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The Method of Weighted Residuals in The Time Domain Applied to Nonlinear Vibrations

Tetsuhiko UEDA and Yuji MATSUZAKI

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The Method of Weighted Residuals in The Time Domain Applied to Nonlinear Vibrations*

Tetsuhiko UEDA** and Yuji MATSUZAKI**

ABSTRACT

A method to solve nonlinear differential equations, governing periodic motion, is presented. The approximation is based on the method of weighted residuals. As an example problem, nonlinear vibrations of an infinitely long cylinder are examined, and their frequencies are calculated, using various types of assumed solutions and weighting functions. The numerical results show that the Galerkin type of finite element procedure gives a good approximation.

概　　要

周期運動を支配する非線形微分方程式を解く方法を示す。近似は重みつき残差法に基づいている。例題に無限長円筒殻の非線形振動を解き，種々の異なる試験関数と重み関数を用いてその振動数を計算した。その結果，ガラーキン型の有限要素法が優れた近似法であることが判った。

1. INTRODUCTION

It is seldom to find an exact solution to nonlinear differential equations which govern oscillatory phenomena in physical and engineering fields. We must usually be satisfied with approximate solutions of certain accuracy in solving these equations. The method of weighted residuals (1, 2), abbreviated by MWR, is a general method to solve such nonlinear differential equations. Above all, it is relatively simple and easy to be applied. We may obtain solutions possessing any desired accuracy by using a computer-oriented approach, which is based on the finite element method and iteration procedure.

Application of MWR to the time domain is straight-forward. Here, we shall briefly describe MWR in the time domain. Consider the equation,

$$\mathscr{D}(\boldsymbol{x}) = \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{x}=\boldsymbol{x}(t)$ and $\boldsymbol{f}=\boldsymbol{f}(t)$ are defined over a certain closed time interval T, and $\mathscr{D}$ is a nonlinear operator. The solution $\boldsymbol{x}$ must satisfy prescribed initial and terminal conditions.
Let

$$x_i^* = \sum_{j=1}^{J(i)} c_i^j \phi_j(t), \qquad (i=1, \cdots, M) \tag{2}$$

be an approximation of the ith unknown contained in Eq. (1). The residual,

$$r_m(t) = \mathscr{D}_m(\boldsymbol{x}^*) - f_m, \quad (m=1, \cdots M) \tag{3}$$

would not vanish unless the assumed solution is exact. In Eq. (3), the subscript m corresponds to the mth equation of Eq. (1). We require that the approximate solution should satisfy Eq. (1) in some average sense. The unknown coefficients c_i^j of the trial functions are determined so that the residuals vanish in a weighted average over the interval T.

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$$\int_T r_m(t)\, W_m^n(t)\, dt = 0 \,,\ (n = 1, 2, \cdots, N(m)) \quad (4)$$

or

$$P_{mn}(c_i^j) - f_{mn} = 0 \,, \quad (5)$$

where

$$P_{mn}(c_i^j) = \int_T \mathscr{D}_m(x^*)\, W_m^n(t)\, dt \,, \quad (6)$$

$$f_{mn} = \int_T f_m(t)\, W_m^n(t)\, dt \,, \quad (7)$$

and $W_m^n(t)$ denotes a weighting function.

The coefficients c_i^j are uniquely determined when the condition,

$$\sum_{i=1}^{M} J(i) = \sum_{m=1}^{M} N(m),$$

is satisfied.

Provided that the integrals on the right hands of Eqs. (6) and (7) exist, any function can generally be used as the weighting function although upon an appropriate choice depends the accuracy of the results. Paritcular choices for weighting functions are equivalent to well-established methods, for example, the collocation method, Galerkin's approach, etc.

When the oscillatory system undergoes a periodic motion, the interval T may be assigned for one period or its fraction. It should be noted, however, that in nonlinear systems the period of oscillation depends on the amplitude, and consequently that the interval T itself is an unknown quantity.

As an example problem to illustrate the powerfulness of MWR, we shall treat nonlinear oscillation of an infinitely long cylinder. Large amplitude vibration of thin shells of revolution has been an important subject for the last fifteen years because of its interesting nonlinear characteristics. There still remain unresolved problems (3, 4). Constrained and extensional vibrations of the cylinder are here examined by using several types of trial and weighting functions.

