componet of $\boldsymbol{\lambda}$ normal to the trailing-edge while λ_t is that tangential to the trailing-edge.

Condition I states that $\boldsymbol{\lambda}$ is described by its two components λ_α ahd λ_β which are to satisty Condition II and III. The latter part of Condition III implies that the location of the trailing vortex sheet is unknown a priori. Condition IV serves as the boundary conditions to $\boldsymbol{\lambda}$.

A merit in the formulation by a vortex distribution is that it has a counterpart in the conventional lifiting-surface theories, viz. the 'vortex lattice' method. (e. g. refs. 17 & 18) This method can be considered as a simplified version of our present formulation and hence its accuracy may be conveniently checked against our formulation and calculated results thereof.

6. CONCLUDING REMARKS

A formulation of lifting potential flow field has been attempted by representing the flow field in terms of a surface distribution of doublets.

Analysis of the potential due to a doublet distribution has revealed the following:

(1) The velocity field generated by a surface distribution of doublets is equivalent to that induced by a vortex distribution over the same surface and along its boundary.

(2) The derivatives of the potential due to a surface distribution of doublets cannot be evaluated at a point on the surface without resorting to the rather awkward concept of the 'finite part of a singular integral'. The alternative vortex expression salvages this situation by rendering the surface derivatives obtainable through the concept of Cauchy's principal value, thus making the evaluation much more feasible.

(3) The strength of doublets should vanish at the boundary of the surface over which they are distributed if the flow velocity is to remain finite along the boundary.

(4) The behaviour of the potential has been examined in the neighbourhood of the trailing-edge of a lifting wing yielding the conditions on the strength of doublets at the edge with which the velocity is rendered finite there.

Thus the formulation of the lifting potential flow problems has been accomplished in terms both of a doublet distribution and of a vortex distribution using non-singular expressions of the surface derivatives of potential and embodying the conditions at the trailing-edge as a representative of the Kutta's condition.

Needless to say, the formulation proposed encompasses the two-dimensional situation as a particular case of the three-dimensional flow field. The approach taken here has a precedent (ref. 19) for the two-dimensional case which is powerful and versatile. Even so, our formulation, applied to the two-dimensional case, may retain some significance in view of the detailed exploration of the Kutta's condition and of the clarification of the physical implication of a doublet distribution as a methemacal model.

The present formulation can be applied to the calculation of incompressible unsteady flow field with little modification owing to the fact that the governing differential equation for this case is the same as for the steady flow while the unsteadiness manifests itself only through the boundary conditions. The extention to compressible flow cases is, however, not so straightforward for the unsteady flow as for the steady flow.

The present formulation has been carried out under the assumption that the flow is inviscid and incompressible. Let alone the effects of viscosisty, accurate prediction of compressibility effects for three-dimensional flow is still beyond our capability. It is well known that the Prandtl-Glauert transformation, which accounts for the first-order compressibility effects, becomes increasingly inaccurate as the critical Mach number is approached. Accordingly measures need to be taken to bridge the gap if one wish to make a good prediction of the flow field in the critical Mach number range. For the two-dimensional flow past aerofoils the second-order effects can be calculated using, for instance, the method due to Van Dyke. (ref. 20) This method has led to the compressibility correction factor proposed by Wilby (ref. 21) for the surface velocity on aerofoils. Labrujere, Loeve & Slooff make use of this correction factor to give an alternative expression of the surface velocity on aerofoils in subcritical flow. By applying this two-dimensional formula locally in the direction of the perturbation velocity, they have contrived a higher-order compressibility correction scheme (refs. 22 & 23) for the three-dimensional flow over wings, the direction of the perturbation velocity being approximated by that on the wing in incompressible flow related to the one under consideration via the Prandtl-Glauert transformation. Justification for this scheme is not at all apparent and its accuracy cannot readily be assessed. Therefore it is hoped to develop a more reasonable and efficient method to account for the three-dimensional higher-order compressibility effects.

The ultimate appraisal of the present formulation is to be made by collating the calculated results with experimental ones. The process of the assessment is two-fold: the first concerns the accuracy of the particular numerical procedure adopted to effect the calculation based on the formulation, and the other is whether the formulation is appropriate as a mathematical model of the physics. In view of these and others it is definitely desirable to carry out the numerical computation for several representative cases.

One of the most interesting cases from the practical point of view will be the flow field around a lifting wing-fuselage combination. The wing and the trailing vortex sheet should be represented either by a doublet system or by a vortex system. As for the fuselage, one may replace it by doublets or vortices, too. However, it seems to be more feasible to represent it instead by a surface distribution of sources as has been done by Hess & Smith (ref. 7 & 8) because, for one thing, the source is a scalar whereas the vortex is a vector (the doublet is a scalar but we have to deal with its gradient, which is a vector), and for another, the coefficient matrix of the resulting linear simultaneous algebraic equations is in general more well-behaved with the source system than the doublet or vortex system. A few numerical applications of the present formulation are now being undertaken and the results thereof will be reported later.

