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Comparison of Accuracies of Solutions of Linear Shell Theories for Closed Circular Cylinders Under Edgewise Loading*

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ABSTRACT

The accuracies of the solutions of the classical linear shell theories are compared for static boundary value problems of edgewise-loaded, circumferentially-closed circular cylindrical shells. Emphasis is placed on the influence of the boundary conditions on the accuracies of the particular solutions. It will be shown that the solutions are obtained accurately within the errors inherent to the Kirchhoff-Love hypothesis for any of those well-known classical theories including the Flügge, the Koiter-Sanders, the Novozhilov, and the Love-Reissner theories. Koiter's conclusions on the accuracy of the equations of the general theory of thin shells are thus reconfirmed. In due course of the analysis, discussions will be made on some of the important characteristics of the solutions as well as on the accuracy of Donnell's approximation.

概　　要

周方向に閉じた薄肉弾性円筒シェルに静的端末荷重が作用する境界値問題の解について，種々の古典シェル理論の精度を比較し，現用の一次近似シェル理論はいずれもKirchhoff-Love の仮定に含まれる基本誤差内の精度で一致する解を与えることを理論的に明らかにする。また，系統的なオーダ評価によって実用的な近似解を導き，解の一般的な諸性質およびDonnell理論の解の誤差を明らかにする。解析はほゞ次の順序で進める。まず理論によって異った値をとる単位オーダのパラメタ (theory indicators) を導入して，理論ごとに異なる構成方程式を包括的に一組の式で表示する。平衡方程式をたわみ変位関数wのみで表わし，wに関する八階微分方程式を導く。端末境界条件を与える諸量の関係式をwのみで表わす。Kirchhoff-Loveの仮定に含まれる基本誤差のオーダ評価に基づいてこれらの関係式の合理的な一次近似式を導く。八階微分方程式が一次近似の精度で一組の複素共役な四階微分方程式に分解できることを証明し，その結果，解を固有関数展開で表示するとき固有値が閉形解で与えられることを示す。変形波数nが比較的小さい場合に対して，この閉形解が極めて簡単で実用的な式で与えられることを示す。nが大きい場合には，支配方程式がDonnellの式と一致することを示す。また，境界条件式についても同様の近似を行い，どの場合に対しても近似式には理論の相異を表わすパラメータが現われないことを示す。

NOTATIONS

Geometric and material constants

h	wall-thickness of the shell
R	radius of the circular cylindrical shell
$2L$	length of the cylindrical shell
$2l$	nondimensional form of $2L$; $2l = 2L/R$
k	geometric parameter; $4k^4 = 12(1-\nu^2)(R/h)^2$
δ	geometric parameter; $\delta = h^2/12R^2$
E	Young's modulus
ν	Poisson's ratio
K	extensional rigidity; $K = Eh/(1-\nu^2)$

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D	bending rigidity; $D = Eh^3/12(1-\nu^2)$
G	shear modulus; $G = E/2(1+\nu)$

Coordinates and differential operators

x	axial coordinate
y	nondimensional form of x; $y = x/R$
θ	circumferential coordinate
$(\ )^{\cdot}$	differential operator; $(\ )^{\cdot} = \partial(\ )/\partial\theta$
$(\ )'$	differential operator; $(\ )' = \partial(\ )/\partial y$
$\nabla^2(\ )$	differential operator; $\nabla^2(\ ) = (\ )'' + (\ )^{\cdot\cdot}$
$L(\ )$	differential operator defined in Eq. (13)
$P(\ ), \overline{P}(\ )$	complex conjugate differential operators defined in Eq. (26)

Deformation and stress

n	integer variable representing the circumferential wave number
λ	representative wave length of deformation
λ_θ	circumferential wave length of deformation
u_x, u_θ, w_z	midsurface displacement components
u, v, w	nondimensional forms of u_x, u_θ and w_z, respectively
$\epsilon_x, \epsilon_\theta, \gamma_{x\theta}$	midsurface strain components
κ_x, κ_θ	changes in curvature of the midsurface
τ	torsion of the midsurface
ω_n	rigid-body rotation about the normal to the midsurface
β	lateral rotation of the generator; $\beta = w'$
$N_x, N_\theta, N_{x\theta}$, $N_{\theta x}$	stress resultants
Q_x, Q_θ	lateral shear resultants
$M_x, M_\theta, M_{x\theta}$, $M_{\theta x}$	stress couples
$S_x, T_{x\theta}$	lateral and tangential components, respectively, of the equivalent edge shear
N, M, S, Q	nondimensional forms of N_x, M_x, S_x and $T_{x\theta}$, respectively
W_i, U_i, V_i, B_i, N_i, M_i, S_i, Q_i	coefficients of eigenfunctions expansions of w, u, v, β, N, M, S and Q, respectively
w_e, u_e, v_e, β_e, N_e, M_e, S_e, Q_e	prescribed edge values of w, u, v, β, N, M, S and Q, respectively
E_i, D_j	theory indicators in the constitutive equations
p_i	eigenvalues
ξ_1, η_1	real and imaginary parts, respectively, of the eigenvalues for the global solutions
ξ_2, η_2	real and imaginary parts, respectively, of the eigenvalues for the edge-zone solutions

Coefficients of equations

L_i	coefficients in the differential operator $L(\ )$
G_i, H_i, K_i	coefficients in Eqs. (10)
f_i, g_i	coefficients in Eqs. (11) and (12)
a_{ij}	coefficients in Eq. (14)
n_i, m_i, q_i, s_i	coefficients in Eqs. (15), (16), (17) and (18)
$a_{ij}^0, f_i^0, g_i^0, n_i^0$, m_i^0, s_i^0, q_i^0	leading terms in polynomial expressions of $a_{ij}, f_i, g_i, n_i, m_i, s_i$ and q_i, respectively, whose specific values differ with different theories
$\overline{a}_{ij}$	modified forms of a_{ij}^0 as defined in Eqs. (23)
A, B, C	complex coefficients in the characteristic equation

Real, imaginary, and complex variables

$\boldsymbol{i}$	unit of imaginary number; $\boldsymbol{i} = (-1)^{1/2}$
Δ_0	real parameter defined in Eq. (73); $\Delta_0 = n^2/2k^2$
Z	arbitrary complex variable
X, Y	real variables; $Z = X + \boldsymbol{i}Y$
α	absolute value of Z; $\alpha = (X^2 + Y^2)^{1/2}$
X_0, Y_0	real variables; $B^2 - 4AC = X_0 + \boldsymbol{i}Y_0$
α_0	absolute value of $(X_0 + \boldsymbol{i}Y_0)$
X_1^+, Y_1^+	real variables; $[-B + (B^2 - 4AC)^{1/2}]/2A = X_1^+ + \boldsymbol{i}Y_1^+$
X_1^-, Y_1^-	real variables; $[-B - (B^2 - 4AC)^{1/2}]/2A = X_1^- + \boldsymbol{i}Y_1^-$

α_1^+	absolute value of $(X_1^+ + i\,Y_1^+)$
α_1^-	absolute value of $(X_1^- + i\,Y_1^-)$
α_2	real variable defined in Eq. (67)
$\Delta X_0[n^4]$, $\Delta Y_0[n^4]$	real variables defined in Eqs. (43), whose predominant factor is n^4
$\Delta X_1^-[n^4]$, $\Delta Y_1^+[n^4]$	real variables defined in Eqs. (53), whose predominant factor is n^4
$\Delta F[n^4]$, $\Delta G[n^8]$	real variables defined in Eqs. (48), whose predominant factors are n^4 and n^8, respectively

1. INTRODUCTION

The theory of thin shells is intrinsically approximate. There always exist in the shell theory errors and limitations due to the approximate deductions of the equations from the theory of elasticity. The accuracy and limitation of the shell theory depends on the hypotheses and criteria underlying the approximation. A first complete linear shell theory was formulated by Love [1] as early as in 1888. Love postulated the so-called Kirchhoff-Love hypothesis, which may be reinterpreted as follows:

1) The shell is thin.
2) The normal to the undeformed midsurface remains normal and its length remains the same after deformation.
3) The thicknesswise normal stresses are negligible.

He then derived the equations known as Love's first approximation and showed that the strain energy expression consisted of the terms representing the strain energies due to membrane stretching and lateral bending. The consistency of the approximation of the strain energy expression to the Kirchhoff-Love hypothesis had remained uncertain until Koiter [2] proved in 1959 that it was actually so, and that the additional terms in the strain energy expression were due to the thicknesswise stresses added for the improvement of Kirchhoff-Love's hypothesis. The strain energy expression has since been widely used as a rational means to check the consistency as well as the accuracy of the shell theory. Thus, the so-called classical theories formulated in different forms by various authors including Flügge [3], Novozhilov [4], Reissner [5], Koiter [2], and Sanders [6] have been proved to be consistent first approximations within the errors involved in Kirchhoff-Love's hypothesis. It is now well understood that the accuracy of a shell theory cannot be improved beyond Love's first approximation unless Kirchhoff-Love's hypothesis is disregarded in the formulation.

A number of attempts have been being made to formulating accurate consistent shell theories, linear or nonlinear, through rigorous approximations directly or indirectly from the theory of elasticity. There, the error estimation is a central feature, and researchers have encountered difficulties in achieveing their goals with mathematical rigor and generality. A breakthrough came when John [7] made a rigorous pointwise error estimate for the shell interior equations. Danielson [8], Koiter and Simmonds [9], and many others followed the passage paved by John to bring up a better understanding on the relationship between the shell theory and elasticity. It should be noted here that John's rigorous analysis has proved that the state of stresses in shells is approximately in plane stress under the assumption of smallness of strains, and that the third of the Kirchhoff-Love hypothesis is a natural consequence of the fundamental assumption of linear elasticity. Several sets of simplest possible consistent nonlinear equations have been proposed. Perhaps, Danielson is the first to have derived such equations. Success of Danielson's attempt may be attributed to the fact that a properly defined set of stress resultants and changes in curvature was chosen as dependent variables. As stated by Danielson himself, however, this fact limits the applicability of his equations to such problems that the boundary conditions are prescribed in terms of the changes of curvature and the stress resultants. Abé [10] and the first author [11] of the present paper, therefore, urged the necessity of the formulation in terms of the displacements from the practical point of view.

There are two aspects in the error estimation for the shell theory; the error estimation for the approximate two-dimensional equations, and that for the solutions of these equations. The general results presented in Refs. [7] through [9] are concerned with the error estimation for the nonlinear shell equations. The error estimation for the solutions, on the other hand, has been done mostly on case-to-case basis, specifying the shell configurations and the boundary conditions. There, a particular boundary value problem for a particular shell configuration was solved by a particular shell theory, and the resulting numerical values of the solutions were compared with those of the elasticity solutions if available or of a seemingly more accurate shell theory solutions. A slightly more general analysis was made by Hoff [12] examining the pointwise errors for the interior solutions of Donnell's equations [13] by calculating and comparing the eigenvalues characteristic to Donnell's and Flügge's equations. In the terminologies used in Ref. [9], this may be described as a sort of an intrinsic error estimate for the general solutions of the interior equations. A complete general arguement on the error estimation for the solutions of the shell theory had been left open until Koiter [14] and Simmonds [15] gave estimates for the global errors in the sense of mean square root. Ideally, we still need to achieve precise estimates with general validity for the pointwise errors, which unfortunately seems out of reach at present.

The present paper is concerned with the comparison of accuracies of the solutions of the classical linear shell theories for various boundary value problems of circular cylindrical shells. This is primarily an extension of Hoff's work in that the solutions are assumed in eigenfunctions expansion. But it deals not only with the general solutions for the interior equations as in Hoff's work but also with the particular solutions placing emphasis on the influence of the boundary conditions. Thus, in the terminologies used in Ref. [9], it may be described as a comparison of accuracies based on the intrinsic pointwise error estimates for the solutions. It will be proved in the end that the differences between the solutions of different theories are of order of magnitude of the errors involved in Kirchhoff-Love's hypothesis. Thus, Koiter's conclusion on the consistency of the classical theories drawn from the error estimates for the equations on the strain energy criterion will be reconfirmed.

