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On the Linear Theory of Thin Elastic Shells

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On the Linear Theory of Thin Elastic Shells*

By Susumu TODA** and Tatsuzo KOGA***

概　要

線形シェル理論は Love の1次近似理論が提案されて以来多くの研究者によって研究され，これまでに数多くの理論が提案されている。ところで構造解析を行なう技術者は，どの理論を適用すればよいかという選択の重要性を感じながらも古くから用いられていて，しかも簡潔に表現されている Love の理論または Donnell の理論を習慣的に用いることが多い。したがって各理論相互の関係を明確にし，その長短を明らかにすることは重要な課題であるが，これまでになされた多くのすぐれた試みはいずれもテンソル計算と微分幾何の知識を駆使して，普遍的な一般論を展開するあまりに，直感的には難解なものになっている。本報告の目的は，高度な数学的手法を出来るだけさけて単純な誤差精度の比較により近似の課程に得られる種々の表示式と在来の近似式を関連づけることにより，線形シェル理論の一つの考え方を示すことにある。第1節では基本仮定について述べる。すなわち，本論文では Kirchhoff-Love の仮定に基く線形理論を前提とし，均質等方性で Hooke の法則が成立ち，厚さ h が一様でかつ最小曲率半径 R_m との比 h/R_m が1に比較して十分に小さい薄肉シェルについて考える。第2節には以下の議論の展開に必要となる弾性論からの基礎式を列挙し，第3節で歪と変位の近似関係式を導びき在来の近似式との比較検討を行ない，各理論の相違はせん断ひずみ γ_{12} の表示とくにそのうちの捩り量の定義にあることを示す。第4節では近似構成方程式について考える。すなわち，誤差の次数に従って1次，2次および3次などの近似式を考えることができ，それらを法線まわりのモーメントに関する平衡方程式を恒等的に満足するように修正して4つの近似構成方程式を提案することが出来る。そのうちの3つは各々 Naghdi, Flügge-Lur'e-Byrne, および Koiter の構成方程式と一致するが，修正1次式（II）には対応する在来理論は見当らない。第5節では代表的なシェルの一例としては円筒殻の場合を考える。4節で得た近似構成方程式を用いて，平衡方程式を Donnell の方程式にならい垂直方向変位 w で書きあらわし，これら各式の特性根の間には大きな相違がないことを示す。最後に第6節では誤差の評価をより厳密に行なえる一方法として，正値汎関数の変分による議論を簡単に行なう。すなわちひずみエネルギーを変分することによっても第4節で得たすべての近似構成方程式を得ることができることを示す。

ABSTRACT

The linear theory of thin elastic shells is discussed within the framework of Kirchhoff-Love's hypothesis. Various approximate expressions of the strain-displacement relations and the constitutive equations are derived simply neglecting the terms of the order of magnitude of the ratio of the wall-thickness to the least radius of curvature of the shell in comparison with the other terms within the equations. Identity and equivalence of these expressions with the existing ones are indicated. Four sets of the approximate expressions of the constitutive equations are presented, all of which satisfy the equilibrium equations referring to the balance of moment about the normal to the middle surface. Three of them turn out to be identical to those of the existing theories, whereas the remaining one appears to be new. For each of these constitutive equations, the equilibrium equations are written in terms of the displacement in an analogous manner of Donnell for a circular cylindrical shell. Comparison of the resulting eighth order differential equations is made by calculating eigenvalues for some particular cases of loading. The result shows no appreciable difference among the solutions.

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NOTATION

A	amplitude of the displacement defined in Eq. (47)
A_1, A_2	Lamé coefficients
D, K	elastic constants defined in Eqs. (26)
E, G	Young's modulus and shear modulus, respectively
h, L, R	thickness, length and radius of the shell
k	quantity defined in Eq. (41)
N_1, N_2, N_{12}, N_{21}, Q_1, Q_2	stress resultants
M_1, M_2, M_{12}, M_{21}	stress couples
p	quantity appears in Eq. (45)
q_1, q_2, q_n	components of the surface load
R_1, R_2	principal radii of curvature of the shell
R_m	the least value of R_1 and R_2
u_1, u_2, w_z	components of displacement
x, z	longitudinal and normal coordinates, respectively
w, y	nondimensional form of w_z and x, respectively
α_1, α_2	line of curvature coordinates in the midsurface
$\beta_\alpha, \beta_{t\alpha}$	normal and tangential components of angular change in the tangents to the coordinate lines
$\varepsilon_1, \varepsilon_2, \gamma_{12}$	strain components
$\varepsilon_1{}^0, \varepsilon_2{}^0, \gamma_{12}{}^0$	strain components in the midsurface
$\kappa_1{}^0, \kappa_2{}^0$	changes in curvature
$\sigma_1, \sigma_2, \tau_{12}$, τ_{1n}, τ_{2n}	stress components
$\tau^0, \tau_1, \tau_2, \tau_3$	quantities defined in Eqs. (17), (19), (20)
θ	circumferential coordinate of circular cylinder

1. INTRODUCTION

Since A. E. H. Love[1] developed well-known Love's first approximation of the theory of thin elastic shells, a number of researchmen have presented various theories, proposed modifications on Love's theory, and argued on the validity of the first approximation. Survey of the state of art and comparison of various theories have also been made by many authors. These include the writings of W. Nash[2], P. M. Naghdi[3] and W. T. Koiter[4] The most up-to-date account on the recent development of the shell theory can be found in the report of W. T. Koiter and J. G. Simmonds[5]. With a few exceptions, most authors derived their equations on the basis of the Kirchhoff-Love hypothesis. Wide variety in the resulting equations is due to difference in rigour and manner of approximation in the subsequent analysis.

The purpose of the present paper is to provide yet another look at the linear theory of thin elastic shells. Derivation of the approximate strain-displacement relations and the expressions of the stress resultants and couples, and correlation of them with the existing ones are made, avoiding the use of sophisticated mathematical means, in the hope that they may provide an additional information to the practitioners of the shell theory. The work presented in this paper is made within the framework of Kirchhoff-Love's hypothesis, which may be stated such that (a) plane cross sections normal to the undeformed middle surface remain normal to the deformed middle surface and no change in length takes place in the normal straight line in the cross section, and that (b) the components of stress normal to the middle surface are small and may be neglected in comparison with the other components. In addition, smallness of strains, displacements, and of the ratio of the thickness h to the least radius of curvature R_m is assumed throughout the present paper. It is also assumed that the shell is homogeneous, isotropic and linearly elastic with the elestic constants of Young's modulus E and Poisson's ratio ν. The line of curvature coordinates system (α_1, α_2) is defined in the middle surface. The third coordinate z is so chosen that it measures the normal distance from the middle surface, and that (α_1, α_2, z) forms a triply orthogonal curvilinear coordinates system (see Fig.1).

The assumption of smallness of the thickness to the least radius of curvature implies that

$$|z/R_m| \ll 1 \tag{1}$$

Approximation is made neglecting the small terms of the order of magniture of z/R_m in comparison with the other terms of the order of magnitude of unity.