REFERENCES

1) Thwaites, B. (ed.); Incompressible Aerodynamics, Oxford University Press, 1960.
2) Royal Aeronautical Society; Method for Predicting the Pressure Distribution on Swept Wings with Subsonic Attached Flow, Transonic Data Memorandum 6312, Dec. 1963
3) Multhopp, H.; Methods for Calculating the Lift Distribution of Wings (Subsonic Lifting-surface Theory), A.R.C. R & M No. 2884, 1955.
4) Zandbergen, P. J., Labrujere, Th. E. & Wouters, J. Y.; A New Approach to the Numerical Solution of the Equation of Subsonic Lifting-surface Theory, NLR-TR-G. 49, Nov. 1967
5) Garner, H.C., Hewitt, B.L. & Labrujere, Th. E.; Comparison of Three Methods for the Evaluation of Subsonic Lifting-surface Theory, A.R.C. R & M No. 3597, June 1968.
6) Argyris, J.H.; The Impact of the Digital Computer on Engineering Sciences, The Aeronautical Journal, Vol. 74, No. 1, Jan. 1970.
7) Hess, J.L. & Smith, A.M.O.; Calculation of Non-lifting Potential Flow about Arbitrary Three-dimensional Bodies, Douglas Report No. ES 40622, March 1962.
8) Hess, J.L. & Smith, A.M.O.; Calculation of Potential Flow about Arbitrary Bodies, Progress in Aeronautical Sciences, Vol. 8, 1967.
9) Rubbert, P.E. & Saaris, G.R.; A General Three-dimensional Potential Flow Method Applied to V/STOL Aerodynamics, SAE Paper No. 680304, April 1968.
10) Djojodihardjo, R.H. & Widnall, Sheila E.; A Numerical Method for the Calculation of Nonlinear, Unsteady Lifting Potential Flow Problems, AIAA Journal, Vol. 7, No. 10, Oct. 1969.
11) Woodward, F.A., Tinoco, E.N. & Larsen, J.W.; Analysis and Design of Supersonic Wing-body Combinations Including Flow Properties in the Near Field-Part I-Theory and Application, NASA CR-73106, Aug. 1967.
12) Woodward, F.A.; Analysis and Design of Wing-body Combinations at Subsonic and Supersonic Speeds, Journal of Aircraft, Vol. 5, No. 6, Nov.-Dec. 1968.
13) Carmichael, R.L.; The Use of Finite Element Methods for Predicting the Aerodynamics of Wing-body Combinations, NASA SP-228, 1970.
14) Courant, R. & Hilbert, D.; Methods in Mathematical Physics, Interscience Publishers, Inc., New York, 1962.
15) Kellogg, O.D.; Foundations of Potential Theory, Frederick Unger Publishing Comany, New York.
16) Lamb, H.; Hydrodynamics, Cambridge University Press, 1932 (Sixth Ed.)
17) Hedman, S.V.; Vortex Lattice Method for Calculation of Quasi Steady State Loadings on Thin Elastic Wings in Subsonic Flow, FFA Rep. 105, 1966.
18) Margason, R.J. & Lamar, J.E.; Vortex-lattice Fortran Program for Estimating Subsonic Aerodynamic Characteristics of Complex Planforms, NASA TN D-6142, Feb. 1971.
19) Giesing, J. P.; Nonlinear Two-dimensional Unsteady Potential Flow with Lift, Journal of Aircraft, Vol. 5, No. 2, March-April 1968.
20) Van Dyke, M.D.; Second-order Subsonic Airfoil Theory Including Edge Effects, NACA Report 1274, 1956.
21) Wilby, P.G.; The Calculation of Sub-critical Pressure Distributions on Symmetric Aerofoils at Zero Incidence, NPL Aero Report 1208, 1967.
22) Labrujere, Th.E., Loeve, W. & Slooff, J.W.; An Approximate Method for the Determination of the Pressure Distribution on Wings in the Lower Critical Speed Range, Paper No. 17 of AGARD C.P. No. 35, 1968.
23) Loeve, W.; Computer Programmes in Use at NLR for the Calculation of Flow Details around Wing-body Combinations at Subsonic Speeds, NLR AT-69-02, 1969.

APPENDIX I

Integrations on a Curved Surface S

At the beginning of Section 2 of the main body of this paper an orthogonal curvilinear coordinate system (α, β, ν) was introduced where the α- and β-axes were confined within the surface S and the ν-axis was taken along the normal to S.

Let (ξ, η, ζ) be the Cartesian coordinates of a point P and (α, β, ν) be the expression of P in the α-β-ν system. Further, let F, G and H be the components of the metric tensor of the α-β-ν system:

$$\left.\begin{aligned} F&=\sqrt{\xi_\alpha^2+\eta_\alpha^2+\zeta_\alpha^2} \\ G&=\sqrt{\xi_\beta^2+\eta_\beta^2+\zeta_\beta^2} \\ H&=\sqrt{\xi_\nu^2+\eta_\nu^2+\zeta_\nu^2} \end{aligned}\right\} \qquad \text{(A. I. 1)}$$

where the subscripts in the above and through this Appendix indicate the differentiation with respect to those variables, viz.

$$\xi_\alpha=\frac{\partial\xi}{\partial\alpha},\ \text{etc.}$$

Then we have

$$\begin{pmatrix}\alpha_\xi & \beta_\xi & \nu_\xi\\ \alpha_\eta & \beta_\eta & \nu_\eta\\ \alpha_\zeta & \beta_\zeta & \nu_\zeta\end{pmatrix}=\begin{pmatrix}\frac{\xi_\alpha}{F^2} & \frac{\xi_\beta}{G^2} & \frac{\xi_\nu}{H^2}\\ \frac{\eta_\alpha}{F^2} & \frac{\eta_\beta}{G^2} & \frac{\eta_\nu}{H^2}\\ \frac{\zeta_\alpha}{F^2} & \frac{\zeta_\beta}{G^2} & \frac{\zeta_\nu}{H^2}\end{pmatrix}, \quad \text{(A.I.2)}$$

and hence

$$\left.\begin{aligned}\frac{\partial}{\partial\xi}&=\frac{\xi_\alpha}{F^2}\frac{\partial}{\partial\alpha}+\frac{\xi_\beta}{G^2}\frac{\partial}{\partial\beta}+\frac{\xi_\nu}{H^2}\frac{\partial}{\partial\nu}\\ \frac{\partial}{\partial\eta}&=\frac{\eta_\alpha}{F^2}\frac{\partial}{\partial\alpha}+\frac{\eta_\beta}{G^2}\frac{\eta_\nu}{\partial\beta}+\frac{\eta_\nu}{H^2}\frac{\partial}{\partial\nu}\\ \frac{\partial}{\partial\zeta}&=\frac{\zeta_\alpha}{F^2}\frac{\partial}{\partial\alpha}+\frac{\zeta_\beta}{G^2}\frac{\partial}{\partial\beta}+\frac{\zeta_\nu}{H^2}\frac{\partial}{\partial\nu}\end{aligned}\right\}. \quad \text{(A. I. 3)}$$

Let $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and (l, m, n) be the unit vectors along the α-, β- and ν-axes respectively:

$$\left.\begin{aligned}(\alpha_1, \alpha_2, \alpha_3)&=\frac{1}{F}(\xi_\alpha, \eta_\alpha, \zeta_\alpha)\\ (\beta_1, \beta_2, \beta_3)&=\frac{1}{G}(\xi_\beta, \eta_\beta, \zeta_\beta)\\ (l,\ m,\ n)&=\frac{1}{H}(\xi_\nu, \eta_\nu, \zeta_\nu)\end{aligned}\right\}. \quad \text{(A. I. 4)}$$

Let, further, ds_α, ds_β and ds_ν be the arc-length elements along the respective axes:

$$\left.\begin{aligned}ds_\alpha&=Fd\alpha\\ ds_\beta&=Gd\beta\\ ds_\nu&=Hd\nu\end{aligned}\right\}. \quad \text{(A. I. 5)}$$

Using (A.I.3), (A.I.4) and (A.I.5) we obtain

$$\left.\begin{aligned}m\frac{\partial}{\partial\zeta}-n\frac{\partial}{\partial\eta}&=\beta_1\frac{\partial}{\partial s_\alpha}-\alpha_1\frac{\partial}{\partial s_\beta}\\ n\frac{\partial}{\partial\xi}-l\frac{\partial}{\partial\zeta}&=\beta_2\frac{\partial}{\partial s_\alpha}-\alpha_2\frac{\partial}{\partial s_\beta}\\ l\frac{\partial}{\partial\eta}-m\frac{\partial}{\partial\xi}&=\beta_3\frac{\partial}{\partial s_\alpha}-\alpha_3\frac{\partial}{\partial s_\beta}\end{aligned}\right\} \quad \text{(A. I. 6)}$$

because by definition

$$(l, m, n)\times(\alpha_1, \alpha_2, \alpha_3)=(\beta_1, \beta_2, \beta_3)$$

and

$$(\beta_1, \beta_2, \beta_3)\times(l, \mathrm{m}, n)=(\alpha_1, \alpha_2, \alpha_3).$$