The circular cylindrical shells are assumed to be circumferentially closed and subjected to external static loads only at their axial edges formed by the plane cross sections normal to the axis of the cylinder. The beam-like bending deformations characterized by $n = 1$, n being the circumferential wave number, as well as the axisymmetric deformations with $n = 0$ are excluded in the present analysis leaving detailed expositions of them to a separate paper to follow. In Section 2, the basic equations for small deformations of elastic circular cylindrical shells are formulated on Kirchhoff-Love's hypothesis. A unique feature in these equations is that the set of the constitutive equations is formulated in such a form that it can easily be reduced to any of those formulated in the well-known theories. The use of this form of the constitutive equations enables us to compare the accuracies of different theories on a unified basis. In Section 3, the governing equations are written only in terms of the lateral displacement. Thus, the equilibrium equations are reduced to a single differential equation of the eighth order. Those quantities to be prescribed as boundary conditions at the axial edges are also written in terms of the lateral displacement. A first approximation to these equations is achieved neglecting small terms of order of magnitude of the errors involved in Kirchhoff-Love's hypothesis. In Section 4, the eighth order differential equation is reduced to a pair of complex conjugate differential equations of the fourth order. In Section 5, the general solution of the homogeneous governing equation is obtained by way of eigenfunctions expansion. The reduction of the order of the governing differential equation from eighth to fourth makes it possible to derive the eigenvalues in closed form. The closed form solutions are simplified to yield the explicit expressions valid for deformations with relatively small values of n for which a noticeable difference is most likely to occur between the solutions obtained by different theories. In Section 6, the boundary constraining equations are presented in the form of eigenfunctions expansion. For all the proper combinations of the boundary conditions, the order-of-magnitude comparison is made among the coefficients matrices of the inhomogeneous linear systems. The accuracy of the Donnell theory will be indicated in due course of the analysis.

After the complete manuscript of this paper was submitted to the editorial committee of the National Aerospace Laboratory, the present authors were informed by Professor K. Heki of Faculty of Engineering of Osaka City University that he had derived explicit forms of the eigenvalues for various shell theories in his earlier paper [32] published in September 1954. A review has brought us to acknowledging the importance of his paper in that the comparison of accuracies of the shell theories by

way of the comparison of the eigenvalues had been attempted one year earlier than a similar attempt of Hoff [12] whose paper appeared in September 1955. His paper seems to have been left relatively unknown without receiving worldwide recognition. This may be attributed to the fact that the paper was written in Japanese so that its circulation has been limited to the domestic audience mainly in the architectural society.

2. BASIC EQUATIONS

Let us consider a thin-walled circular cylindrical shell made of a linearly elastic, homogeneous and isotropic material with Young's modulus E and Poisson's ratio ν. Let the thickness of the shell be h, the radius R and the length $2L$. The coordinates x and θ are so chosen on the midsurface of the shell that x measures the axial distance along the generator from the central, normal cross section and θ the circumferential angular extent with $R\theta$ being the arc length. The displacement components u_x, u_θ and w_z, the stress resultants N_x, N_θ, $N_{x\theta}$, $N_{\theta x}$, Q_x and Q_θ, and the stress couples M_x, M_θ, $M_{x\theta}$ and $M_{\theta x}$ are defined on the midsurface. The positive directions of these quantities as well as the shell geometry and the coordinates system are depicted in Fig. 1.

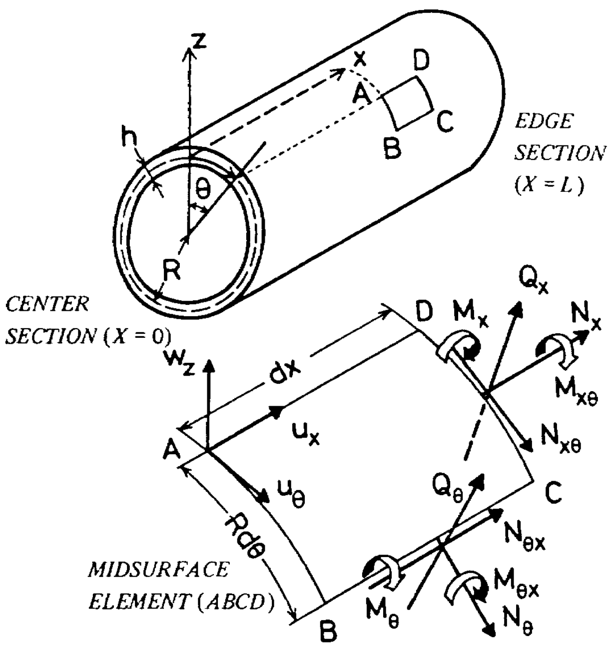

Fig. 1 Shell Geometries, Coordinates System, and Positive Directions of Basic Quantities

For infinitesimal displacements, the midsurface strains ϵ_x, ϵ_θ and $\gamma_{x\theta}$, the changes in curvature κ_x and κ_θ, the relative torsion τ, and the rigid-body rotation ω_n about the outward normal to the midsurface are related with the midsurface displacement components by

$$
\begin{aligned}
\epsilon_x &= \partial u_x/\partial x \\
\epsilon_\theta &= (\partial u_\theta/\partial\theta + w_z)/R \\
\gamma_{x\theta} &= \partial u_\theta/\partial x + (\partial u_x/\partial\theta)/R \\
\kappa_x &= -\partial^2 w_z/\partial x^2 \\
\kappa_\theta &= (\partial u_\theta/\partial\theta - \partial^2 w_z/\partial\theta^2)/R^2 \\
2R\tau &= 3\partial u_\theta/\partial x - (\partial u_x/\partial\theta)/R - 4\partial^2 w_z/\partial x\partial\theta \\
2\omega_n &= \partial u_\theta/\partial x - (\partial u_x/\partial\theta)/R
\end{aligned}
\tag{1}
$$

Here, τ is referred to as the relative torsion so as to distinguish it from a torsion defined by $\tau - \omega_n/R$, which may be referred to as the absolute torsion.

Let it be assumed that the shell is subjected to the external loads only at its cross sectional edges. That is to say that neither surface traction nor concentrated load is applied on the thickness-bounding surface. Then, the equilibrium equations are given by the following six homogeneous equations:

$$
\begin{aligned}
&R\partial N_x/\partial x + \partial N_{\theta x}/\partial\theta = 0 \\
&\partial N_\theta/\partial\theta + R\partial N_{x\theta}/\partial x + Q_\theta = 0 \\
&R\partial M_x/\partial x + \partial M_{\theta x}/\partial\theta - RQ_x = 0 \\
&\partial M_\theta/\partial\theta + R\partial M_{x\theta}/\partial x - RQ_\theta = 0 \\
&R\partial Q_x/\partial x + \partial Q_\theta/\partial\theta - N_\theta = 0 \\
&M_{\theta x} + R(N_{\theta x} - N_{x\theta}) = 0
\end{aligned}
\tag{2}
$$

The constitutive equations formulated in the shell theory based on Kirchhoff-Love's hypothesis can most often be written formally as

$$
\begin{aligned}
N_x &= K(\epsilon_x + \nu\epsilon_\theta + \delta E_1 R\kappa_x) \\
N_\theta &= K\,[\,(1 + \delta E_2)\epsilon_\theta + \nu\epsilon_x - \delta E_1 R\kappa_\theta\,] \\
N_{x\theta} &= Gh\,[\,(1 + \delta E_3/4)\gamma_{x\theta} + \delta E_4 R\tau/2] \\
N_{\theta x} &= Gh\,[\,(1 + 3\delta E_5/4)\,\gamma_{x\theta} - \delta E_6 R\tau/2] \\
M_x &= D(\kappa_x + \nu\kappa_\theta + D_1\epsilon_x/R)
\end{aligned}
\tag{3}
$$

$$M_\theta = D(\kappa_\theta + \nu\kappa_x - D_1\epsilon_\theta/R)$$

$$M_{x\theta} = (Gh^3/12)(\tau - D_4\omega/R + D_2\gamma_{x\theta}/2R)$$

$$M_{\theta x} = (Gh^3/12)(\tau - D_4\omega_n/R - D_3\gamma_{x\theta}/2R)$$

where K, D and G are the elastic constants given by

$$K = Eh/(1-\nu^2)$$

$$D = Eh^3/12(1-\nu^2) \tag{4}$$

$$G = E/2(1+\nu)$$

and δ is a geometric parameter defined by

$$\delta = h^2/12R^2 \tag{5}$$

The parameters $E_i(i = 1, 2, \ldots, 6)$ and $D_j(j = 1, 2, 3, 4)$ are quantities of order of magnitude unity, whose specific values differ with different theories. We shall refer to these parameters as the theory indicators. If E_i and D_j are specified in accordance with Table 1, Eqs. (3) become identical, or at least equivalent, to those constitutive equations formulated in the Flügge [3], the Naghdi [16], the Koiter [2], the Novozhilov [4], and the Love-Reissner [5] theories. Here, Reissner's version of Love's first approximation is referred to as the Love-Reissner theory.

	E_1	E_2	E_3	E_4	E_5	E_6	D_1	D_2	D_3	D_4
Flügge	1	1	1	1	1	1	1	1	1	0
Naghdi	1	0	1	1	1	1	1	1	1	0
Koiter	0	0	0	1	0	1	0	0	0	0
Novezhilov	0	0	2	2	0	0	0	1	-1	0
Love-Reissner	0	0	0	0	0	0	0	0	0	1

Table 1 Values of Theory Indicators E_i and D_j

The sixth of Eqs. (2) is concerned with the balance of moment about the normal to the midsurface. This is a redundant relation, because the left-hand side vanishes identically if the use is made of the integral form of the definition of the stress resultants and couples in terms of the stresses. It should be noted here that the constitutive equations of the Love-Reissner theory fail to fulfil this identity requirement. As a matter of fact, if the use is made of Eqs. (3), the left-hand side of the sixth of Eqs. (2) can be written in terms of $\gamma_{x\theta}$, τ and ω_n as

$$M_{\theta x} + R(N_{\theta x} - N_{x\theta}) = GhR\delta\{[1-(E_4+E_6)/2]R\tau - D_4\omega_n + (3E_5 - E_3 - 2D_3)\gamma_{x\theta}/4\} \tag{6}$$

Specifying the values of E_i and D_j as given in Table 1, we see that the coefficients of the right-hand members of Eq. (6) vanish identically for all the theories but the Love-Reissner. For the Love-Reissner theory, Eq. (6) reads

$$M_{\theta x} + R(N_{\theta x} - N_{x\theta}) = GhR\delta(R\tau - \omega_n) \tag{7}$$

The residual moment on the right-hand side of Eq. (7) is proportional to the absolute torsion $\tau - \omega_n/R$. Thus, the Love-Reissner theory may encounter undesirable situation when the deformations are characterized by large torsion of the midsurface. The magnitude of the errors in the solutions due to this shortcoming of the Love-Reissner theory are yet to be examined.

The following nondimensional quantities and operators are introduced to present the subsequent developments in nondimensional form:

$$u = u_x/R,\ v = u_\theta/R,\ w = w_z/R,\ y = x/R,\ l = L/R$$

$$N = N_x/K,\ M = RM_x/D,\ S = S_x/\delta K,\ Q = T_{x\theta}/K \tag{8}$$

$$(\ )' = \partial(\ )/\partial y,\ (\ )^{\cdot} = \partial(\ )/\partial\theta,\ \nabla^2(\ ) = (\ )'' + (\ )^{\cdot\cdot}$$

where S_x and $T_{x\theta}$ are the equivalent edge-shears defined by

$$S_x = Q_x + (\partial M_{x\theta}/\partial\theta)/R$$

$$T_{x\theta} = N_{x\theta} + M_{x\theta}/R \tag{9}$$

3. GOVERNING EQUATIONS

In this section, the equilibrium equations (2) are reduced to a single eighth order differential equation for w, and the basic quantities to be prescribed as boundary conditions are expressed only in terms of w.