In Section 2, the basic equations are listed. Various approximate expressions of the strain-displacement relations are derived, and identity and equivalence of them with the existing ones are made in Section 3. Approximate expressions of the constitutive equations are presented in Section 4. There, the assumption is made that the strains, $\varepsilon_1{}^0$, $\varepsilon_2{}^0$, $\gamma_{12}{}^0$, are of the

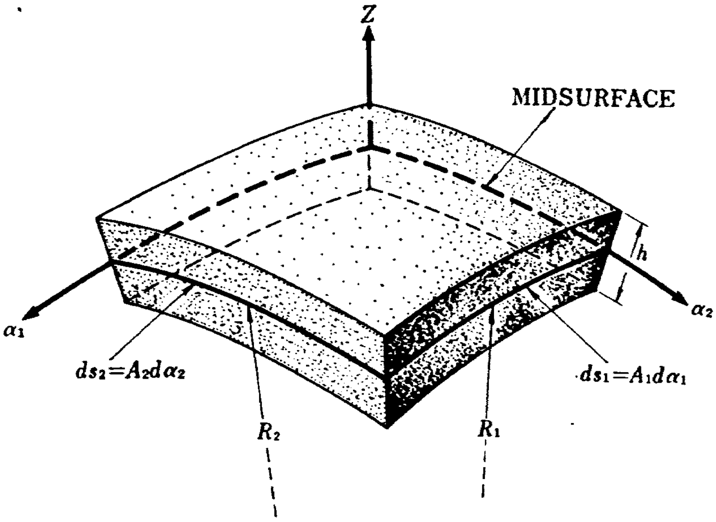

Fig. 1(a) Differential Element of a Shell

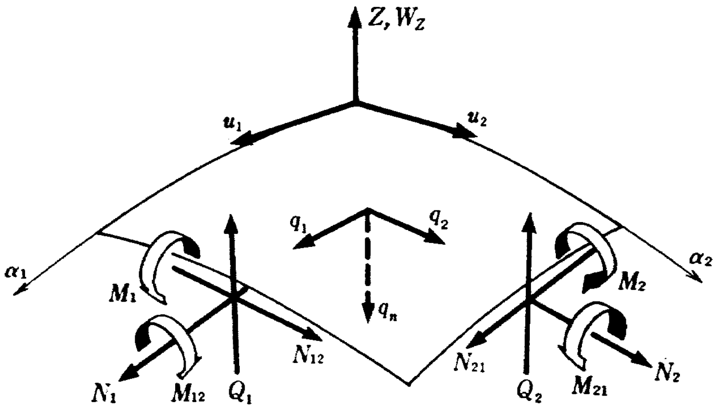

Fig. 1(b) Middle Surface and Notation

same order of magnitude of the change in curvature and the torsion multiplied by the thickness, $h\kappa_1^0$, $h\kappa_2^0$, $h\tau$. The constitutive equations are modified by adding small terms of order of magnitude of errors involved in the original approximate equations, so that the equilibrium equation referring to the balance of moment about the normal to the middle surface is satisfied identically. In Section 5, the equilibrium equations are written in terms of the displacement for a circular cylindrical shell. Eigenvalues are calculated for some particular cases of loading. Finally, in Section 6, a brief discussion is made on the variational approach using the strain energy functionals.

2. BASIC EQUATIONS

For the coordinates system shown in Fig. 1, Hooke's law reads

$$\left.\begin{aligned}&\varepsilon_1=(\sigma_1-\nu\sigma_2)/E, \quad \varepsilon_2=(\sigma_2-\nu\sigma_1)/E\\&\gamma_{12}=\tau_{12}/G\end{aligned}\right\} \tag{2}$$

where ε_i and σ_i $(i=1,2)$ are, respectively, the normal strains and stresses, γ_{12} and τ_{12} are, respectively, the shear strain and stress, and G is the shear modulus.

Let u_1, u_2, w_z be the components of the infinitesimal displacement in the middle surface in the directions of the base vectors tangent to the α_1, α_2 and z, respectively, and let the strains in the middle surface be designated by the superscript 0. Then the strain-displacement relations in the middle surface are given by

$$\left.\begin{aligned}&\varepsilon_1^0=(1/A_1)\partial u_1/\partial\alpha_1+(u_2/A_1A_2)\partial A_1/\partial\alpha_2+w_z/R_1\\&\varepsilon_2^0=(1/A_2)\partial u_2/\partial\alpha_2+(u_1/A_1A_2)\partial A_2/\partial\alpha_1+w_z/R_2\\&\gamma_{12}^0=(1/A_1)\partial u_2/\partial\alpha_1+(1/A_2)\partial u_1/\partial\alpha_2\\&\qquad-(u_1/A_1A_2)\partial A_1/\partial\alpha_2-(u_2/A_1A_2)\partial A_2/\partial\alpha_1\end{aligned}\right\} \tag{3}$$

where A_1 and A_2 are the Lamé coefficients associated with the coordinates α_1 and α_2, and R_1 and R_2 are the principal radii of curvature of the coordinate lines α_1 and α_2, respectively.

The corresponding relations at the distance z from the middle surface are then given in the form

$$\left.\begin{aligned}
\varepsilon_1 = & \frac{1}{A_1(1+z/R_1)}\frac{\partial}{\partial\alpha_1}(u_1+z\beta_1) \\
& + \frac{u_2+z\beta_2}{A_1A_2(1+z/R_1)(1+z/R_2)} \\
& \times \frac{\partial}{\partial\alpha_2}[A_1(1+z/R_1)] + \frac{w_z}{R_1(1+z/R_1)} \\
\gamma_{12} = & \frac{1}{A_2(1+z/R_2)}\frac{\partial}{\partial\alpha_2}(u_1+z\beta_1) \\
& + \frac{1}{A_1(1+z/R_1)}\frac{\partial}{\partial\alpha_1}(u_2+z\beta_2) \\
& - \frac{u_1+z\beta_1}{A_1A_2(1+z/R_1)(1+z/R_2)} \\
& \times \frac{\partial}{\partial\alpha_2}[A_1(1+z/R_1)] \\
& - \frac{u_2+z\beta_2}{A_1A_2(1+z/R_1)(1+z/R_2)} \\
& \times \frac{\partial}{\partial\alpha_1}[A_2(1+z/R_2)]
\end{aligned}\right\} \tag{4}$$

with a similar relation for ε_2, where β_1 and β_2 are the normal components of the angular change in the tangent to the coordinate lines α_1 and α_2, respectively, and they are given by

$$\left.\begin{aligned}
\beta_1 &= u_1/R_1 - (1/A_1)\partial w_z/\partial\alpha_1 \\
\beta_2 &= u_2/R_2 - (1/A_2)\partial w_z/\partial\alpha_2
\end{aligned}\right\} \tag{5}$$

The tangential components of the angular change in the tangent to the coordinate lines may also be defined. They are denoted by β_{t1} and β_{t2}, and are given by

$$\left.\begin{aligned}
\beta_{t1} &= (1/A_1)\partial u_2/\partial\alpha_1 - (u_1/A_1A_2)\partial A_1/\partial\alpha_2 \\
\beta_{t2} &= (1/A_2)\partial u_1/\partial\alpha_2 - (u_2/A_1A_2)\partial A_2/\partial\alpha_1
\end{aligned}\right\} \tag{6}$$

The shear strain γ_{12}^0 and the rotation along the normal ω_n are now written in the form

$$\left.\begin{aligned}
\gamma_{12}^0 &= \beta_{t1} + \beta_{t2} \\
\omega_n &= (\beta_{t1} - \beta_{t2})/2
\end{aligned}\right\} \tag{7}$$

The changes in the normal curvature, κ_1^0 and κ_2^0, of the coordinate lines α_1 and α_2, respectively, are also introduced. They are given by

$$\left.\begin{aligned}
\kappa_1^0 &= (1/A_1)\partial\beta_1/\partial\alpha_1 + (\beta_2/A_1A_2)\partial A_1/\partial\alpha_2 \\
\kappa_2^0 &= (1/A_2)\partial\beta_2/\partial\alpha_2 + (\beta_1/A_1A_2)\partial A_2/\partial\alpha_1
\end{aligned}\right\} \tag{8}$$

The stress resultants and couples may be defined by

$$\begin{Bmatrix} N_1 \\ N_{12} \\ Q_1 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1 \\ \tau_{12} \\ \tau_{1n} \end{Bmatrix} (1+z/R_2)dz \tag{9}$$

$$\begin{Bmatrix} M_1 \\ M_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1 \\ \tau_{12} \end{Bmatrix} (1+z/R_2)zdz \tag{10}$$

with similar expressions for N_2, N_{21}, Q_2, M_2 and M_{21}. Here, the definition of Q_1 and Q_2 in terms of the normal shear stresses τ_{1n} and τ_{2n} have formally been introduced. However, as will become clear in the following development of the equilibrium equations, Q_1 and Q_2 can be eliminated entirely from the equilibrium equations, and, therefore, no further consideration will be made on these quantities.