The surface element dS is given by

$$\begin{aligned}dS&=|(\xi_\alpha, \eta_\alpha, \zeta_\alpha)d\alpha\times(\xi_\beta, \eta_\beta, \zeta_\beta)d\beta|\\ &=F\cdot G\,d\alpha\,d\beta.\end{aligned}$$

Now put

$$\chi_1=\int_S\mu\left(n\frac{\partial}{\partial\eta}-m\frac{\partial}{\partial\zeta}\right)\left(\frac{1}{r}\right)dS,$$

$$\chi_2=\int_S\mu\left(l\frac{\partial}{\partial\zeta}-n\frac{\partial}{\partial\xi}\right)\left(\frac{1}{r}\right)dS,$$

and

$$\chi_3=\int_S\mu\left(m\frac{\partial}{\partial\xi}-l\frac{\partial}{\partial\eta}\right)\left(\frac{1}{r}\right)dS.$$

Let us consider χ_1.

Writing the integral in terms of the α–β coordinates and then invoking the integration by parts χ_1 is transformed as follows:

$$\begin{aligned}\chi_1&=\int_S\mu\left(\alpha_1\frac{\partial}{\partial s_\beta}-\beta_1\frac{\partial}{\partial s_\alpha}\right)\left(\frac{1}{r}\right)F\cdot G\,d\alpha\,d\beta\\ &=\int_S\mu\left(F\alpha_1\frac{\partial}{\partial\beta}-G\beta_1\frac{\partial}{\partial\alpha}\right)\left(\frac{1}{r}\right)d\alpha\,d\beta\\ &=\int_S\mu\left(\xi_\alpha\frac{\partial}{\partial\beta}-\xi_\beta\frac{\partial}{\partial\alpha}\right)\left(\frac{1}{r}\right)d\alpha\,d\beta\\ &=\int_S\frac{1}{r}\left(\xi_\beta\frac{\partial\mu}{\partial\alpha}-\xi_\alpha\frac{\partial\mu}{\partial\beta}\right)d\alpha\,d\beta\\ &\quad-\oint_{\partial S}\frac{\mu}{r}(\xi_\alpha d\alpha+\xi_\beta d\beta)\\ &=\int_S\frac{1}{r}\left(\beta_1\frac{\partial\mu}{\partial s_\alpha}-\alpha_1\frac{\partial\mu}{\partial s_\beta}\right)dS\\ &\quad-\oint_{\partial S}\frac{\mu}{r}(\alpha_1\,ds_\alpha+\beta_1\,ds_\beta).\end{aligned}$$

The symbol $\oint$ indicates that the line integral is in the direction such that the area of integration S is always lying to the left-hand side as one travels along the boundary ∂S.

Since the unit vector (α_1, β_1, l) is in the direction of the ξ-axis, and since the vector $(ds_\alpha, ds_\beta, 0)$ is the representation in the α–β–ν system of the element arc-length ds along ∂S we see that

$$\begin{aligned}\alpha_1 ds_\alpha+\beta_1 ds_\beta&=(\alpha_1, \beta_1, l)\cdot(ds_\alpha, ds_\beta, 0)\\ &=d\xi\end{aligned}$$

where $d\xi$ is the ξ–component of ds.

Thus, together with the similar results of χ_2 and χ_3, we have obtained the following:

$$(\chi_1, \chi_2, \chi_3)=\int_S\frac{\boldsymbol{\lambda}}{r}dS-\oint_{\partial S}\frac{\mu}{r}\boldsymbol{ds}$$

where

$$\boldsymbol{\lambda}=\left(\beta_1\frac{\partial\mu}{\partial s_\alpha}-\alpha_1\frac{\partial\mu}{\partial s_\beta},\ \beta_2\frac{\partial\mu}{\partial s_\alpha}-\alpha_2\frac{\partial\mu}{\partial s_\beta},\ \beta_3\frac{\partial\mu}{\partial s_\alpha}-\alpha_3\frac{\partial\mu}{\partial s_\beta}\right)$$

and

$$\boldsymbol{ds}=(d\xi, d\eta, d\zeta).$$

APPENDIX II

Simplification of Wing Geometry in the Neighbourhood of the Trailing-edge

In this appendix we attempt to offer a mathematical base on which the replacement of integrals over ΔS by those over ΔS_0 is justified.

We devide the content into three: (1) a method of constructing an orthogonal curvilinear coordinate system on S in the neighbourhood of a point T_0 on the trailing-edge, (2) a proof that the integrals over ΔS can be raplaced by those over ΔS_0 with respect to the singularities in $\phi_S(P)$, $\phi_D(P)$, grad ϕ_S and grad ϕ_D which may arise as P tends to T_0, and (3) a proof that the differences between the integrals over ΔS and those over ΔS_0 remain continuous as T_0 is approached.

In any part of the following discussions, the argument is not claimed to be immaculate from the mathematical point of view. The intention is to present just a sketch of a possible line of thought.

(1) An orthogonal curvilinear coordinate system on the wing surface S and its expression in the neighbourhood ΔS of a point T_0 on the trailing-edge

Let $Q(\xi, \eta, \zeta)$ be a point on S.

Suppose that S is described in terms of a surface coordinate system (s, t) as

$$\xi=\xi(s,t), \quad \eta=\eta(s,t) \quad \text{and} \quad \zeta=\zeta(s,t).$$

Our first objective is to find functions u and v:

$$u=u(s,t), \quad v=v(s,t)$$

such that

$$Q_u\cdot Q_v=0$$

identically on S,

where

$$Q_u=\left(\frac{\partial\xi}{\partial u}, \frac{\partial\eta}{\partial u}, \frac{\partial\zeta}{\partial u}\right)$$

and

$$Q_v=\left(\frac{\partial\xi}{\partial v}, \frac{\partial\eta}{\partial v}, \frac{\partial\zeta}{\partial v}\right).$$

Since

$$Q_u=Q_s\cdot s_u+Q_t\cdot t_u$$

and

$$Q_v=Q_s\cdot s_v+Q_t\cdot t_v$$

we have

$$Q_u\cdot Q_v=(Q_s\cdot Q_s)s_u s_v+(Q_s\cdot Q_t)(s_u t_v+s_v t_u) + (Q_t\cdot Q_t)t_u t_v$$

which is rewritten as

$$J^2 Q_u\cdot Q_v=(Q_s\cdot Q_t)(u_s v_t+u_t v_s) - (Q_s\cdot Q_s)u_t v_t-(Q_t\cdot Q_t)u_s v_s$$

where

$$J=u_s v_t-u_t v_s.$$

Hence $Q_u\cdot Q_v=0$ leads to

$$(Q_s\cdot Q_t)(u_s v_t+u_t v_s)=(Q_s\cdot Q_s)u_t v_t + (Q_t\cdot Q_t)u_s v_s. \quad \text{(A. II. 1)}$$

The condition $Q_u\cdot Q_v=0$ does not by itself determine the function $u(s,t)$ and $v(s,t)$. For instance we may arbitrarily fix either u or v and solve the equation (A. II. 1) for the other.