The lateral shears Q_x and Q_θ are eliminated from the first five of Eqs. (2). This reduces the number of equilibrium equations from five to three. These three equations are written in terms of the displacement components with the aid of Eqs. (1) and (3). The result may be written in the form

$$G_1 u'' + G_2 u^{\cdot\cdot} + G_3 v'^{\cdot} + G_4 w''' + G_5 w'^{\cdot\cdot} + G_6 w' = 0$$

$$H_1 v'' + H_2 v^{\cdot\cdot} + H_3 u'^{\cdot} + H_4 w''^{\cdot} + H_5 w^{\cdot\cdot\cdot} + H_6 w^{\cdot} = 0$$

$$K_1 u''' + K_2 u'^{\cdot\cdot} + K_3 u' + K_4 v''^{\cdot} + K_5 v^{\cdot\cdot\cdot} + K_6 v^{\cdot} - \nabla^4 w - K_7 w^{\cdot\cdot} - K_8 w = 0 \qquad (10)$$

where the coefficients G_i, H_i and K_i are constants whose explicit expressions in terms of ν, δ, E_i, and D_j are given in Appendix A.

The first of Eqs. (10) gives $v'^{\cdot}$ in terms of the derivatives of u and w. This is substituted into the second of Eqs. (10) after differentiating it once for each with respect to y and θ to eliminate v. It results in a differential equation for u and w. In a similar manner, u can be eliminated from the first two of Eqs. (10) and a differential equation for v and w is obtained. The result may be written in the form

$$Lu = f_1 w''''' + f_2 w'''^{\cdot\cdot} + f_3 w'^{\cdot\cdot\cdot\cdot} + f_4 w''' + f_5 w'^{\cdot\cdot} \qquad (11)$$

$$Lv = g_1 w''''^{\cdot} + g_2 w''^{\cdot\cdot\cdot} + g_3 w^{\cdot\cdot\cdot\cdot\cdot} + g_4 w''^{\cdot} + g_5 w^{\cdot\cdot\cdot} \qquad (12)$$

where $L(\ )$ is a linear differential operator defined by

$$L(\ ) = L_1(\ )'''' + L_2(\ )''^{\cdot\cdot} + L_3(\ )^{\cdot\cdot\cdot\cdot} \qquad (13)$$

The explicit expressions of the coefficients f_i, g_i and L_i in terms of G_i and H_i are given in Appendix A.

The differential operator $L(\ )$ is applied to the third of Eqs. (10) and the terms in it with Lu and Lv are substituted from Eqs. (11) and (12) to eliminate u and v. As a result, a single eighth order differential equation for w is derived, which may be written formally as

$$a_{81} w'''''''' + a_{82} w''''''^{\cdot\cdot} + a_{83} w''''^{\cdot\cdot\cdot\cdot} + a_{84} w''^{\cdot\cdot\cdot\cdot\cdot\cdot} + a_{85} w^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot} + a_{61} w'''''' + a_{62} w''''^{\cdot\cdot} + a_{63} w''^{\cdot\cdot\cdot\cdot} + a_{64} w^{\cdot\cdot\cdot\cdot\cdot\cdot} + a_{41} w'''' + a_{42} w''^{\cdot\cdot} + a_{43} w^{\cdot\cdot\cdot\cdot} = 0 \qquad (14)$$

where a_{ij} are constant coefficients whose explicit expressions in terms of f_i, g_i, K_i and L_i are presented in Appendix A.

Equation (14) is the governing equation for small deformations of circular cylindrical shells subjected to the edgewise loading. This equation is exact in the sense that no approximation has been made throughout the entire process of derivation starting from the basic equations, Eqs. (1), (2) and (3).

The governing equation is to be solved under a proper set of the boundary conditions. In the present paper, we shall deal only with those circumferentially closed shells whose midsurface boundary lines consist of the two cross sectional circles at $y = \pm\, l$. Thus, the boundary conditions are prescribed only at $y = \pm\, l$, and the quantities to be prescribed consist of appropriate combinations of the following four pairs:

(i) w or S, (ii) w' or M,
(iii) u or N, (iv) v or Q.

If use is made of the third of Eqs. (2), Q_x can be eliminated from the equation defining S. Thus, S is given in terms of the derivatives of M_x, $M_{x\theta}$ and $M_{\theta x}$. All the physical quantities, N, M, Q and S, can now be expressed in terms of u, v and w with the aid of Eqs. (1) and (3). The differential operator $L(\ )$ is applied to these equations and the terms in them with Lu and Lv are substituted from Eqs. (11) and (12). Then, LN, LM, LQ and LS are given only in terms of w. The result may be written in the form

$$LN = n_1 w''''^{\cdot\cdot} + n_2 w''^{\cdot\cdot\cdot\cdot} + n_3 w^{\cdot\cdot\cdot\cdot\cdot\cdot} + n_4 w''^{\cdot\cdot} + n_5 w^{\cdot\cdot\cdot\cdot} \qquad (15)$$

$$LM = m_1 w'''''' + m_2 w''''^{\cdot\cdot} + m_3 w''^{\cdot\cdot\cdot\cdot} + m_4 w^{\cdot\cdot\cdot\cdot\cdot\cdot} + m_5 w'''' + m_6 w''^{\cdot\cdot} + m_7 w^{\cdot\cdot\cdot\cdot} \qquad (16)$$

$$LQ = q_1 w''''^{\cdot} + q_2 w'''^{\cdot\cdot\cdot} + q_3 w'^{\cdot\cdot\cdot\cdot\cdot} + q_4 w'''^{\cdot} + q_5 w'^{\cdot\cdot\cdot} \qquad (17)$$

$$LS = s_1 w''''''' + s_2 w'''''^{\cdot\cdot} + s_3 w'''^{\cdot\cdot\cdot\cdot} + s_4{}'^{\cdot\cdot\cdot\cdot\cdot\cdot} + s_5 w''''' + s_6 w'''^{\cdot\cdot} + s_7 w'^{\cdot\cdot\cdot\cdot} \qquad (18)$$

where n_i, m_i, q_i and s_i are constant coefficients whose explicit expressions in terms of f_i, g_i and L_i are given in Appendix A. We shall refer to Eqs. (15) – (18) together with Eqs. (11) and (12) as the supplemental equations. Once the boundary conditions are prescribed, the supplemental equations impose the boundary constraining conditions on w.

The exact expressions of the coefficients presented in Appendix A can be arranged in the form of polynomials in δ if they are written explicitly in terms of ν, δ, E_i and D_j. For thin shells, therefore, a first approximation may be achieved by taking only the leading terms in these polynomial expressions. The leading terms are designated by the superscript o, such as a_{ij}^{0} for a_{ij}, and their explicit expressions are given in Appendix B.

It is well-known that the followings hold under Kirchhoff-Love's hypothesis:

$$h/R \ll 1 \quad \text{and} \quad (h/\lambda)^2 \ll 1 \tag{19}$$

where λ is the wave length of deformation. The first of these immediately gives

$$\delta \ll 1 \tag{20}$$

A first approximation of the governing and the supplemental equations consistent to the Kirchhoff-Love hypothesis may be achieved, therefore, by replacing the coefficients with their leading terms in the polynomial expressions.

The governing equation, Eq. (14), thus reduces to

$$\nabla^8 w + 2w^{\cdot\cdot\cdot\cdot\cdot\cdot} + (1-\nu^2)w^{\prime\prime\prime\prime}/\delta + w^{\cdot\cdot\cdot\cdot}$$
$$+ a_{61}^{0} w^{\prime\prime\prime\prime\prime\prime} + a_{62}^{0} w^{\prime\prime\prime\prime\cdot\cdot} + a_{63}^{0} w^{\prime\prime\cdot\cdot\cdot\cdot} + a_{42}^{0} w^{\prime\prime\cdot\cdot} = 0 \tag{21}$$

where the coefficients a_{61}^{0}, a_{62}^{0}, a_{63}^{0} and a_{42}^{0} depend on the theory indicators E_i and D_j, so that their values differ with different theories. Since the terms of order of magnitude δ are neglected in Eq. (21), it is admissible to add to or subtract from Eq. (21) terms of order of magnitude δ in order to write it in a convenient form. Here, a term $w^{\prime\prime\prime\prime}$ is added to the left-hand members of Eq. (21) to write it in the form

$$\nabla^4(\nabla^2+1)^2 w + 4k^4 w^{\prime\prime\prime\prime}$$
$$+ \bar{a}_{61} w^{\prime\prime\prime\prime\prime\prime} + \bar{a}_{62} w^{\prime\prime\prime\prime\cdot\cdot} + \bar{a}_{63} w^{\prime\prime\cdot\cdot\cdot\cdot} + \bar{a}_{42} w^{\prime\prime\cdot\cdot} = 0 \tag{22}$$

where

$$\begin{aligned} \bar{a}_{61} &= a_{61}^{0} - 2, & \bar{a}_{62} &= a_{62}^{0} - 6 \\ \bar{a}_{63} &= a_{63}^{0} - 6, & \bar{a}_{42} &= a_{42}^{0} - 2 \end{aligned} \tag{23}$$

and

$$4k^4 = 12(1-\nu^2)(R/h)^2 = (1-\nu^2)/\delta \tag{24}$$

It is noted that Eq. (22) becomes identical to Morley's [17] equation, if the terms with the coefficients $\bar{a}_{ij}$ are neglected.

Similarly, the first approximation to the supplemental equations is achieved. The result is

$$\begin{aligned}
\nabla^4 u &= -\nu w^{\prime\prime\prime} + w^{\prime\cdot\cdot} + \delta(f_1^{0} w^{\prime\prime\prime\prime\prime} + f_2^{0} w^{\prime\prime\prime\cdot\cdot} + f_3^{0} w^{\prime\cdot\cdot\cdot\cdot}) \\
\nabla^4 v &= -(2+\nu)w^{\prime\prime\cdot} - w^{\cdot\cdot\cdot} \\
&\quad + \delta(g_1^{0} w^{\prime\prime\prime\prime\cdot} + g_2^{0} w^{\prime\prime\cdot\cdot\cdot} + g_3^{0} w^{\cdot\cdot\cdot\cdot\cdot}) \\
\nabla^4 N &= (1-\nu^2)w^{\prime\prime\cdot\cdot} \\
&\quad + \delta(n_1^{0} w^{\prime\prime\prime\prime\cdot\cdot} + n_2^{0} w^{\prime\prime\cdot\cdot\cdot\cdot} + n_3^{0} w^{\cdot\cdot\cdot\cdot\cdot\cdot} + n_5^{0} w^{\cdot\cdot\cdot\cdot}) \\
\nabla^4 Q &= -(1-\nu^2)w^{\prime\prime\prime\cdot} \\
&\quad + \delta(q_1^{0} w^{\prime\prime\prime\prime\prime\cdot} + q_2^{0} w^{\prime\prime\prime\cdot\cdot\cdot} + q_3^{0} w^{\prime\cdot\cdot\cdot\cdot\cdot} + q_5^{0} w^{\prime\cdot\cdot\cdot}) \\
\nabla^4 M &= -\nabla^4(w^{\prime\prime} + \nu w^{\cdot\cdot}) + m_5^{0} w^{\prime\prime\prime\prime} + m_6^{0} w^{\prime\prime\cdot\cdot} - \nu w^{\cdot\cdot\cdot\cdot} \\
\nabla^4 S &= -\nabla^4[w^{\prime\prime\prime} + (2-\nu)w^{\prime\cdot\cdot}] \\
&\quad + s_5^{0} w^{\prime\prime\prime\prime\prime} + s_6^{0} w^{\prime\prime\prime\cdot\cdot} + s_7^{0} w^{\prime\cdot\cdot\cdot\cdot}
\end{aligned} \tag{25}$$

where the coefficients f_i^{0}, g_i^{0}, n_i^{0}, q_i^{0}, m_i^{0} and s_i^{0} depend on the theory indicators E_i and D_j, so that their values differ with different theories.