The equilibrium equations are given by

$$\begin{aligned}
&\partial(N_1A_2)/\partial\alpha_1 + \partial(N_{21}A_1)/\partial\alpha_2 + N_{12}\partial A_1/\partial\alpha_2 \\
&\quad - N_2\partial A_2/\partial\alpha_1 + A_1A_2(Q_1/R_1 + q_1) = 0
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\partial(M_1A_2)/\partial\alpha_1 + \partial(M_{21}A_1)/\partial\alpha_2 + M_{12}\partial A_1/\partial\alpha_2 \\
&\quad - M_2\partial A_2/\partial\alpha_1 - Q_1A_1A_2 = 0
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\partial(Q_1A_2)/\partial\alpha_1 + \partial(Q_2A_1)/\partial\alpha_2 \\
&\quad - A_1A_2(N_1/R_1 + N_2/R_2) - q_nA_1A_2 = 0
\end{aligned} \tag{13}$$

$$M_{21}/R_2 - M_{12}/R_1 + N_{21} - N_{12} = 0 \tag{14}$$

with two additional equations which are obtained by interchanging the indices 1 and 2 in Eqs. (11) and (12). Here, q_1, q_2 and q_n are the components of the applied surface load.

3. STRAIN-DISPLACEMENT RELATIONS

Various approximate expressions of the strain-displacement relations are derived in the present section.

First, the terms z/R_i $(i=1,2)$ are neglected simply in comparison with unity in Eqs. (4) without further manipulations. The result may be written in the form

$$\varepsilon_i = \varepsilon_i^0 + z\kappa_i^0 \quad (i=1,\ 2) \tag{15}$$

$$\gamma_{12} = \gamma_{12}^0 + z\tau^0 \tag{16}$$

where

$$\begin{aligned}
\tau^0 = &(1/A_1)\partial\beta_2/\partial\alpha_1 + (1/A_2)\partial\beta_1/\partial\alpha_2 \\
&- (\beta_2/A_1A_2)\partial A_2/\partial\alpha_1 - (\beta_1/A_1A_2)\partial A_1/\partial\alpha_2
\end{aligned} \tag{17}$$

Eqs. (15)–(17) are nothing but the strain-displacement relations of Reissner's version of Love's first approximation.

Instead of neglecting the terms z/R_i directly in Eqs. (4), differentiation is first performed and then the equations are put into the form

$$\left.\begin{aligned}
\varepsilon_1 &= (\varepsilon_1^0 + z\kappa_1^0)/(1+z/R_1) \\
\gamma_{12} &= (\gamma_{12}^0 + z\tau_1 + z^2\tau_2)/[(1+z/R_1)(1+z/R_2)]
\end{aligned}\right\} \tag{18}$$

with a similar relation for ε_2. Here τ_1 and τ_2 are given by

$$\left.\begin{aligned}
\tau_1 &= \tau + (1/R_2 + 1/R_1)\gamma_{12}^0/2 \\
\tau_2 &= (1/R_2 + 1/R_1)\tau/2 + (1/R_2 - 1/R_1)^2\gamma_{12}^0/4
\end{aligned}\right\} \tag{19}$$

with

$$\tau=\tau^0+(1/R_2-1/R_1)\omega_n \qquad (20)$$

If the terms z/R_i are neglected in comparison with unity, and if the term $z^2\tau_2$ is neglected simply because it is a higher order term in z, Eqs. (18) become

$$\left.\begin{array}{l}\varepsilon_i=\varepsilon_i^0+z\kappa_i^0 \quad (i=1,\ 2)\\ \gamma_{12}=\gamma_{12}^0+z\tau_1\end{array}\right\} \qquad (21)$$

The above equations are the first approximation of the strain-displacement relations proposed by V. V. Novozhilov (Eqs. 4.23 in Ref. 6), K. Mizoguchi[7] and by K. Washizu[8].

If the right hand members of Eqs. (18) are expanded into the Taylor series and the terms higher order than the second in z are neglected in the resulting series expressions, Eqs. (18) reduce to

$$\left.\begin{array}{l}\varepsilon_i=\varepsilon_i^0+z\kappa_i^0 \quad (i=1,\ 2)\\ \gamma_{12}=\gamma_{12}^0+z\tau_3\end{array}\right\} \qquad (22)$$

where

$$\tau_3=\tau-(1/R_2+1/R_1)\gamma_{12}^0/2 \qquad (23)$$

Eqs. (22) are identical to the strain-displacement relations presented by V.V. Novozhilov (Eqs. 4.25 in Ref. 6), except for κ_i^0, which should read $(\kappa_i^0-\varepsilon_i^0/R_i)$ in Novozhilov's equations.

An examination of Eqs. (19) reveals that τ_2 is the quantity of order of magnitude τ_1 divided by R_m, and that the term $z^2\tau_2$ is negligible in comparison with the term $z\tau_1$ in the last equation of Eqs. (18). It can also be recognized that the last term on the right hand side of the expression of τ_1 is of order of magnitude γ_{12}^0 divided by R_m, and that it may be negligible compared to the term γ_{12}^0 in the last Eqs. (18). In this manner terms of order of magnitude z/R_m are neglected in comparison with unity in Eqs. (18). The result is

$$\left.\begin{array}{l}\varepsilon_i=\varepsilon_i^0+z\kappa_i^0 \quad (i=1,\ 2)\\ \gamma_{12}=\gamma_{12}^0+z\tau\end{array}\right\} \qquad (24)$$

Eqs. (24) are identical to the strain-displacement relations proposed by J. L. Sanders, Jr.[5] and by W. T. Koiter[4].

The strain-displacement relations are listed as a summary in Table-1. It is noted that the expressions of ε_1 and ε_2 are the same for all four sets of the strain-displacement relations derived above, whereas those of γ_{12} are slightly different from each other. One of the notable differences in the expressions of γ_{12} is that the term with ω_n is missing in Love's first approximation. Omission of the term with ω_n attributes to the well-known inconsistency of Love's first approximation that it does not satisfy the requirement of vanishing strains for rigid body rotation. It can easily be proved that the remaining three sets of the strain-displacement relations satisfy this requirement. The expressions of γ_{12} of the last three sets of the strain-displacement relations are different from each other in the sence that the terms of order of magnitude γ_{12}^0 divided by R_m are added to, or subtracted, or omitted from the form of γ_{12} which is in common with all of these expressions. As has been shown by Koiter in Ref. 4, these are considered to be of the same order of magnitude of errors underlying the foundation of the first approximation theory of thin elastic shells. Consequently, it may be stated that the last three sets of the strain-displacement relations are equivalent to each other within the frame-

Table-1 Strain-displacement relations

	Normal Strain	Shear Strain	Torsion
Love Reissner	$\varepsilon_i=\varepsilon_i^0+z\kappa_i^0$	$\gamma_{12}=\gamma_{12}^0+z\tau^0$	$\tau^0=(A_2/A_1)\partial(\beta_2/A_2)/\partial\alpha_1+(A_1/A_2)\partial(\beta_1/A_1)/\partial\alpha_2$
Novozhilov(1) Mizoguchi Washizu	$\varepsilon_i=\varepsilon_i^0+z\kappa_i^0$	$\gamma_{12}=\gamma_{12}^0+z\tau_1$	$\tau_1=\tau+(1/R_2+1/R_1)\gamma_{12}^0/2$
Novozhilov(2)	$\varepsilon_i=\varepsilon_i^0+z\kappa_i^0$	$\gamma_{12}=\gamma_{12}^0\pm z\tau_3$	$\tau_3=\tau-(1/R_2+1/R_1)\gamma_{12}^0/2$
Koiter Sanders	$\varepsilon_i=\varepsilon_i^0+z\kappa_i^0$	$\gamma_{12}=\gamma_{12}^0+z\tau$	$\tau=\tau^0+(1/R_2-1/R_1)\omega_n$

Note: Novozhilov(1) = Eqs. 4.23 of Ref. 6
Novozhilov(2) = Eqs. 4.25 of Ref. 6, if κ_i^0 are replaced by $(\kappa_i^0-\varepsilon_i^0/R_i)$

work of the first approximation.