Let v be given by

$$v(s,t)=t. \quad \text{(A. II. 2)}$$

u is then the solution of

$$(Q_s\cdot Q_t)u_s=(Q_s\cdot Q_s)u_t. \quad \text{(A. II. 3)}$$

Let a function $f(s,t)$ be such that the equation $f(s,t)=C$ for arbitrary constant C is the solution of

$$\frac{ds}{dt}=-\frac{(Q_s\cdot Q_t)}{(Q_s\cdot Q_s)}. \quad \text{(A. II. 4)}$$

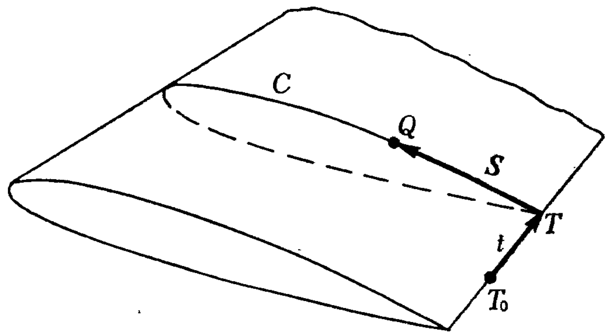
Then u is given by

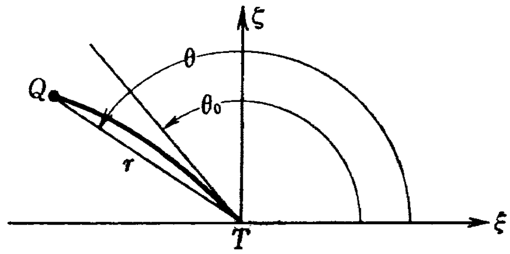
$$u=F(f(s,t)) \quad \text{(A. II. 5)}$$

with arbitrary function F.

Now suppose that the point Q is in the neighbourhood ΔS of a point T_0 on the trailing-edge which is assumed to be straight in the proximity of T_0.

Consider a plane N passing through Q and normal to the trailing-edge. Let C be the intersection of the wing surface S with N and let s be the distance along C between Q and T, where T is the point at which N meets the trailing-edge. Let, further, t be the distance between T and T_0, and r be the distance between Q and T.





Taking T_0 as the origin of coordinates, we have

$$\left.\begin{aligned}\xi&=r(s,t)\cos\theta(s,t)\\ \eta&=t\\ \zeta&=r(s,t)\sin\theta(s,t)\end{aligned}\right\} \quad \text{(A. II. 6)}$$

which leads to

$$Q_s\cdot Q_t=r_s r_t+r^2\theta_s\theta_t$$
$$Q_s\cdot Q_s=r_s^2+(r\theta_s)^2.$$

Assume that the surface S is such that in ΔS ξ_s and ζ_s are Hölder-continuous while ξ_t and ζ_t vanish like s^β, $\beta>0$ along the trailing-edge, i. e. along the line $s=0$.

Let ΔS be such that within it s and t are of same order of magnitude. Then ξ and ζ are expanded as

$$\xi=(\xi_s)_{T0}\cdot s+(\xi_t)_{T0}\cdot t+0(s^{1+\alpha})$$
$$\zeta=(\zeta_s)_{T0}\cdot s+(\zeta_t)_{T0}\cdot t+0(s^{1+\alpha})$$

leading to an estimation

$$\left.\begin{aligned}\xi&=s\cos\theta_0+0(s^{1+\alpha})\\ \zeta&=s\sin\theta_0+0(s^{1+\alpha})\end{aligned}\right\} \quad \text{(A. II. 7)}$$

with a positive constant α, where the symbol $0(s^\alpha)$ represents a term which vanishes at most like s^α as

s tends to zero.

In (A. II. 7) $\cos\theta_0$ and $\sin\theta_0$ are used in place of $(\xi_s)_{T0}$ and $(\zeta_s)_{T0}$ respectively since s is an arc-length along the lines t=const., i. e. η=const. and hence

$$(ds)^2=(d\xi)^2+(d\eta)^2+(d\zeta)^2=[(\xi_s)^2+(\zeta_s)^2](ds)^2,$$

which means $(\xi_s)_{T0}=\cos\theta_0$ and $(\zeta_s)_{T0}=\sin\theta_0$ in view of (A. II. 6) where $\theta_0=\theta(0,0)$.

Combining (A. II. 6) and (A. II. 7) we obtain the following estimation:

$$\left.\begin{aligned} r_s&=\frac{\partial}{\partial s}\sqrt{\xi^2+\eta^2}=1+0(s^\alpha)\\ r\theta_s&=(\xi\eta_s-\xi_s\eta)/r=0(s^\alpha)\end{aligned}\right\}. \qquad \text{(A. II. 8)}$$

Now put

$$u=r+\varepsilon.$$

Since u is a solution of (A. II. 3), ε should satisfy the following:

$$\varepsilon_t-\frac{(Q_s\cdot Q_t)}{(Q_s\cdot Q_s)}\varepsilon_s=\frac{(r\theta_s)[r_s(r\theta_t)-r_t(r\theta_s)]}{r_s^2+(r\theta_s)^2}.$$

By the definition the left-hand side of the above equation is equal to $d\varepsilon/dt$ along a curve $f(s,t)=C$ given by (A. II. 4).

Hence

$$\varepsilon=\int_{f(s,t)=C}\frac{(r\theta_s)[r_s(r\theta_t)-r_t(r\theta_s)]}{r_s^2+(r\theta_s)^2}dt.$$

Since

$$r(r_s\theta_t-r_t\theta_s)=\xi_s\zeta_t-\xi_t\zeta_s$$

by (A. II. 6) and

$$\xi_s\zeta_t-\xi_t\zeta_s=0(s^\beta)$$

in ΔS as has been postulated, we have

$$\varepsilon\sim\int 0(s^{\alpha+\beta})dt\sim 0(s^{1+\alpha+\beta})$$

because $t\sim 0(s)$ in ΔS.