4. REDUCTION OF THE 8TH ORDER DIFFERENTIAL EQUATION TO THE 4TH ORDER

Let a complex differential operator $P(\ )$ be defined by

$$P(\ ) = \nabla^2(\nabla^2+1)(\ )$$
$$+ i\{2k^2(\ )^{\prime\prime} + [\bar{a}_{61}(\ )^{\prime\prime\prime\prime} + \bar{a}_{62}(\ )^{\prime\prime\cdot\cdot} + \bar{a}_{63}(\ )^{\cdot\cdot\cdot\cdot}$$
$$+ \bar{a}_{42}(\ )^{\cdot\cdot}]/4k^2\} \tag{26}$$

where i is unit of imaginary number. Let the complex conjugate to $P(\ )$ be denoted by $\bar{P}(\ )$. Then, a simultaneous application of $P(\ )$ and $\bar{P}(\ )$ upon w results in

$$P\bar{P}w = \nabla^4(\nabla^2+1)^2 w + 4k^4 w^{\prime\prime\prime\prime}$$
$$+ \bar{a}_{61} w^{\prime\prime\prime\prime\prime\prime} + \bar{a}_{62} w^{\prime\prime\prime\prime\cdot\cdot} + \bar{a}_{63} w^{\prime\prime\cdot\cdot\cdot\cdot} + \bar{a}_{42}{}^{\prime\prime\cdot\cdot}$$
$$+ \delta[\bar{a}_{61}^2 w^{\prime\prime\prime\prime\prime\prime\prime\prime} + 2\bar{a}_{61}\bar{a}_{62} w^{\prime\prime\prime\prime\prime\prime\cdot\cdot}$$
$$+ (\bar{a}_{62}^2 + 2\bar{a}_{61}\bar{a}_{63})w^{\prime\prime\prime\prime\cdot\cdot\cdot\cdot} + 2\bar{a}_{62}\bar{a}_{63} w^{\prime\prime\cdot\cdot\cdot\cdot\cdot\cdot}$$
$$+ \bar{a}_{63}^2 w^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot} + 2\bar{a}_{61}\bar{a}_{42} w^{\prime\prime\prime\prime\cdot\cdot} + 2\bar{a}_{62}\bar{a}_{42} w^{\prime\prime\cdot\cdot\cdot\cdot}$$
$$+ 2\bar{a}_{63}\bar{a}_{42} w^{\cdot\cdot\cdot\cdot\cdot\cdot} + \bar{a}_{42}^2 w^{\cdot\cdot\cdot\cdot}]/4(1-\nu^2) \tag{27}$$

The terms in the square blackets multiplied by δ are negligible within the errors of the present approximation. The right-hand side of Eq. (27) thus becomes identical to the left-hand side of Eq. (22). Consequently, the governing equation, Eq. (21) or (22), can be written as a first approximation in the form

$$P\bar{P}w = 0 \tag{28}$$

Since $P(\ )$ and $\bar{P}(\ )$ are complex conjugate to each other, the solution of Eq. (28) is given as the sum of the solution of either $Pw = 0$ or $\bar{P}w = 0$ and its conjugate. Thus, we only need to solve either one of these equations. Take, for instance, $Pw = 0$. Then, the governing equation reads

$$\nabla^2(\nabla^2 + 1)w + \boldsymbol{i}\,[2k^2 w'' + (\bar{a}_{61} w'''' + \bar{a}_{62} w''^{\cdot\cdot} + \bar{a}_{63} w^{\cdot\cdot\cdot\cdot} + \bar{a}_{42} w^{\cdot\cdot})/4k^2] = 0 \tag{29}$$

The order of the differential equation is thus reduced from the eighth to the fourth, which substantially eases the solution of the governing equation.

The reduction of the order of the governing differential equations has been a subject of shell research since 1912 when H. Reissner [18] derived for the axisymmetric deformations of spherical shells a set of two simultaneous differential equations of the second order in terms of a stress function and the meridional rotation. A number of researchers have worked out the analyses aiming at this goal in various ways for various shell configurations. Their analyses may be divided into the following three groups depending on the modes of approach:

1) The governing differential equation of the eighth order is reduced to a set of two simultaneous differential equations of the fourth order in terms of a displacement function and a stress function. Reissner's analysis on the axisymmetric deformations of spherical shells belongs to this group. Meissner [19] extended Reissner's analysis to be valid for the axisymmetric deformations of general shells of revolution. Both Reissner's and Meissner's analyses are well-known classical examples of the attempts to reducing the order of the governing differential equations, though their reductions are from the fourth to the second due to the limitations of their analyses to the axisymmetric deformations. More recently, Wan [20], Simmonds [21], and Latta and Simmonds [22] followed the same approach to work out on the differential equations for more general deformations of shells with various midsurface configurations.

2) A complex displacement-stress function is introduced combining a displacement function and a stress function in the form of a complex variable. The governing equation of the eighth order can be reduced to a single complex differential equation of the fourth order in terms of the complex displacement-stress function. One can find some of the analyses belonging to this group in the literatures authored by Novozhilov [4], Ichino and Takahashi [23], and Simmonds [24].

3) The eighth order differential equation is decomposed into a pair of complex conjugate differential equations of the fourth order. The approach followed in the present paper belongs to this group. Cheng [25] has shown that such a decomposition is possible for the Flügge theory for circular cylindrical shells. The decomposition of the differential equation also implies that, when the dependent variables are expanded into eigenfunctions, the characteristic equation of the eighth degree can be decomposed into a pair of complex conjugate algebraic equations of the fourth degree, and that the eigenvalues are calculated in closed form. Mizoguchi [26] has shown that the characteristic equation for circular cylindrical shells can be decomposed within the theory developed by himself and in essence identical to the Novozhilov theory. Similarly, Koga and Toda [27] has achieved the decomposition of the characteristic equations for various shell theories for circular cylindrical shells. The most important feature in the simplicity in form of Donnell's and Morley's equations lies in the fact that they can easily be decomposed into a pair of complex conjugate differential equations of the fourth order. As a matter of fact, we have

$$\nabla^4 w \pm \boldsymbol{i}\,2k^2 w'' = 0 \tag{30}$$

for Donnell's equation, and

$$\nabla^2(\nabla^2 + 1)w \pm \boldsymbol{i}\,2k^2 w'' = 0 \tag{31}$$

for Morley's equation, so that their characteristic equations can also be decomposed and the eigenvalues be calculated in closed form.

The result of the present analysis, Eq. (29), indicates that such a decomposition of the governing differential equation for circular cylindrical shells can be achieved for all the classical theories including the Flügge, the Koiter-Sanders, the Novozhilov, and the Love-Reissner theories, and that the eigenvalues can be calculated in closed form.

5. EIGENVALUES FOR $n \geq 2$

Let it be assumed that the quantities to be prescribed as boundary conditions at the edges $y = \pm l$ are continuous along the circumference, so that the functional representations of these quantities can be expanded into the Fourier series in θ. Then, the n-th Fourier component of w may be given in the form of eigenfunctions expansion;

$$w = \sum_{i=1}^{8} W_i e^{p_i y} \cos n\theta \tag{32}$$

where W_i are the as yet unknown coefficients and p_i the eigenvalues. The summation is carried over from $i = 1$ to 8, because there exist eight eigenvalues, p_i ($i = 1, 2, \ldots, 8$), for the eighth order governing differential equation, Eq. (22).

Let the circumferential wave length of the n-th Fourier component of deformation be denoted by λ_θ. Then, we have

$$\lambda_\theta = 2\pi R/n \tag{33}$$

which gives

$$\begin{aligned}(h/\lambda_\theta)^2 &= (3/\pi^2)\,\delta n^2 \\ &= [3(1-\nu^2)/\pi^2]\; n^2/4k^4\end{aligned} \tag{34}$$

It follows from the second of the fundamental assumptions, Eqs. (19), that the range of n is limited by

$$\delta n^2 \ll 1 \text{ (or, equivalently, } n^2/4k^4 \ll 1) \tag{35}$$

Substitution from Eq. (32) to Eq. (29) yields the characteristic equation in the form of a quartic algebraic equation for p_i

$$A p_i^4 + B p_i^2 + C = 0 \tag{36}$$

where

$$\begin{aligned}A &= 1 + i\,\bar{a}_{61}/4k^2 \\ B &= 1 - 2n^2 + i\,(2k^2 - \bar{a}_{62} n^2/4k^2) \\ C &= n^2(n^2-1) + i\,(\bar{a}_{63}n^2 - \bar{a}_{42})n^2/4k^2\end{aligned} \tag{37}$$

It can be shown easily that, for $n = 1$, C vanishes identically except for the Love-Reissner theory. When $C = 0$, Eq. (36) has vanishing double roots, $p_i^2 = 0$, so that w is represented not in the exponential function as given in Eq. (32) but in a polynomial function in y. A special consideration is needed, therefore, for dealing with the case of $n = 1$. Similarly, it is readily seen that $C = 0$ for $n = 0$. But, since $(\ )^{\cdot} \equiv 0$ in this case, we must restore our analysis to the basic equations specified for the axisymmetric deformations and derive a governing differential equation of the fourth order. In the present paper, we shall deal only with those cases characterized by $n \geq 2$ leaving detailed analyses for $n = 0$ and 1 to a separate paper to follow.

For $n \geq 2$, formal solution of Eq. (36) is a trivial matter. It immediately gives

$$p_i = \pm \left\{ [-B \pm (B^2 - 4AC)^{1/2}]/2A \right\}^{1/2} \tag{38}$$

The right-hand members of Eq. (38) are calculated with the aid of the following formulae for the square root of complex variables: Let a complex variable Z be denoted by

$$Z = X + i\,Y \tag{39}$$

where X and Y are real. Let the absolute value of Z be denoted by α, so that

$$\alpha = (X^2 + Y^2)^{1/2} \tag{40}$$

Then, the square root of Z is calculated by

$$Z^{1/2} = \pm \left(\frac{\alpha + X}{2}\right)^{1/2} + i\left(\frac{\alpha - X}{2}\right)^{1/2} \tag{41}$$

where the + and − signs should be assigned depending on $Y > 0$ and $Y < 0$, respectively.

Let us first apply the above formulae for calculating the square root of $B^2 - 4AC$ by writing

$$B^2 - 4AC = X_0 + i\,Y_0$$

Direct substitution of A, B and C from Eqs. (37) gives

$$\begin{aligned}X_0 &= 1 + \bar{a}_{62}n^2 - 4k^4 + \Delta X_0[n^4]/k^4 \\ Y_0 &= -k^2[4(2n^2-1) + \Delta Y_0[n^4]/k^4]\end{aligned} \tag{42}$$

with

$$\Delta X_0[n^4] = [(\bar{a}_{61}\bar{a}_{63} - \bar{a}_{62}^2/4)n^4 - \bar{a}_{61}\bar{a}_{42}n^2]/4$$

$$\Delta Y_0[n^4] = (\bar{a}_{63} + \bar{a}_{61} - \bar{a}_{62})n^4 + (\bar{a}_{62}/2 - \bar{a}_{42} - \bar{a}_{61})n^2 \tag{43}$$

Here, the notations $\Delta X_0[n^4]$ and $\Delta Y_0[n^4]$ are used to indicate that the largest terms in them are proportional to n^4, and that $\Delta X_0[n^4]/k^4$ and $\Delta Y_0[n^4]/k^4$ are small quantities in comparison with unity except for relatively large values of n.

The absolute value of $X_0 + i\,Y_0$ is calculated as

$$\begin{aligned}\alpha_0 &= (X_0^2 + Y_0^2)^{1/2}\\ &= 4k^4\{1 + [2(2n^2-1)^2 - (\bar{a}_{62}n^2+1)]/2k^4\\ &\quad + [(\bar{a}_{62}n^2+1)^2/8 - \Delta X_0 + (2n^2-1)\Delta Y_0]/2k^8\\ &\quad + (\Delta Y_0)^2/16k^{12}\}^{1/2}\end{aligned} \tag{44}$$

where the terms of order of magnitude $(n^2/4k^4)^3$ in the square root have been neglected.