4. STRESS RESULTANTS AND COUPLES

If the terms z/R_i are neglected in comparison with unity in the definition of the stress resultants and couples, Eqs. (9) and (10), and if use is made of Reissner's version of Love's first approximation of the strain-displacement relations, Eqs. (15) and (16), integration in Eqs. (9) and (10) can easily be carried out. It yields the following expression of the stress resultants and couples which is commonly known as those of Reissner's version of Love's first approximation ([10]):

Love-Reissner:

$$\left.\begin{aligned} N_1&=K(\varepsilon_1{}^0+\nu\varepsilon_2{}^0)\\ M_1&=D(\kappa_1{}^0+\nu\kappa_2{}^0)\\ N_{12}&=N_{21}=G h\gamma_{12}{}^0\\ M_{12}&=M_{21}=(Gh^3/12)\tau^0 \end{aligned}\right\}\qquad(25)$$

with similar equations for N_2 and M_2. Here, K and D are given by

$$K=Eh/(1-\nu^2),\quad D=Eh^3/12(1-\nu^2)\qquad(26)$$

A natural extension of the above may be to make use of the exact relations of the strains and displacements, Eqs. (18), in the definition of the stress resultants and couples, Eqs. (9) and (10). The integrands in Eqs. (9) and (10) are then expanded into the Taylor series about $z=0$. Subsequently, the integration is carried out term by term to yield

$$\left.\begin{aligned} N_1&=K\{\varepsilon_1{}^0+\nu\varepsilon_2{}^0+(h^2/12)(1/R_2-1/R_1)\\ &\qquad\times(\kappa_1{}^0-\varepsilon_1{}^0/R_1)[1+O(h^2/R_m{}^2)]\}\\ M_1&=D\{\kappa_1{}^0+\nu\kappa_2{}^0+(1/R_2-1/R_1)\varepsilon_1{}^0\\ &\qquad-(3h^2/20R_1)(1/R_2-1/R_1)\\ &\qquad\times(\kappa_1{}^0-\varepsilon_1{}^0/R_1)[1+O(h^2/R_m{}^2)]\}\\ N_{12}&=Gh\{\gamma_{12}{}^0+(h^2/24)(1/R_2-1/R_1)\\ &\qquad\times[\tau+(\gamma_{12}{}^0/2)(1/R_2-3/R_1)]\\ &\qquad\times[1+O(h^2/R_m{}^2)]\}\\ M_{12}&=(Gh^3/12)\{\tau+(\gamma_{12}{}^0/2)(1/R_2-1/R_1)\\ &\qquad-(3h^2/40R_1)(1/R_2-1/R_1)\\ &\qquad\times[\tau+(\gamma_{12}{}^0/2)(1/R_2-3/R_1)]\\ &\qquad\times[1+O(h^2/R_m{}^2)]\} \end{aligned}\right\}\qquad(27)$$

with similar results for N_2, M_2, N_{21} and M_{21}.

Order of magnitude comparison is made on the basis of the assumption that the middle surface strains $\varepsilon_1{}^0$, $\varepsilon_2{}^0$ and $\gamma_{12}{}^0$ are of the same order of magnitude of $h\kappa_1{}^0$, $h\kappa_2{}^0$ and $h\tau$. First, the terms of order of magnitude $(h/R_m)^3$ are neglected and the resulting approximate expressions are referred to as the third order approximation. In a similar manner, the second order and the first order approximations can be defined by neglecting terms of order of magnitude $(h/R_m)^2$ and h/R_m, respectively. The third, second and first order approximations are, thus, considered to involve errors of order of magnitude $(h/R_m)^3$, $(h/R_m)^2$ and h/R_m, respectively. The result is listed in the following:

Third order approximation:

$$N_1=K\{\varepsilon_1{}^0+\nu\varepsilon_2{}^0+(h^2/12)(1/R_2-1/R_1)(\kappa_1{}^0-\varepsilon_1{}^0/R_1)\}\qquad(28.\text{a})$$

$$M_1=D\{\kappa_1{}^0+\nu\kappa_2{}^0+(1/R_2-1/R_1)\times[\varepsilon_1{}^0-(3h^2/20R_1)\kappa_1{}^0]\}\qquad(28.\text{b})$$

$$N_{12}=Gh\{\gamma_{12}{}^0+(h^2/24)(1/R_2-1/R_1)\times[\tau+(\gamma_{12}{}^0/2)(1/R_2-3/R_1)]\}\qquad(28.\text{c})$$

$$M_{12}=(Gh^3/12)\{\tau+(1/R_2-1/R_1)\times[\gamma_{12}{}^0/2-(3h^2/40R_1)\tau]\}\qquad(28.\text{d})$$

Second order approximation:

$$N_1=K\{\varepsilon_1{}^0+\nu\varepsilon_2{}^0+(h^2/12)(1/R_2-1/R_1)\kappa_1{}^0\}\qquad(29.\text{a})$$

$$M_1=D\{\kappa_1{}^0+\nu\kappa_2{}^0+(1/R_2-1/R_1)\varepsilon_1{}^0\}\qquad(29.\text{b})$$

$$N_{12}=Gh\{\gamma_{12}{}^0+(h^2/24)(1/R_2-1/R_1)\tau\}\qquad(29.\text{c})$$

$$M_{12}=(Gh^3/12)\{\tau+(1/R_2-1/R_1)\gamma_{12}{}^0/2\}\qquad(29.\text{d})$$

First order approximation: (Love)

$$N_1=K(\varepsilon_1{}^0+\nu\varepsilon_2{}^0)\qquad(30.\text{a})$$

$$M_1=D(\kappa_1{}^0+\nu\kappa_2{}^0)\qquad(30.\text{b})$$

$$N_{12}=N_{21}=Gh\gamma_{12}{}^0\qquad(30.\text{c})$$

$$M_{12}=(Gh^3/12)\tau\qquad(30.\text{d})$$

with similar result for N_2, M_2, N_{21} and M_{21}.

Love's original version of the first approximation is identical to the first order approximation, except for the appearance of τ_3 in place of τ in Eq. (30.d). Equivalence of τ and τ_3 has, however, been discussed in the preceding section. The first order approximation may, therefore, be considered equivalent to Love's first approximation, and thus Love's name is indicated in the parenthesis.

It has been pointed out by many authors that the stress resultants and couples of Reissner's version of Love's first approximation, Eqs. (25), do not satisfy the sixth equilibrium equation, Eq. (14). In connection with this, N_{12}, N_{21}, M_{12} and M_{21} as given in Eqs. (28)-(30) are substituted in Eq. (14). It turns out that none of these approximations listed above satisfy Eq. (14) identically. Therefore, modification is made on Eqs. (29)-(30) by employing the higher order approximations for those quantities which are essential to meet the requirement of satisfying Eq. (14) identically. Modification in this manner may be justified by the similar argument made in the preceding section on the equivalence of τ_1, τ_3 and τ such that terms of order of magnitude of errors involved in the approximation may be added to the approximate equations. It should be emphasized, however, that the modification doesn't necessarily imply refinement of the original approximate equations, that is to say that the modified equations involve errors of the same order of magnitude of the unmodified ones.