Thus we are led to an estimation that

$$\begin{aligned} u&=r+0(s^{1+\alpha+\beta})\\ &=s+0(s^{1+\alpha}), \end{aligned} \qquad \text{(A. II. 9)}$$

since

$$r=s+0(s^{1+\alpha})$$

in view of (A. II. 8), and that

$$\left.\begin{aligned} \xi&=u\cos\theta_0+0(u^{1+\alpha})\\ \eta&=v\\ \zeta&=u\sin\theta_0+0(u^{1+\alpha})\end{aligned}\right\} \qquad \text{(A. II. 10)}$$

by virtue of (A. II. 7) and (A. II. 2).

As a final step, expressions of the distance r between $P(x,y,z)$ and $Q(\xi,\eta,\zeta)$ are considered for the later reference:

$$r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2.$$

Putting

$$x=D\cdot\cos\varphi \quad\text{and}\quad z=D\cdot\sin\varphi \qquad \text{(A. II. 11)}$$

and using (A. II. 10), r^2 is written as

$$\begin{aligned} r^2&=[D\cos\varphi-u\cos\theta_0+0(u^{1+\alpha})]^2+(y-v)^2\\ &\quad+[D\sin\varphi-u\sin\theta_0+0(u^{1+\alpha})]^2\\ &=r_0^2+2[A(D\cos\varphi-u\cos\theta_0)\\ &\quad+B(D\sin\varphi-u\sin\theta_0)]\cdot 0(u^{1+\alpha}) \end{aligned} \qquad \text{(A. II. 12)}$$

where A and B are constants and

$$r_0^2=u^2-2uD\cdot\cos(\varphi-\theta_0)+D^2+(y-v)^2. \qquad \text{(A. II. 13)}$$

Since

$$\begin{aligned}[][u\sin(\varphi-\theta_0)]^2&\leq[u\sin(\varphi-\theta_0)]^2\\ &\quad+[D-u\cos(\varphi-\theta_0)]^2+(y-v)^2=r_0^2\end{aligned} \qquad \text{(A. II. 14)}$$

and

$$|A(D\cos\varphi-u\cos\theta_0)+B(D\sin\varphi-u\sin\theta_0)|\leq\sqrt{A^2+B^2}\,r_0$$

the follwing estimation follows from (A. II. 12):

$$r=r_0[1+0(u^\alpha)] \qquad \text{(A. II. 15)}$$

provided

$$\sin(\varphi-\theta_0)\neq 0. \qquad \text{(A. II. 16)}$$

Taking ΔS sufficiently small, we can make the factor $1+0(u^\alpha)$ always positive in ΔS.

Then the following inequality holds in ΔS under the condition (A. II. 16) with a positive constant κ:

$$\kappa\cdot r_0\leq r. \qquad \text{(A. II. 17)}$$

(2) Singularities in grad $\phi_D(P)$, etc. at a point T_0 on the trailing-edge

Let $R(A)$ denote the singular terms arising in quantity $A(x,y,z)$ when the point $P(x,y,z)$ approaches T_0.

In this section we intend to show the following:

$$R\left[\int_{\Delta S}\frac{\sigma}{r}dS\right]=R\left[\sigma(T_0)\int_{\Delta S_0}\frac{1}{r}dS\right], \qquad \text{(A. II. 18)}$$

$$R\left[\text{grad}\left(\int_{\Delta S}\frac{\sigma}{r}dS\right)\right]=R\left[\sigma(T_0)\,\text{grad}\left(\int_{\Delta S_0}\frac{1}{r}dS\right)\right], \qquad \text{(A. II. 19)}$$

$$R\left[\int_{\Delta S}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS\right]=R\left[\mu(T_0)\int_{\Delta S_0}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS\right], \qquad \text{(A. II. 20)}$$

and

$$\begin{aligned} R\left[\text{grad}\left\{\int_{\Delta S}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS\right\}\right] &= R\left[\text{rot}\left\{\boldsymbol{\lambda}(T_0)\int_{\Delta S_0}\frac{1}{r}dS\right\}\right.\\ &\quad\left.-\text{rot}\left\{\mu(T_0)\oint_{\partial\Delta S_0}\frac{1}{r}d\boldsymbol{s}\right\}\right] \end{aligned} \qquad \text{(A. II. 21)}$$

where in the last expression we have used the relation:

$$\text{grad}\left\{\int_{\Delta S}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS\right\}=\text{rot}\left\{\int_{\Delta S}\frac{\boldsymbol{\lambda}}{r}dS-\oint_{\partial\Delta S}\frac{\mu}{r}d\boldsymbol{s}\right\}$$

in which $\partial\Delta S$ denotes the boundary of ΔS.

Let ΔS be described as

$$0<u<\varepsilon,\quad -\varepsilon<v<\varepsilon$$

in terms of the coordinates (u,v) defined in the previous section. Likewise let ΔS_0 be given by

$$0<\bar{u}<\varepsilon,\quad -\varepsilon<\bar{v}<\varepsilon$$

where $(\bar{u},\bar{v})$ is such that a point $Q_0(\bar{\xi},\bar{\eta},\bar{\zeta})$ on ΔS_0 is expressed as

$$\left.\begin{aligned}\bar\xi&=\bar u\cos\theta_0\\ \bar\eta&=\bar v\\ \bar\zeta&=\bar u\sin\theta_0\end{aligned}\right\}$$

where θ_0 is as defined in the paragraph following (A.II.7).

In this section we assume that $P(x, y, z)$ is located outside ΔS and ΔS_0.

Without loss of generality we may put

$$y=0.$$

Let (x, z) be given by (A.II.11).

From the above assumption we have

$$\varphi\neq\theta_0.$$

Put

$$u=\rho\sin\omega \quad\text{and}\quad v=\rho\cos\omega,\quad 0<\omega<\pi.$$

Then

$$\begin{aligned}r_0^2&=u^2+v^2+D^2-2uD\cos(\varphi-\theta_0)\\ &=\rho^2+D^2-2\rho D\sin\omega\cos(\varphi-\theta_0)\\ &=[\rho-D\sin\omega\cos(\varphi-\theta_0)]^2\\ &\quad+D^2[1-\sin^2\omega\cos^2(\varphi-\theta_0)].\end{aligned}$$

It can easily be shown that as long as $\varphi\neq\theta_0$, there exist two positive constants C_0 and $\sin\phi$ such that

$$C_0[(\rho-D\cos\phi)^2+(D\sin\phi)^2]<r_0^2$$

and hence, by virtue of (A.II.17),

$$C_1[(\rho-D\cos\phi)^2+(D\sin\phi)^2]<r^2$$

where $0<C_1=C_0\kappa^2$.