Let it be assumed that

$$(2n^2/k^2)^2 \ll 1 \tag{45}$$

Then, the square root on the right-hand side of Eq. (44) can be expanded into an infinite series, which may be truncated neglecting small terms of order of magnitude $(4n^4/k^4)^3$. As a result, we have

$$\begin{aligned}\alpha_0 &= 4k^4\{1 + [2(2n^2-1)^2 - (\bar{a}_{62}n^2+1)]/4k^4\\ &\quad + [-(2n^2-1)^4/2 + (\bar{a}_{62}n^2+1)(2n^2-1)^2/2\\ &\quad - \Delta X_0 + (2n^2-1)\Delta Y_0]/4k^8\}\end{aligned} \tag{46}$$

It follows that

$$\begin{aligned}\left(\frac{\alpha_0+X_0}{2}\right)^{1/2} &= (2n^2-1)[1 + \Delta F[n^4]/4k^4]^{1/2}\\ \left(\frac{\alpha_0-X_0}{2}\right)^{1/2} &= 2k^2\{1 + [(2n^2-1)^2 - (\bar{a}_{62}n^2+1)]\\ &\quad /4k^4 + \Delta G[n^8]/8k^8\}^{1/2}\end{aligned} \tag{47}$$

with

$$\begin{aligned}\Delta F[n^4] &= -(2n^2-1)^2 + \bar{a}_{62}n^2 + 1 + 2\Delta Y_0/(2n^2-1)\\ \Delta G[n^8] &= -(2n^2-1)^4/2 + (\bar{a}_{62}n^2+1)(2n^2-1)^2/2\\ &\quad - 2\Delta X_0 + (2n^2-1)\Delta Y_0\end{aligned} \tag{48}$$

Here, the notations $\Delta F[n^4]$ and $\Delta G[n^8]$ are used to indicate that the largest terms in them are proportional to n^4 and n^8, respectively, and that $\Delta F[n^4]/k^4$ and $\Delta G[n^8]/k^8$ are small quantities in comparison with unity under the assumption stated in Eq. (45).

Again, the square roots in the right-hand members of Eqs. (47) can be expanded into infinite series. The result may be written within the accuracy of the present approximation in the form

$$\begin{aligned}\left(\frac{\alpha_0+X_0}{2}\right)^{1/2} &= (2n^2-1)\,[1 + \Delta F/8k^4 - (\Delta F)^2/128k^8]\\ \left(\frac{\alpha_0-X_0}{2}\right)^{1/2} &= 2k^2\{1 + [(2n^2-1)^2 - (\bar{a}_{62}n^2+1)]/8k^4\\ &\quad + \{\Delta G - [(2n^2-1)^2 - (\bar{a}_{62}n^2+1)]^2/8\}/16k^8\}\end{aligned} \tag{49}$$

With these results available, we can calculate the square root of B^2-4AC as

$$(B^2-4AC)^{1/2} = -\left(\frac{\alpha_0+X_0}{2}\right)^{1/2} + i\left(\frac{\alpha_0-X_0}{2}\right)^{1/2} \tag{50}$$

We now proceed to calculating the square root of $[-B \pm (B^2-4AC)^{1/2}]/2A$ by writing

$$\begin{aligned}[-B + (B^2-4AC)^{1/2}]/2A &= X_1^+ + i\,Y_1^+\\ [-B - (B^2-4AC)^{1/2}]/2A &= X_1^- + i\,Y_1^-\end{aligned} \tag{51}$$

The left-hand members of Eqs. (51) may be written out explicitly with the aid of Eqs. (37), (49) and (50). After an appropriate approximation, we have

$$\begin{aligned}X_1^+ &= [(2n^2-1)^2\bar{a}_{61} - \bar{a}_{61} - 2(2n^2-1)\Delta F[n^4]]/32k^4\\ Y_1^+ &= n^2(n^2-1)\,[1 + \Delta Y_1^+[n^4]/k^4]/2k^2\\ X_1^- &= (2n^2 - 1 - \bar{a}_{61}/2)\,[1 + \Delta X_1^-[n^4]/k^4]\\ Y_1^- &= -2k^2 + [2\bar{a}_{62}n^2 - 2\bar{a}_{61}(2n^2-1) - 4n^4\\ &\quad + 4n^2]/8k^2\end{aligned} \tag{52}$$

where

$$\begin{aligned}\Delta Y_1^+[n^4] &= \{\Delta G[n^8] - [(2n^2-1)^2 - (\bar{a}_{62}n^2+1)]^2/8\\ &\quad + \bar{a}_{61}(2n^2-1)\Delta F[n^4]/4\}/8n^2(n^2-1)\\ \Delta X_1^-[n^4] &= [\bar{a}_{61}\bar{a}_{62}n^2/2 + (2n^2-1)\Delta F[n^4]\\ &\quad - \bar{a}_{61}(2n^2-1)^2/2 + \bar{a}_{61}(\bar{a}_{62}n^2+1)/2]\end{aligned} \tag{53}$$

$/16(2n^2 - 1 - \bar{a}_{61}/2)$

Proceeding as before, we can calculate the absolute values of $X_1^+ + i\,Y_1^+$ and $X_1^- + i\,Y_1^-$. The results are

$$\begin{aligned}\alpha_1^+ &= [(X_1^+)^2 + (Y_1^+)^2]^{1/2}\\ &= n^2(n^2-1)\Big\{1 + \{\Delta Y_1^+[n^4] + [(2n^2-1)^2\bar{a}_{61}\\ &\quad - 2(2n^2-1)\Delta F[n^4] - \bar{a}_{61}]^2/256n^4(n^2-1)^2\}\\ &\quad /k^4\Big\}/2k^2\\ \alpha_1^- &= [(X_1^-)^2 + (Y_1^-)^2]^{1/2}\\ &= 2k^2\{1 + [6n^4 - 6n^2 + 1 - \bar{a}_{62}n^2 + \bar{a}_{61}^2/4]/8k^4\}\end{aligned} \tag{54}$$

It follows then that

$$\begin{aligned}\left(\frac{\alpha_1^+ \pm X_1^+}{2}\right)^{1/2} &= \frac{n}{2k}(n^2-1)^{1/2}\\ &\times\left\{1 \pm \frac{1}{8k^2}[2(2n^2-1) - (\bar{a}_{63}n^2 - \bar{a}_{42})/(n^2-1)]\right\}\\ \left(\frac{\alpha_1^- \pm X_1^-}{2}\right)^{1/2} &= k\{1 \pm (2n^2 - 1 - \bar{a}_{61}/2)/4k^2\}\end{aligned} \tag{55}$$

If the terms of order of magnitude $1/2k^2$ are neglected in accordance with the fundamental assumptions, Eqs. (19), Eqs. (55) are simplified to

$$\begin{aligned}\left(\frac{\alpha_1^+ \pm X_1^+}{2}\right)^{1/2} &= \frac{n^2}{2k}(1 - 1/n^2)^{1/2}\left[1 \pm \frac{n^2}{2k^2}\right]\\ \left(\frac{\alpha_1^- \pm X_1^-}{2}\right)^{1/2} &= k\left[1 \pm \frac{n^2}{2k^2}\right]\end{aligned} \tag{56}$$

We thus obtain the explicit expressions for Eqs. (38) by the formula given in Eq. (41). If the complex conjugates are added, the complete eight eigenvalues p_i are given in the form

$$\begin{aligned}p_1, p_2, p_3, p_4 &= \pm(\xi_1 \pm i\,\eta_1)\\ p_5, p_6, p_7, p_8 &= \pm(\xi_2 \pm i\,\eta_2)\end{aligned} \tag{57}$$

with

$$\begin{aligned}\xi_1 &= \frac{n^2}{2k}(1 - 1/n^2)^{1/2}(1 + \frac{n^2}{2k^2})\\ \eta_1 &= \frac{n^2}{2k}(1 - 1/n^2)^{1/2}(1 - \frac{n^2}{2k^2})\\ \xi_2 &= k(1 + \frac{n^2}{2k^2})\\ \eta_2 &= k(1 - \frac{n^2}{2k^2})\end{aligned} \tag{58}$$

A similar result has been obtained by Gol'denweizer [28] by the method of asymptotic integration.

For a number of problems of practical importance, n takes relatively small values so that the following holds:

$$2n^2/k^2 \ll 1 \tag{59}$$

In this case, we have the simplest form of the eigenvalues as

$$\begin{aligned}\xi_1, \eta_1 &= \frac{n^2}{2k}(1 - 1/n^2)^{1/2}\\ \xi_2, \eta_2 &= k\end{aligned} \tag{60}$$

The numerical values of ξ_1, η_1, ξ_2 and η_2 are calculated with the aid of Eqs. (58) and (60) for various values of n and k. The results are compared in Table 2 with the rigorous numerical solutions obtained from the exact form of the governing equation, Eq. (14), specified for the Flügge theory. A satisfactory agreement is observed between the exact and the approximate solutions in the range of n/k specified by Eq. (59). In Table 2 are also included the solutions for $n = 1$ to show the approximate formulae, Eqs. (58) and (60), are valid even for that special case.

The most important feature in Eqs. (58) is that the terms with $\bar{a}_{61}$, $\bar{a}_{62}$, $\bar{a}_{63}$ and $\bar{a}_{42}$ have disappeared entirely as small quantities of order of magnitude h/R and δn^2. This indicates that the differences in the eigenvalues p_i calculated with different theories are of order of magnitude of the errors involved in the fundamental assumptions. In other words, as far as the eigenvalues p_i are concerned, any one of those classical theories will provide valid solutions within the accuracy of the first approximation. Another important feature is that ξ_2 and η_2 are proportional to k, whereas ξ_1 and η_1 are inversely proportional to it. Since k is a large quantity, this indicates that p_5, p_6, p_7 and p_8 represent the so-called edge-zone solutions which decay out rapidly as the distance from the edge increases, whereas p_1, p_2, p_3 and p_4 the global solutions which vary gradually over the entire surface of the shell.

k	n	Theories	ξ_1	η_1	ξ_2	η_2
5	1	Exact	0	0	5.0852	4.9153
		Approx. I	0	0	5.1000	4.9000
		Approx. II	0	0	5.0000	5.0000
	2	Exact	0.3645	0.3220	5.4034	4.6398
		Approx. I	0.3741	0.3187	5.4000	4.6000
		Approx. II	0.3464	0.3464	5.0000	5.0000
	3	Exact	0.9298	0.6865	5.9665	4.2775
		Approx. I	1.0013	0.6958	5.9000	4.1000
		Approx. II	0.8485	0.8485	5.0000	5.0000
	(4)	Exact	1.6996	1.0260	6.7356	3.9386
		Approx. I	2.0449	1.0535	6.6000	3.4000
		Approx. II	1.5492	1.5492	5.0000	5.0000
10	1	Exact	0	0	10.0425	9.9575
		Approx. I	0	0	10.0500	9.9500
		Approx. II	0	0	10.0000	10.0000
	2	Exact	0.1758	0.1704	10.1952	9.8104
		Approx. I	0.1767	0.1697	10.2000	9.8000
		Approx. II	0.1732	0.1732	10.0000	10.0000
	4	Exact	0.8227	0.7088	10.8407	9.2734
		Approx. I	0.8366	0.7126	10.8000	9.2000
		Approx. II	0.7746	0.7746	10.0000	10.0000
	6	Exact	1.9551	1.4102	11.9728	8.5722
		Approx. I	2.0943	1.4554	11.8000	8.2000
		Approx. II	1.7748	1.7748	10.0000	10.0000
	(8)	Exact	3.4910	2.0631	13.5087	7.9194
		Approx. I	4.1909	2.1589	13.2000	6.8000
		Approx. II	3.1749	3.1749	10.0000	10.0000
50	1	Exact	0	0	50.0085	49.9915
		Approx. I	0	0	50.0100	49.9900
		Approx. II	0	0	50.0000	50.0000
	2	Exact	0.0347	0.0346	50.0385	49.9615
		Approx. I	0.0347	0.0346	50.0400	49.9600
		Approx. II	0.0346	0.0346	50.0000	50.0000
	4	Exact	0.1554	0.1544	50.1590	49.8420
		Approx. I	0.1554	0.1544	50.1600	49.8400
		Approx. II	0.1549	0.1549	50.0000	50.0000
	6	Exact	0.3574	0.3524	50.3610	49.6440
		Approx. I	0.3575	0.3524	50.3600	49.6400
		Approx. II	0.3550	0.3550	50.0000	50.0000
	8	Exact	0.6428	0.6268	50.6463	49.3697
		Approx. I	0.6431	0.6269	50.6400	49.3600
		Approx. II	0.6350	0.6350	50.0000	50.0000
	10	Exact	1.0139	0.9745	51.0174	49.0220
		Approx. I	1.0149	0.9751	51.0000	49.0000
		Approx. II	0.9950	0.9950	50.0000	50.0000