Modification is first made on the second order approximation by replacing Eq. (29.c) with the corresponding equation of the third order approximation, Eq. (28.c). The modified approximation is referred to as the modified second order (I). Further modification on the second order approximation is made by replacing Eq. (29.a) with Eq. (28.a) in addition to the replacement of Eq. (29.c) with Eq. (28.c). The modified approximation thus obtained is referred to as the modified second order (II). In a similar manner, the modified first order (I) and (II) are established by replacing Eq. (30.c) with Eq. (29.c), and Eqs. (30.a,c) with Eqs. (29.a,c), respectively. The modified approximations are listed in the following:

Modified second order (I): (Naghdi–1957)

$$N_1=K\{\varepsilon_1{}^0+\nu\varepsilon_2{}^0+(h^2/12)(1/R_2-1/R_1)\kappa_1{}^0\} \tag{31.a}$$

$$M_1=D\{\kappa_1{}^0+\nu\kappa_2{}^0+(1/R_2-1/R_1)\varepsilon_1{}^0\} \tag{31.b}$$

$$N_{12}=Gh\{\gamma_{12}{}^0+(h^2/24)(1/R_2-1/R_1)\times[\tau+(\gamma_{12}{}^0/2)(1/R_2-3/R_1)]\} \tag{31.c}$$

$$M_{12}=(Gh^3/12)\{\tau+(1/R_2-1/R_1)\gamma_{12}{}^0/2\} \tag{31.d}$$

Modified second order (II): (Flügge-Lur'e-Byrne)

$$N_1=K\{\varepsilon_1{}^0+\nu\varepsilon_2{}^0+(h^2/12)\times(1/R_2-1/R_1)(\kappa_1{}^0-\varepsilon_1{}^0/R_1)\} \tag{32.a}$$

$$M_1=D\{\kappa_1{}^0+\nu\kappa_2{}^0+(1/R_2-1/R_1)\varepsilon_1{}^0\} \tag{32.b}$$

$$N_{12}=Gh\{\gamma_{12}{}^0+(h^2/24)(1/R_2-1/R_1)\times[\tau+(1/R_2-3/R_1)\gamma_{12}{}^0/2]\} \tag{32.c}$$

$$M_{12}=(Gh^3/12)\{\tau+(1/R_2-1/R_1)\gamma_{12}{}^0/2\} \tag{32.d}$$

Modified first order (I): (Koiter)

$$N_1=K(\varepsilon_1{}^0+\nu\varepsilon_2{}^0) \tag{33.a}$$

$$M_1=D(\kappa_1{}^0+\nu\kappa_2{}^0) \tag{33.b}$$

$$N_{12}=Gh\{\gamma_{12}{}^0+(h^2/24)(1/R_2-1/R_1)\tau\} \tag{33.c}$$

$$M_{12}=(Gh^3/12)\tau \tag{33.d}$$

Modified first order (II):

$$N_1=K\{\varepsilon_1{}^0+\nu\varepsilon_2{}^0+(h^2/12)(1/R_2-1/R_1)\kappa_1{}^0\} \tag{34.a}$$

$$M_1=D(\kappa_1{}^0+\nu\kappa_2{}^0) \tag{34.b}$$

$$N_{12}=Gh\{\gamma_{12}{}^0+(h^2/24)(1/R_2-1/R_1)\tau\} \tag{34.c}$$

$$M_{12}=(Gh^3/12)\tau \tag{34.d}$$

with similar results for N_2, M_2, N_{21} and M_{21}.

It can easily be proved that all four sets of the stress resultants and couples listed here satisfy the sixth equilibrium equation, Eq. (14), identically.

It turns out that the first three of these approximations are identical to those derived by the authors whose names are indicated in the parenthesis: The modified second order approximation (I) is identical to the stress resultants and couples derived by P. M. Naghdi[11] in 1957. W. Flügge[12], A. I. Lur'e[13], and R. Byrne[14] independently derived the stress resultants and couples which are identical to Eqs. (32). The stress resultants and couples of the modified first order (I) are identical to those proposed by W. T. Koiter[4]. The modified first order approximation (II), Eqs. (34), appears to be new. Similarity in the manner of modification indicates that the modified first order (II) may be interpreted as the first order version of Függe-Lur'e-Byrne's theory.

Possibility of making other modifications may be considered; for example, by employing the higher order approximations for M_1 instead of, or together with those for N_{12} and N_1. Examination of the equilibrium equations and of the definition of the stress resultants and couples reveals, however, that the stress resultant and couples appear in the same order in the equilibrium equations and that the leading terms in the expression of N_1 and N_{12} are the strains multiplied by h, whereas those of M_1 and M_{12} are the change in curvature and the torsion multiplied by h^3. This indicates that, if refinement is made on N_{12}, which is essential for the requirement of satisfying the sixth equilibrium equation, refinement on N_1 is meaningful, whereas that on M_1 is somewhat meaningless. This also implies that Flügge-Lur'e-Byrne' and the modified first order (II) approximations may be considered as belonging to the classes of approximation of the third and the second order, respectively.

Although it is not the purpose of the present paper to discuss on many other existing theories proposed by various authors, a brief mention should be made on the theory of Sanders[5] and on that of Novozhilov[6]. These theories are similar to each other in the sence that they

Table-2 Constitutive Equations

	N_1/K	N_{12}/Gh	M_1/D	$M_{12}/(Gh^3/12)$
Flügge-Lur'e-Byrne	$\varepsilon_1{}^0+\nu\varepsilon_2{}^0$ $+(h^2/12)(1/R_2-1/R_1)\kappa_1{}^0$ $-(h^2/12)(1/R_2-1/R_1)$ $\times\varepsilon_1{}^0/R_1$	$\gamma_{12}{}^0$ $+(h^2/24)(1/R_2-1/R_1)\tau$ $+(h^2/24)(1/R_2-1/R_1)$ $\times(1/R_2-3/R_1)\gamma_{12}{}^0/2$	$\kappa_1{}^0+\nu\kappa_2{}^0$ $+(1/R_2-1/R_1)$ $\times\varepsilon_1{}^0$	τ $+(1/R_2-1/R_1)$ $\times\gamma_{12}{}^0/2$
Naghdi (1957)	$\varepsilon_1{}^0+\nu\varepsilon_2{}^0$ $+(h^2/12)(1/R_2-1/R_1)\kappa_1{}^0$	$\gamma_{12}{}^0$ $+(h^2/24)(1/R_2-1/R_1)\tau$ $+(h^2/24)(1/R_2-1/R_1)$ $\times(1/R_2-3/R_1)\gamma_{12}{}^0/2$	$\kappa_1{}^0+\nu\kappa_2{}^0$ $+(1/R_2-1/R_1)$ $\times\varepsilon_1{}^0$	τ $+(1/R_2-1/R_1)$ $\times\gamma_{12}{}^0/2$
1st. order (II)	$\varepsilon_1{}^0+\nu\varepsilon_2{}^0$ $+(h^2/12)(1/R_2-1/R_1)\kappa_1{}^0$	$\gamma_{12}{}^0$ $+(h^2/24)(1/R_2-1/R_1)\tau$	$\kappa_1{}^0+\nu\kappa_2{}^0$	τ
Koiter	$\varepsilon_1{}^0+\nu\varepsilon_2{}^0$	$\gamma_{12}{}^0$ $+(h^2/24)(1/R_2-1/R_1)\tau$	$\kappa_1{}^0+\nu\kappa_2{}^0$	τ
Love-Reissner	$\varepsilon_1{}^0+\nu\varepsilon_2{}^0$	$\gamma_{12}{}^0$	$\kappa_1{}^0+\nu\kappa_2{}^0$	τ^0
Sanders	$\varepsilon_1{}^0+\nu\varepsilon_2{}^0$	$\gamma_{12}{}^0$	$\kappa_1{}^0+\nu\kappa_2{}^0$	τ
Novozhilov	$\varepsilon_1{}^0+\nu\varepsilon_2{}^0$	$\gamma_{12}{}^0$ $+(h^2/12R_2)\tau_1$	$\kappa_1{}^0+\nu\kappa_2{}^0$	τ_1

Note: The shear strain and the torsion of the Sanders theory should be related to the average stress resultant and couples, $\overline{N}_{12}=(N_{21}+N_{12})/2$ and $\overline{M}_{12}=(M_{12}+M_{21})/2$

introduce new functions in such a manner that Eq. (14) be satisfied identically. Sanders introduced the average stress resultants and couples and the corresponding average displacements, and he formulated the theory with the aid of the principle of virtual work. Novozhilov introduced arbitrary stress functions so that the equilibrium equations are satisfied identically.

The constitutive equations are listed in Table-2, in which the shear strain and the torsion in Sanders' theory should be related to the average stress resultants and couples, $\overline{N}_{12}=(N_{12}+N_{21})/2$ and $\overline{M}_{12}=(M_{12}+M_{21})/2$.