We first consider an integral given as

$$\begin{aligned}I&=\int_{\Delta S}\frac{\lambda(u,v)}{r^n}dS-\int_{\Delta S_0}\frac{\lambda(0,0)}{r^n}dS\\ &=I_1+\lambda(0,0)I_2\end{aligned}$$

where

$$I_1=\int_{\Delta S}\frac{\lambda(u,v)-\lambda(0,0)}{r^n}dS$$

and

$$I_2=\int_{\Delta S}\frac{1}{r^n}dS-\int_{\Delta S_0}\frac{1}{r^n}dS.$$

Assume that the density $\lambda(u, v)$ is Hölder-continuous on ΔS with index β:

$$|\lambda(u,v)-\lambda(u_0,v_0)|\leq K\sqrt{(u-u_0)^2+(v-v_0)^2}^{\,\beta},\qquad \beta>0.$$

Then

$$\begin{aligned}|I_1|&\leq\int_0^\epsilon du\int_{-\epsilon}^\epsilon dv\frac{K\sqrt{u^2+v^2}^{\,\beta}}{r^n}\\ &\leq\frac{K}{C_1'}\int_0^{\sqrt2\epsilon}\rho d\rho\int_0^\pi d\omega\frac{\rho^\beta}{\sqrt{(\rho-D\cos\phi)^2+(D\sin\phi)^2}^{\,n}}.\end{aligned}$$

The last inequality is obvious since the integrand is positive and the area of integration is larger in (ρ, ω) than in (u, v).

Putting

$$\tan\chi=\frac{\rho-D\cos\phi}{D\sin\phi}$$

the last integral can be written as

$$\begin{aligned}&\int_0^{\sqrt2\epsilon}\rho d\rho\int_0^\pi d\omega\frac{\rho^\beta}{\sqrt{(\rho-D\cos\phi)^2+(D\sin\phi)^2}^{\,n}}\\ &=\pi\frac{D^{2+\beta-n}}{(\sin\phi)^{n-1}}\int_{\chi_0}^{\chi_1}[\cos(\chi-\phi)]^{1+\beta}(\cos\chi)^{n-(3+\beta)}d\chi\end{aligned}$$

where

$$\chi_0=\mathrm{Tan}^{-1}\left(-\frac{\cos\phi}{\sin\phi}\right)=\phi-\frac{\pi}{2},$$

and

$$\chi_1=\mathrm{Tan}^{-1}\left(\frac{\sqrt2\epsilon-D\cos\phi}{D\sin\phi}\right).$$

Since $-\pi/2<\chi_0$, $\chi_1<\pi/2$, the integral on the right-hand side is positive. (i. e. does not vanish for all D.) We see from the above result that $|I_1|$ is likely to diverge as $D\to0$ when $2+\beta<n$.

Therefore we restrict ourselves to cases where $n<2+\beta$.

Since

$$\begin{aligned}&(\rho-D\cos\phi)^2+(D\sin\phi)^2\\ &\quad=(\rho\sin\phi)^2+(D-\rho\cos\phi)^2>(\rho\sin\phi)^2\end{aligned}$$

we see that

$$\begin{aligned}&\int_0^{\sqrt2\epsilon}\frac{\rho^{1+\beta}d\rho}{\sqrt{(\rho-D\cos\phi)^2+(D\sin\phi)^2}^{\,n}}\\ &\quad<\frac{1}{(\sin\phi)^n}\int_0^{\sqrt2\epsilon}\rho^{1+\beta-n}d\rho\\ &\quad=\frac{1}{(\sin\phi)^n}\cdot\frac{1}{2+\beta-n}(\sqrt2\epsilon)^{2+\beta-n}.\end{aligned}$$

Hence

$$|I_1|<C\cdot\epsilon^{2+\beta-n}$$

regardless the value of D as long as $\varphi\neq\theta_0$ and $n<2+\beta$, which implies

$$R\left[\int_{\Delta S}\frac{\lambda(u,v)}{r^n}dS\right]=R\left[\int_{\Delta S}\frac{\lambda(0,0)}{r^n}dS\right]$$

under the same assumptions.

Let us next proceed to the examination of I_2.

In view of the preceding result, we consider only the cases $n=1$ and $n=2$.

For $n=1$ the two integrals composing I_2 are themselves finite whence no singularities in I_2 as $D\to0$. Because:

$$\left|\int_{\Delta S}\frac{dS}{r}\right|<\frac{\pi}{\sqrt{C_1}}\int_0^{\sqrt2\epsilon}\frac{\rho d\rho}{\sqrt{(\rho-D\operatorname{dos}\phi)^2+(D\sin\phi)^2}}\qquad\text{(A.II.22)}$$

and the integrand of the right-hand side are uniformly continuous in the region of integration as $D\to0$.

For $n=2$: we have, in general,

$$\begin{aligned}&\int_{\Delta S_0}\frac{1}{r^n}dS\\ &=\int_0^\epsilon d\bar u\int_{-\epsilon}^\epsilon d\bar v\frac{1}{\sqrt{\bar u^2+\bar v^2+D^2-2\bar uD\cos(\varphi-\theta_0)}^{\,n}}\\ &=\int_0^\epsilon du\int_{-\epsilon}^\epsilon dv\frac{1}{\sqrt{u^2+v^2+D^2-2uD\cos(\varphi-\theta_0)}^{\,n}}.\end{aligned}$$

That is, the integration over ΔS_0 is expressed in terms of the variables (u, v) as is the integration over the corresponding ΔS.

Put

$$K=\frac{1}{r_2}-\frac{1}{r_0^2}$$

where r and r_0 are given by (A.II.12) and (A.II.13), respectively, by putting $y=0$.

r^2 can be expressed as

$$r^2=r_0^2+D\cdot 0(u^{1+\alpha})+0(u^{2+\alpha})$$

whence by virtue of (A.II.17)

$$|K|\leq\frac{A\cdot D\cdot u^{1+\alpha}+B\cdot u^{2+\alpha}}{\kappa[(u-Dc)^2+(Ds)^2+v^2]^2}$$

provided $\varphi\neq\theta_0$, where A and B are positive constants, and

$$c=\cos(\varphi-\theta_0) \text{ and } s=|\sin(\varphi-\theta_0)|>0.$$

I_2 is then estimated as

$$|I_2|<\frac{A}{\kappa}\int_0^\varepsilon du\int_{-\varepsilon}^{\varepsilon}dv\frac{D\cdot u^{1+\alpha}}{[(u-Dc)^2+(Ds)^2+v^2]^2}+\frac{B}{\kappa}\int_0^\varepsilon du\int_{-\varepsilon}^{\varepsilon}dv\frac{u^{2+\alpha}}{[(u-Dc)^2+(Ds)^2+v^2]^2}.$$

Since

$$s^2(u^2+v^2)\leq(u-Dc)^2+(Ds)^2+v^2$$

the secoud integral is bounded as

$$\int_0^\varepsilon du\int_{-\varepsilon}^{\varepsilon}\frac{u^{2+\alpha}}{[(u-Dc)^2+(Ds)^2+v^2]^2}<\frac{2\pi}{s^2}\int_0^{\sqrt{2}\varepsilon}\rho^{\alpha-1}d\rho<C\varepsilon^\alpha.$$

For cases where $\alpha>1$, the first integral is reduced to the second one except for the factor D. For $\alpha=1$ the integration is performed in terms of elementary functions showing that it vanishes like D as D tends to zero.