2. EQUATION OF NONLINEAR OSCILLATION

When the radial displacement of an infinitely long cylinder (See Figure.) is assumed in the form of

$$w(y, t) = A(t) \cos \frac{\ell y}{R} + B(t), \quad (8)$$

the modal equations are given as (5)

$$\frac{d^2 a}{d\tau^2} + a + 12a\left(b + \frac{a^2}{4}\right) = 0 \,, \quad (9.1)$$

$$\varepsilon \frac{d^2 b}{d\tau^2} + 12\left(b + \frac{a^2}{4}\right) = 0 \,, \quad (9.2)$$

where

$$a = \frac{A}{h} \,,$$

$$b = \frac{R}{\ell^2 h} \frac{B}{h} \,,$$

$$\tau = \omega_0 t \,,$$

$$\omega_0^2 = \frac{E h^2 \ell^4}{12(1 - \nu^2)\rho R^4} \,,$$

$$\varepsilon = \left(\frac{\ell^2 h}{R}\right)^2 \,,$$

h = uniform thickness,
R = radius of the cylinder,
E = Young's modulus,
ν = Poisson's ratio,
ρ = density of the material,
ℓ = circumferential wave number.

The conditions specifying the motion w(y,t) are

$$\frac{da}{d\tau} = \frac{db}{d\tau} = 0 \quad \text{at} \quad \tau = 0 \,, \quad (10.1)$$

$$a = \frac{db}{d\tau} = 0 \quad \text{at} \quad \tau = \bar{\tau} \,, \quad (10.2)$$

where $\bar{\tau}$ denotes a quarter period.

3. CONSTRAINED VIBRATION

If the shell is constrained to vibrate with no radial displacement at the node of $\cos\frac{\ell y}{R}$, that is, B(t)=0, then a set of Eq. (9) reduces to (5)

$$\frac{d^2 a}{d\tau^2} + a + \delta a^3 = 0 \,, \quad (11)$$

where $\delta=3$. When $\delta=3(1-\nu^2)/4$, Eq. (11) corresponds to Eq. (10) of Ref. 5. It is noted that Eq. (11) agrees with the equation governing the motion of a mass on a cubic spring.

3.1 Approximation using harmonic functions

Two-term solution: We first assume a solution satisfying Eqs. (10),

$$a^{*}=\bar{a}\{(1-\zeta)\cos\Omega\tau+\zeta\cos 3\Omega\tau\}\ , \qquad (12)$$

where $\Omega=\pi/(2\bar{\tau})$. The asterisk for an approximation will be omitted hereafter since no confusion would take place. The weighting functions are selected to correspond to the trial function, i.e.

$$W_1^1=\cos\Omega\tau,\ \text{and}\ W_1^2=\cos 3\Omega\tau.$$

The time interval of the integration can be taken as $[0, \bar{\tau}]$. Substituting Eq. (12) into Eq. (11), multiplying the left hand side of Eq.(11) by W_1^1 or W_1^2 and integrating from 0 to $\bar{\tau}$, we obtain

$$\Omega^2=1+\frac{3}{4}\beta(1-\zeta+2\zeta^2)\ , \qquad (13.1)$$

$$9\zeta\Omega^2=\zeta+\frac{1}{4}\beta(1+3\zeta-9\zeta^2)\ , \qquad (13.2)$$

where $\beta=\delta\,\bar{a}^2$. (14)

Elimination of Ω^2 from Eqs. (13) yields

$$46\beta\zeta^3-18\beta\zeta^2+(24\beta+32)\zeta-\beta=0\ . \qquad (15)$$

For a specific value of β, ζ is numerically solved by Eq. (15). Substituting these values of ζ and β into Eq. (13.1), we can evaluate the square of nonlinear frequency Ω^2.

One-term solution: When the trial function is

$$a=\bar{a}\cos\Omega\tau, \qquad (16)$$

that is, $\zeta=0$ in Eq.(12), we obtain a simple relation between the frequency and the amplitude from Eq. (13.1).

$$\Omega^2=1+\frac{3}{4}\beta. \qquad (17)$$

It should be noted that, when $\delta=3$, Eq. (17) agrees with Eq. (9) of Ref. 5 where Evensen employed the method of harmonic balance (MHB) with the same assumed solution of Eq. (16). Since both results are equivalent, we conclude that in this example MHB corresponds to MWR in which the weighting function is set to have the same form as the trial function.

3.2 Approximation by the finite element method

The time interval is divided into p small elements with a length of $\Delta_k=\tau_{k+1}-\tau_k$. Choosing the Hermitian polynomials for the trial function in each element, we apply the finite element method to determine the unknown nodal values of the function.