5. EQUILIBRIUM EQUATIONS FOR CIRCULAR CYLINDRICAL SHELLS

A circular cylindrical shell of uniform thickness h, radius R and length L is considered. The coordinates system (x,θ) is defined in the middle surface of the shell, such that x measures the length in the axial direction and θ is the circumferential coordinate mesured in radians (see Fig. 2), and that α_1 and α_2 are identified with θ ard x, respectively. It follows then that $1/R_2=0$, $1/R_1=1/R$, $A_2=1$ and $A_1=R$. The equations derived in the preceding sections are specialized for the circular cylindrical shell, and, for each of the approximate expressions of the stress resultants and couples, the equilibrium equations are expressed in terms of the displacement in a form analogous to the Donnell equations. The result is listed in nondimensional form in the following:

Naghdi (1957):

$$\nabla^4(\nabla^2+1)^2w+4k^4w^{\prime\prime\prime\prime}+2(1-\nu)\times(w^{\prime\prime ::}-w^{\prime\prime\prime\prime\prime\prime}+w^{\prime\prime\cdot\cdot})+2w^{\prime\prime\prime\prime\cdot\cdot}=0 \tag{35}$$

Flügge-Lur'e-Byrne:

$$\nabla^4(\nabla^2+1)^2w+4k^4w^{\prime\prime\prime\prime}+2(1-\nu)\times(w^{\prime\prime ::}-w^{\prime\prime\prime\prime\prime\prime}+w^{\prime\prime\cdot\cdot})=0 \tag{36}$$

Koiter-Sanders:

$$\nabla^4(\nabla^2+1)^2w+4k^4w^{\prime\prime\prime\prime}+2(w^{\prime\prime ::}-w^{\prime\prime\prime\prime\prime\prime}+w^{\prime\prime\cdot\cdot})=0 \tag{37}$$

Modified first order (II):

$$\nabla^4(\nabla^2+1)^2w+4k^4w^{\prime\prime\prime\prime}+(2-\nu)(w^{\prime\prime ::}-w^{\prime\prime\prime\prime\prime\prime}+w^{\prime\prime\cdot\cdot})=0 \tag{38}$$

Novozhilov—Mizoguchi:

$$\nabla^4(\nabla^2+1)^2w+4k^4w^{\prime\prime\prime\prime}+2(w^{\prime\prime ::}-w^{\prime\prime\prime\prime\prime\prime}+w^{\prime\prime\cdot\cdot})+2(1-\nu^2)w^{\prime\prime\prime\prime\cdot\cdot}=0 \tag{39}$$

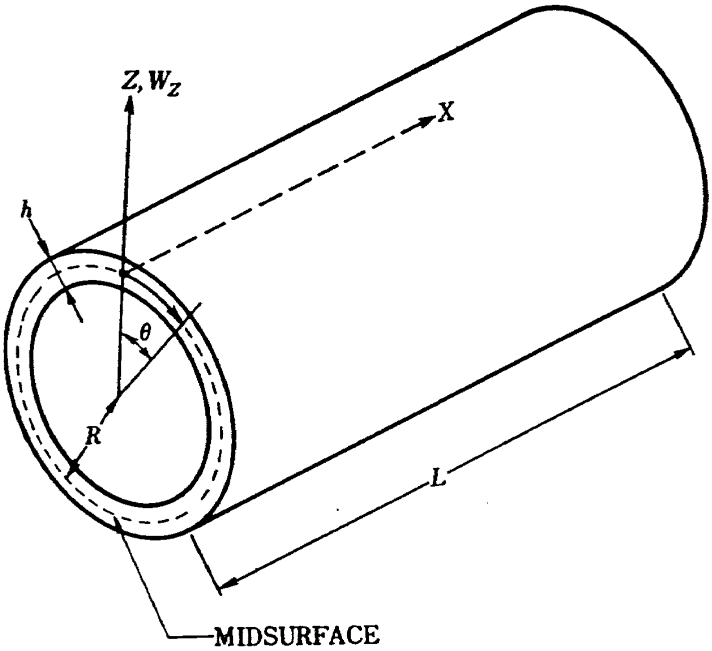

Fig. 2 Geometry and Coordinates of Circular Cylindrical Shell

Love-Reissner:

$$\nabla^4(\nabla^2+1)^2 w+4k^4 w''''+(5+3\nu)w''^{\cdot\cdot}/2$$
$$+2[\nu w''^{\cdot\cdot\cdot\cdot}-w''''''-(1-\nu)w''''^{\cdot\cdot}]=0 \qquad (40)$$

where w is the outward-normal displacement w_z divided by R, the dot and the prime as superscript indicate differentiation with respect to θ and the nondimensional coordinate y defined by x divided by R, respectively, and

$$4k^4=12(1-\nu^2)(R/h)^2 \qquad (41)$$

$$\nabla^2(\ )=(\ )''+(\ )^{\cdot\cdot} \qquad (42)$$

The equation obtained by Novozhilov's theory is also presented in the above for the sake of comparison. It should be noted here that the equation identical to Eq. (39) has already been derived by Mizoguchi[7], who has independently developed a linear theory of thin elastic shells which turned out to be identical, in essence, to that formulated by Novozhilov. The Sanders theory yields the equation which is identical to that obtained with the aid of the constitutive equations of Koiter. Sanders' name is thus indicated together with Koiter's. Eq. (36) designated by Flügge-Lur'e-Byrne is the one which has been derived by J. Kempner[15]. A close resembrance in form should be noticed among the equations of Flügge-Lur'e-Byrne, Koiter-Sanders and of the modified first order (II), and among those of Naghdi and Novozhilov—Mizoguchi.

A similar equation can also be derived with the aid of the equations defining exactly the stress resultants and couples. The stress resultants and couples as given in Eqs. (9) and (10) are substituted into the equilibrium equations. As is possible for the circular cylindrical shell of uniform thickness, the order of differentiation and integration is interchanged. The integrands are then expanded into the Taylor series. Subsequent integration term by term and rather involved mathematical manipulations yield an eighth order differential equation in w. It turns out, after neglecting terms of order of magnitude $(h/R_m)^2$ in comparison with unity, to be identical to that of Függe-Lur'e-Byrne.

Approximate equations which take much simpler form than Eqs. (35)-(40) have been proposed by L. S. D. Morley[16] and by L. H. Donnell[17]. They are given in the form

Morley:

$$\nabla^4(\nabla^2+1)^2 w+4k^4 w''''=0 \qquad (43)$$

Donnell:

$$\nabla^8 w+4k^4 w''''=0 \qquad (44)$$

N. J. Hoff[18] investigated the accuracy of the Donnell equation. Hoff utilized the result of earlier studies made by himself and his collaborators[19),20] that solutions to many equilibium problems can be obtained with the aid of eigenfunctions trigonometric either along the longitude or along the circumference. He thus assumed w in the form

$$w=e^{py}\cos n\theta \quad (n=0,1,2,\cdots) \qquad (45)$$

when the shell is free of surface loading and loads are applied at edges of constant x, and

$$w=e^{p\theta}\cos ny \quad (n=m\pi R/L;\ m=0,1,2\cdots) \tag{46}$$

when the shell is free of surafce loading and loads are applied at edges of constant θ, and in the form

$$w=A\cos sy\cos n\theta \quad (s=m\pi R/L;\ m,n=0,1,2,\cdots) \tag{47}$$

when the shell is subjejcted to a radial pressure which can be expressed in the Fourier series of the same form as Eq. (47). Comparison was then made of the numerical values of p and A obtained and the Donnell equation and for the Függe-Lur'e-Byrne equation.