For $\alpha<1$ we have

$$\int_0^\varepsilon du\int_{-\varepsilon}^{\varepsilon}dv\frac{Du^{1+\alpha}}{[(u-Dc)^2+(Ds)^2+v^2]^2}<\frac{D^\alpha}{s^2}\int_{\theta_0}^{\theta_1}[\cos(\theta-\phi_0)]^{1+\alpha}\left[\frac{\delta\cos^{1-\alpha}\theta}{\delta^2+\cos^2\theta}+\frac{\pi}{2}(\cos\theta)^{-\alpha}\right]d\theta$$

where

$$\theta_0=\mathrm{Tan}^{-1}\left(-\frac{c}{s}\right),\quad \theta_1=\mathrm{Tan}^{-1}\left(\frac{\varepsilon-Dc}{Ds}\right),$$

$$\phi_0=\mathrm{Tan}^{-1}\left(\frac{s}{c}\right) \text{ and } \delta=\frac{Ds}{\varepsilon}.$$

Hence in this case the first integral vanishes like D^α as $D\to 0$.

Thus in any event we have

$$\lim_{D\to 0}\left\{D\int_0^\varepsilon du\int_{-\varepsilon}^{\varepsilon}dv\frac{u^{1+\alpha}}{[(u-Dc)^2+(Ds)^2+v^2]^2}\right\}=0$$

and hence

$$|I_2|<C\varepsilon^\alpha$$

which, together with the result about I_1, leads to

$$R\left[\int_{\Delta S}\frac{\lambda(u,v)}{r^n}dS\right]=R\left[\int_{\Delta S_0}\frac{\lambda(0,0)}{r^n}dS\right] \quad \text{(A.II.23)}$$

for both $n=1$ and $n=2$ provided the point $P(x,y,z)$ is outside of ΔS and λ is Hölder-continuous on ΔS.

The assertions (A.II.18), (A.II.19), and (A.II.20) are reduced to cases where the kernels in the integrands are at most of type of $1/r^2$ and hence a sweeping proof of those assertions is provided by (A.II.23) under the assumption that σ and μ are Hölder-continuous on ΔS.

As for (A.II.21) we must show

$$R\left[\mathrm{rot}\oint_{\partial\Delta S}\frac{\mu}{r}ds\right]=R\left[\mathrm{rot}\left\{\mu(T_0)\oint_{\partial\Delta S_0}\frac{1}{r}ds\right\}\right]$$

in addition to (A.II.23) to assure its validity.

The singularities in these line integrals as the point T_0 is approached arise solely from the parts of the trailing-edge contained in $\partial\Delta S$ and $\partial\Delta S_0$ respectively. Since the trailing-edge is assumed to be straight, these two parts of the trailing-edge coincide with each other by virtue of the manner in which ΔS_0 is constructed from ΔS. Moreover, the density μ is constant and equal to $\mu(T_0)$ along this part of the integration path because of the condition established in Section 5 of the main text. (formula (5.18)) Hence the portions of the above two line integrals responsible for the singularity are identical with each other thus confirming our assertion.

(3) Continuity Properties of ϕ_S and ϕ_D at a point T_0 on the trailing-edge

What to be shown in this section is that the integrals:

$$I_S(P)=\int_{\Delta S}\frac{\sigma}{r}dS-\sigma(T_0)\int_{\Delta S_0}\frac{1}{r}dS$$

and

$$I_D(P)=\int_{\Delta S}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS-\mu(T_0)\int_{\Delta S_0}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS,$$

when the point P lies on ΔS, are not responsible for the possible discontinuities in ϕ_S and ϕ_D respectively which may arise as P approaches a point T_0 on the trailing-edge. To prove this assertion it is sufficient to show that $|I_S(P)|$ and $|I_D(P)|$ are uniformly bounded in ΔS and tend to zero as ΔS, and correspondingly ΔS_0, shrinks to T_0.

Let (ξ_p,η_p,ζ_p) be the coordinates of a point p lying on, say, ΔS_U. As a consequence of (A.II.10) we have

$$\left.\begin{aligned}\xi-\xi_p&=(u-u_p)\cos\theta_0+0(\rho_p^{1+\alpha})\\ \eta-\eta_p&=v-v_p\\ \zeta-\zeta_p&=(u-u_p)\sin\theta_0+0(\rho_p^{1+\alpha})\end{aligned}\right\}$$

for $Q(\xi,\eta,\zeta)$ belonging to ΔS_U and hence

$$r^2=(\xi_p-\xi)^2+(\eta_p-\eta)^2+(\zeta_p-\zeta)^2=\rho_p^2[1+0(\rho_p^\alpha)]$$

for the integration over ΔS_U where

$$\rho_p^2=(u-u_p)^2+(v-v_p)^2.$$

Let us consicer I_s first.

In fact the integral $\int_{\Delta S}(\sigma/r)dS$ itself is shown to be uniformly bounded provided the density σ is so, because: put

$$J=\int_{\Delta S}\frac{dS}{r}=\int_{\Delta S_U}\frac{dS}{r}+\int_{\Delta S_L}\frac{dS}{r}.$$

By adopting a polar coordinate system (ρ_p, ω_p) with p as its origin, the first integral in J is seen to be bounded as follows:

$$0<\int_{\Delta S_U}\frac{dS}{r}<\int\rho_p d\rho_p\frac{2\pi}{\rho_p[1+0(\rho_p{}^\alpha)]}$$
$$<C\int_0^{\sqrt{5}\,\varepsilon}d\rho_p<C\cdot\varepsilon$$

where C is a positive constant.

As for the second integral in J, the point p is outside of ΔS_L and the boundedness of J by an order of magnitude of ε is already shown in the preceding section. (cf. (A. II. 22))

Next the integral I_D is considered.