Table 2 Numerical Comparison of ξ_i and η_i (Exact: from Eq. (14) for Flügge, Approx. I: from Eqs. (58), Approx. II: from Eqs. (60), (n): for which $2n^2/k^2 < 1$ doesn't hold)

We have thus far derived the eigenvalues in explicit closed form under the assumption $(2n^2/k^2)^2 \ll 1$ and $2n^2/k^2 \ll 1$. On the other hand, if we assume that

$$2n^2/k^2 = 0\,(1) \tag{61}$$

so that n and k are of the same order and

$$n^2 \gg 1 \tag{62}$$

the governing equation, Eq. (22), is simplified further to the form

$$\nabla^8 w + 4k^4 w'''' = 0 \tag{63}$$

The reduced fourth order differential equation becomes

$$\nabla^4 w + i\, 2k^2 w'' = 0 \tag{64}$$

Here, again, the terms with $\overline{a}_{61}$, $\overline{a}_{62}$, $\overline{a}_{63}$ and $\overline{a}_{42}$ are not present, indicating that the eigenvalues are not affected by the differences between the theories within the accuracy of the first approximation.

The characteristic equation resulting from the substitution from Eq. (32) into Eq. (64) reads

$$p_i^4 - 2(n^2 - i\,k^2)\,p_i^2 + n^4 = 0 \tag{65}$$

which gives

$$p_i^2 = n^2 - i\,k^2 \pm i\,k^2(1 + i\,2n^2/k^2)^{1/2} \tag{66}$$

If we write

$$\alpha_2 = [1 + (2n^2/k^2)^2]^{1/2} \tag{67}$$

the square root in the right-hand members of Eq. (66) can be evaluated with the aid of Eq. (41) and, consequently, Eq. (66) is rewritten as

$$p_i^2 = k^2\left\{\left[n^2/k^2 \mp \left(\frac{\alpha_2 - 1}{2}\right)^{1/2}\right] - i\left[1 \mp \left(\frac{\alpha_2 + 1}{2}\right)^{1/2}\right]\right\} \tag{68}$$

This indicates that the absolute values of p_i are of order of magnitude of k.

It is important for the subsequent developments to notice that the following relation holds in the entire range of n:

$$\delta\,|p_i^2| \ll 1 \tag{69}$$

Equation (63) is identical in form to Donnell's equation. Donnell's equation, however, has been derived on a more relaxed assumption than that stated by Eq. (62), such that a characteristic deformation wave is confined in a shallow portion of the shell, and it has frequently been applied as a convenient tool for the approximate analyses of deformations with relatively small values of n. Since we may assume that Eq. (59) holds for relatively small values of n, we can derive the explicit expressions of the eigenvalues for Donnell's equation in a similar manner to that used in the derivation of Eqs. (60). The result is

$$\begin{aligned} \xi_1, \eta_1 &= \frac{n^2}{2k} \\ \xi_2, \eta_2 &= k \end{aligned} \tag{70}$$

Comparing Eqs. (70) with Eqs. (60), we see that Eqs. (70) lack the factor $(1 - 1/n^2)^{1/2}$ in ξ_1 and η_1. This clearly shows the well-known characteristics of Donnell's equation that it provides reliable results for large values of n, whereas the results become increasingly inaccurate as n takes on smaller values. Since ξ_2 and η_2 are the same in both Eqs. (70) and (60), we may conclude that Donnell's equation is accurate enough to estimate the edge-zone solutions for any values of n.

6. BOUNDARY CONDITIONS

All the dependent variables to be prescribed as boundary conditions are expanded into Fourier series in θ. For each Fourier component, they may be written in the form

$$\begin{aligned} (u, \beta, N, M, S) &= \sum_{i=1}^{8} (U_i, B_i, N_i, M_i, S_i)\, e^{p_i y} \cos n\theta \\ (v, Q) &= \sum_{i=1}^{8} (V_i, Q_i)\, e^{p_i y} \sin n\theta \end{aligned} \tag{71}$$

where U_i, B_i, N_i, M_i, S_i, V_i and Q_i are as yet undetermined coefficients, and β is the axial rotation of the generator defined by

$$\beta = w' \tag{72}$$

All the coefficients on the right-hand side of Eqs.

(71) can be expressed in terms of W_i with the aid of Eqs. (25). Let us assume for simplicity that the values of n are limited by Eq. (45), and write

$$\Delta_0 = n^2/2k^2 \tag{73}$$

Then, an approximation consistent to that employed in the derivation of Eqs. (57) and (58) yields

For the global solutions ($i = 1, 2, 3, 4$):

$$
\begin{aligned}
U_i &= \mp (1/2k)(1 - 1/n^2)^{1/2} \left\{ [1 - (1+\nu)\Delta_0] \pm \boldsymbol{i}\,[1 + (1+\nu)\Delta_0] \right\} W_i \\
B_i &= \pm (n^2/2k)(1 - 1/n^2)^{1/2} [(1 + \Delta_0) \pm \boldsymbol{i}\,(1 - \Delta_0)]\, W_i \\
V_i &= -(1/n)\,[1 \mp \boldsymbol{i}\,\nu \Delta_0]\, W_i \\
N_i &= -(n^2/2k^2)(1 - 1/n^2)\,[\pm \boldsymbol{i}\,(1 - \nu^2)]\, W_i \\
M_i &= n^2(1 - 1/n^2)\,[\nu \mp \boldsymbol{i}\,\Delta_0]\, W_i \\
S_i &= \pm (n^4/2k)(1 - 1/n^2)^{1/2} \left\{ [2 - \nu + s_7^0/n^2 + (3-\nu)\Delta_0] \pm \boldsymbol{i}\,[2 - \nu + s_7^0/n^2 - (3-\nu)\Delta_0] \right\} W_i \\
Q_i &= \mp (1 - \nu^2)(n^3/4k^3)(1 - 1/n^2)^{3/2} [(1 - \Delta_0) \mp \boldsymbol{i}\,(1 + \Delta_0)]\, W_i
\end{aligned} \tag{74}
$$

For the edge-zone solutions ($i = 5, 6, 7, 8$):

$$
\begin{aligned}
U_i &= \mp (1/2k) \left\{ [\nu - (1+\nu)\Delta_0] \mp \boldsymbol{i}\,[\nu + (1+\nu)\Delta_0] \right\} W_i \\
B_i &= \pm k\, [(1 + \Delta_0) \pm \boldsymbol{i}\,(1 - \Delta_0)]\, W_i \\
V_i &= (n/2k^2)\,[\Delta_0 \mp \boldsymbol{i}\,(2 + \nu)]\, W_i \\
N_i &= -(n^2/2k^2)\,[\mp \boldsymbol{i}\,(1 - \nu^2)]\, W_i \\
M_i &= -2k^2\,[(2 - \nu)\Delta_0 \pm \boldsymbol{i}\,]\, W_i \\
S_i &= \pm 2k^3 \left\{ [1 - (1+\nu)\Delta_0] \mp \boldsymbol{i}\,[1 + (1+\nu)\Delta_0] \right\} W_i \\
Q_i &= \pm (1 - \nu^2)(n/2k)\,[(1 - \Delta_0) \mp \boldsymbol{i}\,(1 + \Delta_0)]\, W_i
\end{aligned} \tag{75}
$$

where the order of the + and − signs follows that of Eqs. (57).

The terms with s_7^0 in S_i are the only terms in Eqs. (74) and (75) that are affected by the differences between the theories. The explicit expressions of s_7^0 given in Appendix B shows that D_4 is the only theory indicator involved in it. It is anticipated, therefore, that the solutions of the Love-Reissner theory may deviate from those of the other theories if and only if the boundary conditions involve S.

Let us designate the prescribed values at the edges by the subscript e. Then, the boundary condition at one edge of the cylinder, say at $y = +l$, is given by a proper combination of the following four pairs of equations:

$$w\,|_{y=l} = w_e \cos n\theta \quad \text{or} \quad S\,|_{y=l} = S_e \cos n\theta \tag{76a}$$

$$\beta\,|_{y=l} = \beta_e \cos n\theta \quad \text{or} \quad M|_{y=l} = M_e \cos n\theta \tag{76b}$$

$$u\,|_{y=l} = u_e \cos n\theta \quad \text{or} \quad N|_{y=l} = N_e \cos n\theta \tag{76c}$$

$$v\,|_{y=l} = v_e \sin n\theta \quad \text{or} \quad Q\,|_{y=l} = Q_e \sin n\theta \tag{76d}$$

The left-hand members of Eqs. (76) are written in terms of W_i with the aid of Eqs. (74) and (75) specifying $y = l$. The results may be arranged in such a way that the terms representing the edge-zone solutions become of order of magnitude unity. Let it be first assumed that the values of n are limited by Eq. (59), namely, that $2n^2/k^2 \ll 1$. Then, the pairs of the boundary constraining equations at $y = l$, Eqs. (76) read

$$
\left\{
\begin{aligned}
&\sum_{i=1}^{4} e^{p_i l} W_i + \sum_{i=5}^{8} e^{p_i l} W_i = w_e \\
&(n^4/4k^4)(1-1/n^2)^{1/2}(2-\nu+s_7^0/n^2) \sum_{i=1}^{4} [\pm(1 \pm i)] \\
&\quad \times e^{p_i l} W_i + \sum_{i=5}^{8} [\pm(1 \mp \boldsymbol{i})]\, e^{p_i l} W_i = S_e/2k^3
\end{aligned}
\right. \tag{77a}
$$

$$
\left\{
\begin{aligned}
&(n^2/2k^2)(1-1/n^2)^{1/2} \sum_{i=1}^{4} [\pm(1 \pm \boldsymbol{i})]\, e^{p_i l} W_i \\
&\quad + \sum_{i=5}^{8} [\pm(1 \pm \boldsymbol{i})]\, e^{p_i l} W_i = \beta_e/k \\
&(\nu n^2/2k^2)(1-1/n^2) \sum_{i=1}^{4} e^{p_i l} W_i \\
&\quad - \sum_{i=5}^{8} (\pm \boldsymbol{i})\, e^{p_i l} W_i = M_e/2k^2
\end{aligned}
\right. \tag{77b}
$$

$$\left\{\begin{aligned}&(1-1/n^2)^{1/2}\sum_{i=1}^{4}[\mp(1\pm i)]e^{p_i l}W_i\\&\quad+\nu\sum_{i=5}^{8}[\mp(1\mp i)]e^{p_i l}W_i=2ku_e\\&-(1-1/n^2)\sum_{i=1}^{4}(\pm i)e^{p_i l}W_i\\&\quad-\sum_{i=5}^{8}(\mp i)e^{p_i l}W_i=2N_e k^2/n^2(1-\nu^2)\end{aligned}\right. \tag{77c}$$

$$\left\{\begin{aligned}&-(2k^2/n^2)\sum_{i=1}^{4}e^{p_i l}W_i\\&\quad+(2+\nu)\sum_{i=5}^{8}(\mp i)e^{p_i l}W_i=2v_e k^2/n\\&(n^2/2k^2)(1-1/n^2)^{3/2}\sum_{i=1}^{4}[\mp(1\mp i)]e^{p_i l}W_i\\&\quad+\sum_{i=5}^{8}[\pm(1\mp i)]e^{p_i l}W_i=2Q_e k/n(1-\nu^2)\end{aligned}\right. \tag{77d}$$

Another set of pairs of the boundary constraining equations is obtained for $y = -l$ by replacing $+l$ in Eqs. (77) with $-l$.