Comparison of the solutions to Eqs. (35)–(40) is made following an identical procedure

Table-3 Roots when $w=e^{py}\cos n\theta$

k	n	Theories	p_1, p_2	p_3, p_4
5	1	Naghdi	$5.0833\pm4.9132i$	0
		F-L-B	$5.0843\pm4.9143i$	0
		K-S	$5.0990\pm4.8990i$	0
		1st. (II)	$5.0917\pm4.9066i$	0
		N-M	$5.0987\pm4.8986i$	0
		L-R	$5.0983\pm4.8983i$	0.0118
5	3	Naghdi	$5.9588\pm4.2665i$	$0.9318\pm0.6872i$
		F-L-B	$5.9660\pm4.2763i$	$0.9300\pm0.6865i$
		K-S	$5.9789\pm4.2583i$	$0.9280\pm0.6893i$
		1st. (II)	$5.9724\pm4.2673i$	$0.9290\pm0.6879i$
		N-M	$5.9723\pm4.2494i$	$0.9296\pm0.6899i$
		L-R	$5.9999\pm4.2603i$	$0.8605\pm0.7676i$
5	10	Naghdi	$12.4639\pm2.9564i$	$7.5308\pm1.9028i$
		F-L-B	$12.5166\pm3.0368i$	$7.4812\pm1.9264i$
		K-S	$12.5264\pm2.9987i$	$7.4751\pm1.9484i$
		1st. (II)	$12.5215\pm3.0178i$	$7.4781\pm1.9374i$
		N-M	$12.4788\pm2.9244i$	$7.5199\pm1.9279i$
		L-R	$14.3949\pm4.3668i$	$5.2728\pm3.9935i$
10	1	Naghdi	$10.0423\pm9.9573i$	0
		F-L-B	$10.0424\pm9.9574i$	0
		K-S	$10.0499\pm9.9499i$	0
		1st. (II)	$10.0461\pm9.9536i$	0
		N-M	$10.0498\pm9.9498i$	0
		L-R	$10.0498\pm9.9498i$	0.0030
10	3	Naghdi	$10.4570\pm9.5749i$	$0.4400\pm0.4057i$
		F-L-B	$10.4581\pm9.5760i$	$0.4399\pm0.4057i$
		K-S	$10.4653\pm9.5682i$	$0.4396\pm0.4060i$
		1st. (II)	$10.4617\pm9.5721i$	$0.4398\pm0.4058i$
		N-M	$10.4643\pm9.5671i$	$0.4397\pm0.4060i$
		L-R	$10.4665\pm9.5686i$	$0.4299\pm0.4161i$
10	10	Naghdi	$15.2732\pm7.3959i$	$5.2682\pm2.5739i$
		F-L-B	$15.2808\pm7.4080i$	$5.2633\pm2.5744i$
		K-S	$15.2864\pm7.3965i$	$5.2614\pm2.5783i$
		1st. (II)	$15.2836\pm7.4022i$	$5.2623\pm2.5763i$
		N-M	$15.2795\pm7.3855i$	$5.2658\pm2.5779i$
		L-R	$15.7358\pm7.4767i$	$4.5223\pm3.4881i$
50	10	Naghdi	$51.0173\pm49.0219i$	$1.0139\pm0.9745i$
		F-L-B	$51.0174\pm49.0220i$	$1.0139\pm0.9745i$
		K-S	$51.0189\pm49.0205i$	$1.0139\pm0.9745i$
		1st. (II)	$51.0181\pm49.0212i$	$1.0139\pm0.9745i$
		N-M	$51.0188\pm49.0204i$	$1.0139\pm0.9745i$
		L-R	$51.0192\pm49.0203i$	$1.0040\pm0.9847i$

to Hoff's. The result is shown in Table-3, 4 and 5. In Table-5, A_{NA}, A_{FLB}, A_{KS}, A_{II}, A_{NM} and A_{LR} indicate the amplitudes of the displacement function in the form of Eq. (47) for the cases of Naghdi, Flügge-Lur'e-Byrne, Koiter-Sanders, the modified first order (II), Novozhilov—Mizoguchi and Love-Reissner, respectively. The solutions to the Donnell and Morley equations are not listed in the tables, because they are available in the earlier publications (see Refs. 16 and 18, and the book of H. Kraus[23]). The tables show no appreciable difference among these solutions.

Table-4 Roots when $w=e^{p\theta}\cos ny$

k	n	Theories	p_1, p_2	p_3, p_4
5	0.01	Naghdi	$0.0510\pm0.0490i$	$0.0025\pm1.0000i$
		F-L-B	$0.0510\pm0.0490i$	$0.0025\pm1.0000i$
		K-S	$0.0511\pm0.0489i$	$0.0025\pm1.0000i$
		1st. (II)	$0.0510\pm0.0489i$	$0.0025\pm1.0000i$
		N-M	$0.0511\pm0.0489i$	$0.0025\pm1.0000i$
		L-R	$0.0509\pm0.0491i$	$0.0039\pm1.0000i$
5	1.00	Naghdi	$2.5477\pm0.9712i$	$1.0438\pm2.3742i$
		F-L-B	$2.5470\pm0.9695i$	$1.0456\pm2.3750i$
		K-S	$2.5481\pm0.9648i$	$1.0428\pm2.3768i$
		1st. (II)	$2.5475\pm0.9671i$	$1.0441\pm2.3759i$
		N-M	$2.5488\pm0.9664i$	$1.0412\pm2.3762i$
		L-R	$2.5449\pm0.9741i$	$1.0508\pm2.3732i$
5	10.00	Naghdi	$12.4450\pm2.0181i$	$7.6930\pm3.3148i$
		F-L-B	$12.3954\pm1.9458i$	$7.7456\pm3.2946i$
		K-S	$12.3967\pm1.9146i$	$7.7474\pm3.3219i$
		1st. (II)	$12.3959\pm1.9301i$	$7.7466\pm3.3081i$
		N-M	$12.4419\pm1.9822i$	$7.6995\pm3.3398i$
		L-R	$12.3894\pm1.9589i$	$7.7589\pm3.2957i$
10	0.01	Naghdi	$0.1014\pm0.0985i$	$0.0100\pm1.0002i$
		F-L-B	$0.1014\pm0.0985i$	$0.0100\pm1.0002i$
		K-S	$0.1015\pm0.0985i$	$0.0100\pm1.0002i$
		1st. (II)	$0.1014\pm0.0985i$	$0.0100\pm1.0002i$
		N-M	$0.1015\pm0.0985i$	$0.0100\pm1.0002i$
		L-R	$0.1013\pm0.0986i$	$0.0104\pm1.0002i$
10	1.00	Naghdi	$3.5372\pm1.4100i$	$1.4610\pm3.4144i$
		F-L-B	$3.5371\pm1.4097i$	$1.4613\pm3.4146i$
		K-S	$3.5376\pm1.4083i$	$1.4602\pm3.4152i$
		1st. (II)	$3.5373\pm1.4090i$	$1.4608\pm3.4149i$
		N-M	$3.5377\pm1.4085i$	$1.4599\pm3.4151i$
		L-R	$3.5364\pm1.4114i$	$1.4631\pm3.4139i$
10	10.00	Naghdi	$14.5458\pm3.4322i$	$7.0446\pm7.1010i$
		F-L-B	$14.5389\pm3.4210i$	$7.0534\pm7.1011i$
		K-S	$14.5396\pm3.4130i$	$7.0534\pm7.1064i$
		1st. (II)	$14.5392\pm3.4170i$	$7.0534\pm7.1037i$
		N-M	$14.5458\pm3.4233i$	$7.0454\pm7.1063i$
		L-R	$14.5366\pm3.4258i$	$7.0596\pm7.1003i$
50	10.00	Naghdi	$26.2696\pm9.5160i$	$10.9565\pm22.8162i$
		F-L-B	$26.2695\pm9.5158i$	$10.9566\pm22.8163i$
		K-S	$26.2696\pm9.5154i$	$10.9563\pm22.8165i$
		1st. (II)	$26.2696\pm9.5156i$	$10.9565\pm22.8164i$
		N-M	$26.2697\pm9.5155i$	$10.9562\pm22.8164i$
		L-R	$26.2693\pm9.5163i$	$10.9571\pm22.8161i$