Put

$$I_D(p)=J_U(p)+J_L(p)+J_W(p)$$

where

$$J_U(p)=\int_{\Delta S_U}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS-\mu_U\int_{\Delta S_{0U}}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS,$$
$$J_L(p)=\int_{\Delta S_L}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS-\mu_L\int_{\Delta S_{0L}}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS,$$

and

$$J_W(p)=\int_{\Delta S_W}\mu\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS-\mu_W\int_{\Delta S_{0W}}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS.$$

Since

$$\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)=\frac{l(\xi_P-\xi)+m(\eta_p-\eta)+n(\zeta_p-\zeta)}{r^3}$$

and

$$\left.\begin{aligned}l&=-\sin\theta_0+0(\rho_p{}^\alpha)\\ m&=0(\rho_p{}^\alpha)\\ n&=\cos\theta_0+0(\rho_p{}^\alpha)\end{aligned}\right\}$$

for $Q(\xi,\eta,\zeta)$ on ΔS_U, we have

$$\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)=\frac{0(\rho_p{}^{1+\alpha})}{\rho_p{}^3\sqrt{1+0(\rho_p{}^\alpha)}^3}=A\rho_p{}^{\alpha-2}$$

for Q on ΔS_U where A is a function of ρ_p and ω_p which is uniformly bounded in ΔS_U.

Since all the terms expressed by the symbol $0(\rho_p{}^\alpha)$ are absent on ΔS_0, we have

$$\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)=0$$

for ΔS_{0U}.

Hence

$$|J_U(p)|<C\int_0^{\sqrt{5}\,\varepsilon}\rho_p{}^{\alpha-1}d\rho_p<C\varepsilon^\alpha$$

provided μ is bounded on ΔS_U.

Next we evaluate $|J_L(p)|$ for $p\in\Delta S_U$.

On ΔS_L we have

$$\left.\begin{aligned}\xi&=u\cos\theta_L+0(u^{1+\alpha})\\ \eta&=v\\ \zeta&=u\sin\theta_L+0(u^{1+\alpha})\end{aligned}\right\}$$

and

$$\left.\begin{aligned}l&=-\sin\theta_L+0(u^\alpha)\\ m&=0(u^\alpha)\\ n&=\cos\theta_L+0(u^\alpha)\end{aligned}\right\}.$$

Put

$$\left.\begin{aligned}\xi_p&=D\cos\varphi\\ \eta_p&=0\\ \zeta_p&=D\sin\varphi\end{aligned}\right\}.$$

Then

$$\begin{aligned}r^2&=(\xi_p-\xi)^2+(\eta_p-\eta)^2+(\zeta_p-\zeta)^2\\ &=u^2+v^2+D^2-2uD\cos(\varphi-\theta_L)\\ &\quad+D\cdot 0(\rho^{1+\alpha})+0(\rho^{2+\alpha})\end{aligned}$$

and

$$\begin{aligned}N&\equiv l(\xi_p-\xi)+m(\eta_p-\eta)+n(\zeta_p-\zeta)\\ &=D[\sin(\varphi-\theta_L)+0(\rho^\alpha)]+0(\rho^{1+\alpha})\end{aligned}$$

where

$$\rho=\mu^2+v^2.$$

The corresponding expressions r_0 and N_0 for the point (ξ_0,η_0,ζ_0) on ΔS_{0L} corresponding to (ξ,μ,ζ) are derived from the above by suppressing all the terms represented by the symbols $0(\rho^\alpha)$, etc.

Thus

$$r_0{}^2=u^2+v^2+D^2-2uD\cos(\varphi-\theta_L)$$

and

$$N_0=D\sin(\varphi-\theta_L).$$

Now

$$J_L(p)=\int_{\Delta S_L}(\mu-\mu_L)\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS$$
$$+\mu_L\left\{\int_{\Delta S_L}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS-\int_{\Delta S_{0L}}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS\right\}.$$

Assume that μ is Hölder-continuous on ΔS_L with index β. Since p is located outside ΔS_L, i.e. $\varphi\neq\theta_L$, we have the following estimation:

$$\left|\int_{\Delta S_L}(\mu-\mu_L)\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS\right|<C_1\int_0^\varepsilon du\int_{-\varepsilon}^\varepsilon dv\sqrt{u^2+v^2}^\beta\frac{|D[\sin(\varphi-\theta_L)+0(\rho^\alpha)]+0(\rho^{1+\alpha})|}{\sqrt{u^2+v^2+D^2-2uD\cos(\varphi-\theta_L)+D\cdot 0(\rho^{1+\alpha})+0(\rho^{2+\alpha})}^3}$$
$$<C_2\int_0^{\sqrt{2}\,\varepsilon}\rho d\rho\frac{\rho^\beta[D|\sin(\varphi-\theta_L)|+A\rho^{1+\alpha}]}{\sqrt{(\rho-D\cos\varphi)^2+(D\sin\varphi)^2}^3}$$

which, by a similar procedure to that taken in the preceding section (cf. the estimation of I_2), leads to

$$\left|\int_{\Delta S_L}(\mu-\mu_L)\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS\right|<C\varepsilon^\alpha. \qquad \text{(A. II. 24)}$$

Finally

$$\int_{\Delta S_L}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS-\int_{\Delta S_{0L}}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS=\int_0^\varepsilon du\int_{-\varepsilon}^\varepsilon dv\left[\frac{D\{\sin(\varphi-\theta_L)+0(\rho^\alpha)\}+0(\rho^{1+\alpha})}{\sqrt{u^2+v^2+D^2-2uD\cos(\varphi-\theta_L)+D\cdot 0(\rho^{1+\alpha})+0(\rho^{2+\alpha})}^3}\right.$$
$$\left.-\frac{D\sin(\varphi-\theta_L)}{\sqrt{u^2+v^2+D^2-2uD\cos(\varphi-\theta_L)}^3}\right]$$

$$=\int_0^\varepsilon du\int_{-\varepsilon}^{\varepsilon} dv\frac{K}{\sqrt{u^2+v^2+D^2-2uD\cos(\varphi-\theta_L)}^3}$$

where

$$K=[1+0(\rho^\alpha)][D\{\sin(\varphi-\theta_L)+0(\rho^\alpha)\}+0(\rho^{1+\alpha})]-D\sin(\varphi-\theta_L).$$

Hence we obtain an estimation:

$$\left|\int_{\Delta S_L}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS-\int_{\Delta S_{0L}}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS\right|<C_1\int_0^\varepsilon du\int_{-\varepsilon}^{\varepsilon} dv\frac{D\cdot\rho^\alpha+\rho^{1+\alpha}}{\sqrt{u^2+v^2+D^2-2uD\cos(\varphi-\theta_L)}^3}<C\varepsilon^\alpha \quad \text{(A. II. 25)}$$

in the analogous manner to the preceding one.

Combining (A. II. 24) and (A. II. 25) we see that $J_L(p)$ is uniformly bounded by an order of ε^α.

A similar estimation is effected with $J_W(p)$ resulting in the boundedness of $J_W(p)$ by $C\cdot\varepsilon^\alpha$, thus confirming our assertion for the case where p lies on ΔS_U. The result, however, is obviously unaltered if p is located on ΔS_L or, for I_D, on ΔS_W, which completes the proof.

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