It is important to recognize that the terms with s_7^0 in the part of the global solutions in the second of Eqs. (77a) are present as small quantities of order of magnitude $(n^2/2k^2)^2$, whereas all the terms involved in Eqs. (77c) are of order of magnitude unity. This indicates that the terms with s_7^0 are always negligible in the calculations of the determinants and their co-factors of the coefficients matrices. Let us assume, for instance, that u_e and S_e are prescribed as boundary conditions. Then, the boundary constraining equations contain the second of Eqs. (77a) and the first of Eqs. (77c). The latter is multiplied by $(n^2/2k^2)^2(2-\nu+s_7^0/n^2)$ and added to the former to eliminate from it the part of the global solutions. We thus have

$$\begin{aligned}&[1-\nu(n^2/2k^2)^2(2-\nu+s_7^0/n^2)]\sum_{i=5}^{8}[\pm(1\mp i)]e^{p_i l}W_i\\&=S_e/2k^3+2k(n^2/2k^2)^2(2-\nu+s_7^0/n^2)u_e\end{aligned} \tag{78}$$

The term $(n^2/2k^2)^2(2-\nu+s_7^0/n^2)$ in the left-hand members of Eq. (78) is negligible in comparison with unity. The second term in the right-hand members is also negligible in comparison with $2ku_e$ on the right-hand side of the first of Eqs. (77c), because they always appear in linear combination in the calculations of the inhomogeneous terms. Consequently, we may approximate the second of Eqs. (77a) by

$$\sum_{i=5}^{8}[\pm(1\mp i)]e^{p_i l}W_i=S_e/2k^3 \tag{79}$$

It can easily be shown that the same is true in the case where N_e and S_e are prescribed as boundary conditions. Those terms that are affected by the differences between the theories have thus been completely neglected in the boundary constraining equations as well as in the eigenvalues, in the case where n is limited by Eq. (45), namely, $(2n^2/k^2)^2 \ll 1$.

On the other hand, if n takes such large values that Eqs. (61) and (62) hold, namely, $n^2/k^2 = 0(1)$ and $n^2 \gg 1$, it can be shown with due consideration of Eqs. (35) and (69), that the supplemental equations, Eqs. (25), reduce to

$$\begin{aligned}&\nabla^4 u=-\nu w'''+w'^{\cdot\cdot}\\&\nabla^4 v=-(2+\nu)w''^{\cdot}-w^{\cdot\cdot\cdot}\\&\nabla^4 N=(1-\nu^2)w''^{\cdot\cdot}\\&\nabla^4 Q=-(1-\nu^2)w'''^{\cdot}\\&\nabla^4 M=-\nabla^4(w''+\nu w^{\cdot\cdot})\\&\nabla^4 S=-\nabla^4[w'''+(2-\nu)w'^{\cdot\cdot}]\end{aligned} \tag{80}$$

Here, again, none of those terms that are affected by the differences between the theories is present.

It may now be stated, therefore, that the solutions for $n \geq 2$ of the boundary value problems of edgewise-loaded, circumferentially-closed circular cylindrical shells are obtained accurately within the errors involved in Kirchhoff-Love's hypothesis for any of those classical theories listed in Table 1, including the Love-Reissner.

It is interesting to see how the boundary constraining equations are affected by the use of the Donnell theory when n takes relatively small values. Approximation consistent to the Donnell theory yields a set of pairs of the boundary constraining equations similar to Eqs. (77). But, now, the factor

$(1 - 1/n^2)^{1/2}$ in Eqs. (77) are replaced by unity. The solutions of the Donnell theory thus become increasingly less accurate as n takes on smaller values. When $n = 2$, the factor $(1 - 1/n^2)^{1/2}$ is calculated as 0.866 and the value becomes much smaller as its power increases. Comparing the powers of this factor in the coefficients matrices of all the possible combinations of proper boundary conditions, we may anticipate that the errors involved in the solutions of the Donnell theory are most likely to be greatest when the boundary conditions are prescribed entirely by the physical quantities, namely, S, M, N and Q, and smallest when they are prescribed entirely by the geometric quantities, namely, by w, β, u and v. Furthermore, since the eigenvalues p_i (i= 1, 2, 3, 4) for the global solutions also contain the factor $(1 - 1/n^2)^{1/2}$, the values of $\exp(p_i l)$ calculated by the Donnell theory may deviate substantially from those calculated by other theories for large values of l. It may therefore be stated that the solutions of the Donnell theory become increasingly less accurate as n takes on smaller values and l becomes greater.

7. CONCLUSION

Some of the important conclusions of the present analysis may be summarized as follows:

1. The classical shell theories including the Flügge, the Koiter-Sanders, the Novozhilov, and the Love-Reissner theories, yield valid solutions for small deformations characterized by $n \geq 2$ of the edgewise-loaded, circumferentially-closed circular cylindrical shells. The relative deviation in the numerical values of the solutions from one theory to another is of order of magnitude of the errors inherent to the Kirchhoff-Love hypothesis.

2. Since none of those coefficients which depend on the theory indicators E_i and D_j is present at the final stage of the approximation in both the eigenvalues and the boundary constraining equations, all the terms with those coefficients may be neglected at arbitrary stages of the developments. We can thus derive the simplest expressions of the constitutive equations, the governing equations, and the supplemental equations as follows:

(a) Constitutive equations;

$$N_x = K(\epsilon_x + \nu\epsilon_\theta), \qquad N_\theta = K(\epsilon_\theta + \nu\epsilon_x)$$

$$N_{x\theta} = N_{\theta x} = Gh\,\gamma_{x\theta}$$

$$M_x = D(\kappa_x + \nu\kappa_\theta), \qquad M_\theta = D(\kappa_\theta + \nu\kappa_x)$$

$$M_{x\theta} = M_{\theta x} = (Gh^3/12)\,\tau$$

Except for τ in place of $(\tau - \omega_n/R)$ in $M_{x\theta}$ and $M_{\theta x}$, these are identical to the constitutive equations of the Love-Reissner theory.

(b) Governing equation;

$$\nabla^4(\nabla^2 + 1)^2 w + 4k^4 w^{\prime\prime\prime\prime} = 0$$

This is the well-known Morley equation. It is nowadays often referred to as the Morley-Koiter equation, because Koiter [29] has proved the consistency of Morley's approximation.

(c) Supplemental equations;

$$\nabla^4 u = -\nu w^{\prime\prime\prime} + w^{\prime\cdot\cdot}$$

$$\nabla^4 v = -(2+\nu)w^{\prime\prime\cdot} - w^{\cdot\cdot\cdot}$$

$$\nabla^4 N = (1-\nu^2)w^{\prime\prime\cdot\cdot}$$

$$\nabla^4 Q = -(1-\nu^2)w^{\prime\prime\prime\cdot}$$

$$\nabla^4 M = -\nabla^4(w^{\prime\prime} + \nu w^{\cdot\cdot}) - \nu w^{\cdot\cdot\cdot\cdot}$$

$$S = -w^{\prime\prime\prime} - (2-\nu)w^{\prime\cdot\cdot}$$

3. If w is given in the form

$$w = \sum_{i=1}^{8} W_i e^{p_i y} \cos n\theta$$

and if n is assumed such that

$$(2n^2/k^2)^2 \ll 1,$$

the eigenvalues p_i are calculated by

$$p_i = \pm(\xi_1 \pm i\eta_1)\,; \quad i = 1, 2, 3, 4$$

$$p_i = \pm(\xi_2 \pm i\eta_2)\,; \quad i = 5, 6, 7, 8$$

with

$$\xi_1 = (n^2/2k)(1 - 1/n^2)^{1/2}(1 + n^2/2k^2)$$

$$\eta_1 = (n^2/2k)(1 - 1/n^2)^{1/2}(1 - n^2/2k^2)$$

$$\xi_2 = k(1 + n^2/2k^2)$$

$$\eta_2 = k(1 - n^2/2k^2)$$

If n is assumed much smaller, so that

$$2n^2/k^2 \ll 1,$$

we have the simplest expressions of the eigenvalues

$$\xi_1 = \eta_1 = (n^2/2k)(1 - 1/n^2)^{1/2}$$

$$\xi_2 = \eta_2 = k$$

It should be noted that the absolute values of the eigenvalues for the edge-zone solutions ($i = 5, 6, 7, 8$) are proportional to k, whereas those for the global solutions ($i = 1, 2, 3, 4$) are inversely proportional to it.

4. If n is assumed such that

$$n^2 \gg 1 \text{ and } 2n^2/k^2 = 0(1)$$

the absolute values of the eigenvalues p_i are of order of magnitude k for both the global and the edge-zone solutions.

In this case, the governing equation becomes identical to Donnell' equation;

$$\nabla^8 w + 4k^4 w'''' = 0$$

Donnell's equation is, therefore, a consistent first approximation for $n^2 \gg 1$.

5. If Donnell's equation is applied for the case of

$$2n^2/k^2 \ll 1,$$

the eigenvalues are calculated as

$$p_i = \pm (1 \pm \boldsymbol{i})(n^2/2k)\ ; \quad i = 1, 2, 3, 4$$

$$p_i = \pm (1 \pm \boldsymbol{i})k\ ; \quad i = 5, 6, 7, 8$$

The solutions of the Donnell theory become increasingly inaccurate as n becomes smaller and the length-to-radius ratio l greater. The errors are greatest when the boundary conditions are prescribed entirely by the physical quantities, and smallest when they are prescribed entirely by the geometric quantities. These imply that the Donnell theory is not suited for the analyses of inextensional deformations.

6. The following relations hold under Kirchhoff-Love's hypothesis:

$$\delta n^2 \ll 1 \text{ and } \delta |p_i^2| \ll 1$$

These are very useful relations for achieving a consistent approximation in the theory of thin circular cylindrical shells. If we knew these relations a priori, we could have achieved the approximations in the preceding sections in a more straightforward manner.