Table 5 Coefficients for $w=A\cos sy \cos n\theta$

k	n	s	A_{NA}/A_{FLB}	A_{KS}/A_{FLB}	A_{II}/A_{FLB}	A_{NM}/A_{FLB}	A_{LR}/A_{FLB}
5	1	0	—	—	—	—	—
5	2	0	1.0000	1.0000	1.0000	1.0000	1.0000
5	3	0	1.0006	1.0006	1.0006	1.0006	1.0006
5	1	0.5	1.0008	1.0000	1.0000	1.0007	0.9957
5	2	0.5	1.0015	1.0053	1.0027	1.0066	0.9854
5	3	0.5	1.0000	1.0021	1.0000	1.0021	0.9965
5	1	1.0	1.0006	0.9994	0.9994	1.0000	0.9981
5	2	1.0	1.0028	1.0023	1.0012	1.0048	0.9913
5	3	1.0	1.0017	1.0041	1.0021	1.0057	0.9914
5	10	1.0	1.0000	1.0000	1.0000	1.0000	1.0000
10	1	0	—	—	—	—	—
10	2	0	1.0000	1.0000	1.0000	1.0000	1.0000
10	3	0	1.0006	1.0006	1.0006	1.0006	1.0006
10	1	0.5	1.0000	1.0000	1.0000	1.0000	0.9997
10	2	0.5	1.0001	1.0007	1.0003	1.0009	0.9981
10	3	0.5	1.0000	1.0010	1.0010	1.0019	0.9981
10	1	1.0	1.0000	1.0000	1.0000	1.0000	0.9999
10	2	1.0	1.0003	1.0002	1.0002	1.0005	0.9995
10	3	1.0	1.0005	1.0010	1.0005	1.0014	0.9981
10	10	1.0	1.0000	1.0001	1.0001	1.0001	1.0000

6. CONCLUSION AND DISCUSSION

Various approximate expressions of the strain-displacement relations and constitutive equations have been derived after a rather intuitive discussion on the order of magnitude of the errors involved. Identity and equivalence of these expressions with the existing ones has been indicated. One of the approximate expressions of the constitutive equations, the first order approximation (II), appears to be new. The basic equations with four different sets of the constitutive equations were specialized for a circular cylindrical shell and the equilibrium equations were written in terms of the displacement in an analogous manner of Donnell's equation. Comparison of the numerical values of the eigenvalues indicates that there is no appreciable difference among the solutions to these equations for those particular cases of loading considered.

It has often been stated that the eigenvalues can be obtained in an explicit form only to the Donnell and Morley equations, and that those to other equations are obtained only through numerical computations (see, for example, Ref. 21 and a brief note of S. H. Iyer and S. H. Simmonds[2)]). It can be shown, however, that the eigenvalues as a closed form solution can be obtained for the boundary value problems governed by any of Eqs. (35)-(40). This can be accomplished by adding to or subtracting from the characteristic equations appropriate terms of the order of magnitude of errors involved in the basic equations, so that the roots of the characteristic equations are found with the aid of the available explicit formulas of the algebraic equations. This method has been suggested by Mizoguchi who obtained the eigenvalues for the equation (39) designated by Novozhilov—Mizoguchi.

If the displacement function w is assumed in the form of Eq. (46), for example, the characteristic equation resulting from Eq. (36), designated by Flügge-Lur'e-Byrne, may be written in the form

$$p^8+2(1-2n^2)p^6+[(6+H^2)n^4-2(4-\nu)n^2+1]p^4 -2n^2(2n^4-3n^2+2-\nu)p^2+n^4(n^4-2\nu n^2+4k^4)=0 \tag{48}$$

where

$$H^2=4(1-\nu)^2/[4k^4+4(1-\nu)n^2-(2-\nu)^2] \tag{49}$$

which is of the order of magnitude of errors involved in the basic equations, and thus the addition of which in the manner of Eq. (48) is permissible. It can easily be proved that the solutions to Eq. (48) satisfy the following fourth order algebraic equation and its conjugate:

$$p^4+(1-2n^2-in^2H)p^2 + n^2[n^2-2+\nu+i2(1-\nu)/H]=0 \qquad (50)$$

where $i=\sqrt{-1}$.

The solution of Eq. (50) can be found as in a closed form. In a similar manner, by adding small terms similar to H^2, the closed form solutions are obtained to the characteristic equations resulting from the other differential equations. The foregoing arguements and the obscurity underlying the process of approximation appear to reduce the importance of Morley's equation. It should be noted here, however, that C. P. Mangelsdorf[23] has recently shown that the equation identical to Morley's can be derived, by way of the variational method, from the modified Koiter's energy functional which had been suggested by W. T. Koiter[24].

Throughout the preceding discussions, approximation has been made simply comparing the order of magnitude of terms within the equations. This, however, overlooks the possibility that there may be yet other terms which are of the same order of magnitude as the terms used for comparison, and that these larger terms could produce a sum which was of the same order of magnitude as the term to be considered negligible.

In general, the only rational basis for the order of magnitude comparison would be the sum of all the terms in the equation. If the sum of all the terms of the expression is positive and if it is possible to divide the terms into groups, the sum of which are, by themselves, positive, each group of the terms could provide a proper basis for comparison.

The functional defining the strain energy of deformation is positive and the terms in the functional expression can be divided into groups which are also positive for $\nu>1$. The Koiter energy functional for the circular cylindrical shell, for example, can be written in the form

$$P_K=(K/2)\int[[u'+\nu(v^{\cdot}+w)]^2+(1-\nu^2)(v^{\cdot}+w)^2 + (1-\nu)(v'+u^{\cdot})^2/2+(h^2/12R^2) \times\{[w''+\nu(w^{\cdot\cdot}-v^{\cdot})]^2+(1-\nu^2)(w^{\cdot\cdot}-v^{\cdot})^2 + 2(1-\nu)(w'^{\cdot}-3v'/4+u^{\cdot}/4)^2\}]ds \qquad (51)$$

where the integration is over the middle surface within the boundary, and u ane v are the nondimensional form of the displacements in the axial and circumferential directions, u_x and u_θ, defined by $u=u_x/R$ and $v=u_\theta/R$, respectively. Each term in the integrand of Eq. (51) and the sum of any combination of them can be used as the basis for comparison. Mangelsdolf thus reasoned the addition of small terms to P_K to present a modified Koiter energy functional P_K'

$$P_K'=P_K+P_m$$

where P_m indicates the group of small terms added for modification and is given by

$$P_m=(K/2)\int(h^2/12R^2)[-2(1-\nu)u'(w^{\cdot\cdot}-v^{\cdot}) + (v^{\cdot}+w)^2+2(v^{\cdot}+w)w'' + 2(v^{\cdot}+w)(w^{\cdot\cdot}-v^{\cdot})-(3/8)(1-\nu)(u^{\cdot}+v')^2 + (1-\nu)(u^{\cdot}+v')(w'^{\cdot}+u^{\cdot}/4-3v'/4]ds \qquad (52)$$

Variation of P_K' with respect to u, v, and w yields the Euler equations which subsequently reduce to the equation identical to Morley's, Eq. (43).

In a similar manner, by way of the variational method, the equations identical to Flügge-Lur'e-Byrne's, Eq. (36), and Naghdi's, Eq. (35), can be derived. In these cases, P_m should take the form

$$P_m=(K/2)\int(h^2/12R^2)[-2u'w''+(v^{\cdot}+w)^2 + 2(v^{\cdot}+w)(w^{\cdot\cdot}-v^{\cdot}) + (3/8)(1-\nu)(u^{\cdot}+v')^2]ds \qquad (53)$$

for Flügge-Lur'e-Byrne's equation, and

$$P_m=(K/2)\int(h^2/12R^2)[-2u'w''+2(v^{\cdot}+w)(w^{\cdot\cdot}-v^{\cdot}) + (3/8)(1-\nu)(u^{\cdot}+v')^2]ds \qquad (54)$$

for Naghdi's equation.

All the terms in Eq. (53) and (54), except for the term $u'w''$, are included in Eq. (52) and the negligibility of them in comparison with the terms of P_k has already been proved by Mangelsdorf. Following the same reasoning offered by Mangelsdorf for the negligibility of the term $u'(w^{\cdot\cdot}-v^{\cdot})$, we can show that the term $u'w''$ is also negligible.

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