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APPENDIX A. EXACT EXPRESSIONS OF COEFFICIENTS

G_i:

$$G_1 = 1$$

$$G_2 = \frac{1-\nu}{2}\left[1+\frac{\delta}{4}(3E_5+E_6)\right]$$

$$G_3 = \frac{1}{2}\left[1+\nu+\frac{3\delta}{4}(1-\nu)(E_5-E_6)\right] \qquad \text{(A.1)}$$

$$G_4 = -\delta E_1$$

$$G_5 = \frac{\delta}{2}(1-\nu)E_6$$

$$G_6 = \nu$$

H_i:

$$H_1 = \frac{1-\nu}{2}\left[1+\frac{\delta}{2}\left(3-D_4+D_2+\frac{1}{2}E_3+\frac{3}{2}E_4\right)\right]$$

$$H_2 = 1+\delta(1-D_1+E_2-E_1)$$

$$H_3 = \frac{1+\nu}{2}+\frac{\delta}{4}(1-\nu) \times \left(-1+D_4+D_2+\frac{1}{2}E_3-\frac{1}{2}E_4\right) \qquad \text{(A.2)}$$

$$H_4 = -\delta\left[1+\frac{1-\nu}{2}E_4\right]$$

$$H_5 = -\delta(1-E_1)$$

$$H_6 = 1+\delta(E_2-D_1)$$

K_i:

$$K_1 = D_1$$

$$K_2 = \frac{1-\nu}{2}\left[-1+D_4+\frac{1}{2}(D_2-D_3)\right]$$

$$K_3 = -\frac{\nu}{\delta}$$

$$K_4 = \frac{1}{2}\left[3-\nu-(1-\nu)D_4+\frac{1-\nu}{2}(D_2-D_3)\right] \qquad \text{(A.3)}$$

$$K_5 = 1-D_1$$

$$K_6 = \frac{1}{\delta}\left[-1+\delta(E_1-E_2)\right]$$

$$K_7 = D_1+E_1$$

$$K_8 = \frac{1}{\delta}(1+\delta E_2)$$

f_i:

$$f_1 = -H_1G_4$$

$$f_2 = -H_1G_5-H_2G_4+H_4G_3$$

$$f_3 = -H_2G_5+H_5G_3 \qquad \text{(A.4)}$$

$$f_4 = -H_1G_6$$

$$f_5 = -H_2G_6+H_6G_3$$

g_i:

$$g_1 = -H_4G_1+H_3G_4$$

$$g_2 = -H_5G_1-H_4G_2+H_3G_5$$

$$g_3 = -H_5G_2 \qquad \text{(A.5)}$$

$$g_4 = -H_6G_1+H_3G_6$$

$$g_5 = -H_6G_2$$

L_i:

$$L_1 = H_1G_1$$

$$L_2 = H_1G_2+H_2G_1-H_3G_3 \qquad \text{(A.6)}$$

$$L_3 = H_2G_2$$

a_{ij}:

$$a_{81} = K_1f_1-L_1$$

$$a_{82} = K_1f_2+K_2f_1+K_4g_1-L_2-2L_1$$

$$a_{83} = K_1f_3+K_2f_2+K_4g_2+K_5g_1-L_3-2L_2-L_1$$

$$a_{84} = K_2f_3+K_4g_3+K_5g_2-2L_3-L_2$$

$$a_{85} = K_5g_3-L_3$$

$$a_{61} = K_1f_4+K_3f_1 \qquad \text{(A.7)}$$

$$a_{62} = K_1f_5+K_2f_4+K_3f_2+K_4g_4+K_6g_1-K_7L_1$$

$$a_{63} = K_2f_5+K_3f_3+K_4g_5+K_5g_4+K_6g_2-K_7L_2$$

$$a_{64} = K_5g_5+K_6g_3-K_7L_3$$

$$a_{41} = K_3f_4-K_8L_1$$

$$a_{42} = K_3f_5+K_6g_4-K_8L_2$$

$$a_{43} = K_6g_5-K_8L_3$$

n_i:

$$
\begin{aligned}
n_1 &= f_2 + \nu g_1 - \delta E_1 L_2 \\
n_2 &= f_3 + \nu g_2 - \delta E_1 L_3 \\
n_3 &= n_5 = \nu g_3 \\
n_4 &= f_5 + \nu g_4 + \nu L_2
\end{aligned}
\tag{A.8}
$$

m_i:

$$
\begin{aligned}
m_1 &= D_1 f_1 - L_1 \\
m_2 &= D_1 f_2 + \nu g_1 - L_2 - \nu L_1 \\
m_3 &= D_1 f_3 + \nu g_2 - L_3 - \nu L_2 \\
m_4 &= m_7 = \nu g_5 \\
m_5 &= D_1 f_4 \\
m_6 &= D_1 f_5 + \nu g_4
\end{aligned}
\tag{A.9}
$$

q_i:

$$
\begin{aligned}
q_1 &= K_u f_1 + K_v g_1 - \delta(1-\nu)(1+\frac{1}{2}E_4)L_1 \\
q_2 &= K_u f_2 + K_v g_2 - \delta(1-\nu)(1+\frac{1}{2}E_4)L_2 \\
q_3 &= K_u f_3 + K_v g_3 - \delta(1-\nu)(1+\frac{1}{2}E_4)L_3 \\
q_4 &= K_u f_4 + K_v g_4 \\
q_5 &= K_u f_5 + K_v g_5
\end{aligned}
\tag{A.10}
$$

where

$$
\begin{aligned}
K_u &= \frac{1-\nu}{2}\left[1+\frac{\delta}{2}(-1+D_2+D_4+\frac{1}{2}E_3-\frac{1}{2}E_4)\right] \\
K_v &= \frac{1-\nu}{2}\left[1+\frac{\delta}{2}(3+D_2-D_4+\frac{1}{2}E_3+\frac{3}{2}E_4)\right]
\end{aligned}
\tag{A.11}
$$

s_i:

$$
\begin{aligned}
s_1 &= D_1 f_1 - L_1 \\
s_2 &= D_1 f_2 + K_t g_1 + K_s f_1 - L_2 - (2-\nu)L_1 \\
s_3 &= D_1 f_3 + K_t g_2 + K_s f_2 - L_3 - (2-\nu)L_2 \\
s_4 &= K_t g_3 + K_s f_3 - (2-\nu)L_3 \\
s_5 &= D_1 f_4 \\
s_6 &= K_t g_4 + K_s f_4 + D_1 f_5 \\
s_7 &= K_t g_5 + K_s f_5
\end{aligned}
\tag{A.12}
$$

where

$$
\begin{aligned}
K_s &= (1-\nu)\,[D_4 - 1 + (D_2 - D_3)/2]\,/2 \\
K_t &= [3-\nu-(1-\nu)(2D_4 - D_2 + D_3)/2]\,/2
\end{aligned}
\tag{A.13}
$$

APPENDIX B. LEADING TERMS OF THE COEFFICIENTS

$a_{ij} \doteqdot -\frac{1-\nu}{2}\, a_{ij}^0$:

$$
\begin{aligned}
& a_{81}^0 = 1,\ a_{82}^0 = 4,\ a_{83}^0 = 6 \\
& a_{84}^0 = 4,\ a_{85}^0 = 1 \\
& a_{61}^0 = \nu(E_1 + D_1) \\
& a_{62}^0 = 5+\nu-\frac{1-\nu}{2}\,[2D_4-(1+\nu)(D_2-D_3) \\
& \qquad -(2+\nu)E_4+\nu E_6] \\
& a_{63}^0 = 7+\nu-(1-\nu)D_4-\nu D_1-\nu E_1 \\
& \qquad + \frac{1-\nu}{2}(E_6+E_4) \\
& a_{64}^0 = 2 \\
& a_{41}^0 = \frac{1-\nu^2}{\delta} \\
& a_{42}^0 = 3+\nu-\nu E_1-\nu D_1+\frac{1-\nu}{2}(E_4+E_6-D_4) \\
& a_{43}^0 = 1
\end{aligned}
\tag{B.1}
$$

$L_1 \doteqdot \frac{1-\nu}{2}\, L_i^0$:

$$
L_1^0 = 1,\ L_2^0 = 2,\ L_3^0 = 1 \tag{B.2}
$$

so that

$$
L^0(\ ) = \nabla^4(\ ) \tag{B.3}
$$

$f_1 \doteqdot \frac{1-\nu}{2}\, \bar{f}_i^0$:

$$
\begin{aligned}
& \bar{f}_1^0 = \delta f_1^0,\ \bar{f}_2^0 = \delta f_2^0,\ \bar{f}_3^0 = \delta f_3^0 \\
& \bar{f}_4^0 = -\nu,\ \bar{f}_5^0 = 1
\end{aligned}
\tag{B.4}
$$

where

$$
\begin{aligned}
& f_1^0 = E_1 \\
& f_2^0 = -\frac{1+\nu}{1-\nu} + \frac{2}{1-\nu}E_1 - \frac{1+\nu}{2}E_4 - \frac{1-\nu}{2}E_6 \\
& f_3^0 = -\frac{1+\nu}{1-\nu}(1-E_1) - E_6
\end{aligned}
\tag{B.5}
$$

$$g_i \doteqdot \frac{1-\nu}{2}\,\overline{g}_i^0:$$

$$\begin{aligned}&\overline{g}_1^0 = \delta g_1^0,\ \overline{g}_2^0 = \delta g_2^0,\ \overline{g}_3^0 = \delta g_3^0\\ &\overline{g}_4^0 = -(2+\nu),\ \overline{g}_5^0 = -1\end{aligned} \tag{B.6}$$

where

$$\begin{aligned}&g_1^0 = \frac{2}{1-\nu} - \frac{1+\nu}{1-\nu}E_1 + E_4\\ &g_2^0 = \frac{3-\nu}{1-\nu} - \frac{2}{1-\nu}E_1 + \frac{1-\nu}{2}E_4 + \frac{1+\nu}{2}E_6\\ &g_3^0 = 1 - E_1\end{aligned} \tag{B.7}$$

$$n_i \doteqdot \frac{1-\nu}{2}\,\overline{n}_1^0:$$

$$\begin{aligned}&\overline{n}_1^0 = \delta n_1^0,\ \overline{n}_2^0 = \delta n_2^0,\ \overline{n}_3^0 = \delta n_3^0\\ &\overline{n}_4^0 = 1-\nu^2,\ \overline{n}_5^0 = \delta n_5^0\end{aligned} \tag{B.8}$$

where

$$\begin{aligned}&n_1^0 = -1+\nu E_1 - \frac{1-\nu}{2}(E_4+E_6)\\ &n_2^0 = -1+\nu+\frac{\nu(1-\nu)}{2}E_4 + \frac{(1-\nu)(2+\nu)}{2}E_6\\ &n_3^0 = \nu(1-E_1)\\ &n_5^0 = \nu(1-E_1)\end{aligned} \tag{B.9}$$

$$m_i \doteqdot \frac{1-\nu}{2}\,\overline{m}_i^0:$$

$$\begin{aligned}&\overline{m}_1^0 = -1,\ \overline{m}_2^0 = -(2+\nu),\ \overline{m}_3^0 = -(1+2\nu)\\ &\overline{m}_4^0 = -\nu,\ \overline{m}_5^0 = m_5^0,\ \overline{m}_6^0 = m_6^0\\ &\overline{m}_7^0 = -\nu\end{aligned} \tag{B.10}$$

where

$$\begin{aligned}&m_5^0 = -\nu D_1\\ &m_6^0 = D_1 - \nu(2+\nu)\end{aligned} \tag{B.11}$$

$$q_i \doteqdot \frac{1-\nu}{2}\,\overline{q}_i^0:$$

$$\begin{aligned}&\overline{q}_1^0 = \delta q_1^0,\ \overline{q}_2^0 = \delta q_2^0,\ \overline{q}_3^0 = \delta q_3^0\\ &\overline{q}_4^0 = -(1-\nu^2),\ \overline{q}_5^0 = \delta q_5^0\end{aligned} \tag{B.12}$$

where

$$\begin{aligned}&q_1^0 = \nu(1-E_1)\\ &q_2^0 = -(1-\nu)(1+\frac{2+\nu}{2}E_4 - \frac{\nu}{2}E_6)\\ &q_3^0 = -1+\nu E_1 - \frac{1-\nu}{2}(E_4+E_6)\\ &q_5^0 = -1+\nu E_1 - \frac{1-\nu}{2}(E_4+E_6-D_4)\end{aligned} \tag{B.13}$$

$$s_i \doteqdot \frac{1-\nu}{2}\,\overline{s}_i^0:$$

$$\begin{aligned}&\overline{s}_1^0 = -1,\ \overline{s}_2^0 = -(4-\nu),\ \overline{s}_3^0 = -(5-2\nu)\\ &\overline{s}_4^0 = -(2-\nu),\ \overline{s}_5^0 = s_5^0,\ \overline{s}_6^0 = s_6^0,\ \overline{s}_7^0 = s_7^0\end{aligned} \tag{B.14}$$

where

$$\begin{aligned}&s_5^0 = -\nu D_1\\ &s_6^0 = -3+D_1+(1-\nu)D_4 - \frac{1-\nu^2}{2}(D_2-D_3)\\ &s_7^0 = -2+\nu+(1-\nu)D_4\end{aligned} \tag{B.15}$$

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