First, we shall confine ourselves to the k-th element. Let us introduce a local time coordinate defined by

$$\xi=(\tau-\tau_k)/\Delta_k, \qquad (18)$$

i.e. $\xi=0$ at $\tau=\tau_k$, and $\xi=1$ at $\tau=\tau_{k+1}$.

The solution is assumed in the form of

$$a^{(k)}(\xi)=\boldsymbol{\phi}^{(k)}\boldsymbol{u}^{(k)}, \qquad (19)$$

where

$$\boldsymbol{u}^{(k)}=(u^{1k}\ u^{2k}\ u^{1k+1}\ u^{2k+1})^T, \qquad (20)$$

$$\boldsymbol{\phi}^{(k)}=[\phi_{1k}\ \phi_{2k}\ \phi_{1k+1}\ \phi_{2k+1}]\ , \qquad (21)$$

$$\left.\begin{aligned}\phi_{1k}&=1-3\xi^2+2\xi^3,\\ \phi_{2k}&=(\xi-2\xi^2+\xi^3)\Delta_k,\\ \phi_{1k+1}&=3\xi^2-2\xi^3,\\ \phi_{2k+1}&=(-\xi^2+\xi^3)\Delta_k\ .\end{aligned}\right\} \qquad (22)$$

The superscript (k) indicates that the quantity is associated with the k-th element. The unknowns u^{sk} for s=1 and 2 are related to the values of the solution and its slope at $\tau=\tau_k$;

$$u^{1k}=a^{(k)}(0)=a^{(k-1)}(1), \qquad (23.1)$$

$$u^{2k}=\frac{1}{\Delta_k}\frac{da^{(k)}(\xi)}{d\xi}\Big|_{\xi=0}=\frac{1}{\Delta_{k-1}}\frac{da^{(k-1)}(\xi)}{d\xi}\Big|_{\xi=1}\ . \qquad (23.2)$$

Let the residual and the weight for the k-th element be denoted by $r^{(k)}$ and $w^{(k)}$, respectively, where

$$\boldsymbol{w}^{(k)}=(\underbrace{0\cdots\cdots 0}_{2(k-1)-1}\ w^{1k}\ w^{2k}\ w^{1k+1}\ w^{2k+1}\underbrace{0\cdots\cdots 0}_{2(P-k)-1})^T. \qquad (24)$$

Then, Eq. (4) are rewritten as

$$\sum_{k=1}^{P}\Delta_k\int_0^1 r^{(k)}\boldsymbol{w}^{(k)}d\xi=0\ , \qquad (25)$$

where

$$\varDelta_k \int_0^1 r^{(k)} w^{(k)} d\xi = -\frac{4}{\pi^2}\Omega^2 D^{(k)} u^{(k)} + K^{(k)} u^{(k)}$$

$$+\delta K_N^{(k)} u^{(k)}, \tag{26}$$

$$D^{(k)} = -\frac{1}{\varDelta_k}\int_0^1 w^{(k)} \frac{d^2\phi^{(k)}}{d\xi^2} d\xi, \tag{27}$$

$$K^{(k)} = \varDelta_k \int_0^1 w^{(k)} \phi^{(k)} d\xi, \tag{28}$$

$$K_N^{(k)} = \varDelta_k \int_0^1 w^{(k)} (\phi^{(k)} u^{(k)})^2 \phi^{(k)} d\xi. \tag{29}$$

W^{sk} (s=1 and 2) should be assigned with respect to u^{sk}. If W^{sk} takes the same shape as has the trial function corresponding to u^{sk}, then Eq. (25) becomes Galerkin's approach. It should be noted that W^{21} and W^{1p+1} vanish in Eq. (24) since the corresponding u^{21} and u^{1p+1} are constrained because of the conditions given by Eqs. (10).

Summing up the left hand side of Eq. (25), we obtain

$$(-\lambda D + K + \delta K_N) u = 0, \tag{30}$$

where

$$\lambda = \frac{4}{\pi^2}\Omega^2$$

$$u = (u^{11}\ u^{12}\ u^{22} \cdots u^{2P}\ u^{2P+1})^T \tag{31.1,2}$$

u^{21} and u^{1p+1} are excluded from u by the constraint conditions at the boundaries. For a specific value of β, Eq. (30) can be treated as an eigen-value problem. We can solve Eq. (30) for λ by an iterative procedure,

$$D^{-1}(K + \delta K_{Nq-1}) u_q = \lambda_q u_q, \tag{32}$$

where the matrix K_{Nq-1} is formed by using the eignevector which is obtained in the (q-1)st step. It should be noted here that the magnitude of the vector u must be kept constant, for instance, $u^{11}=\bar{a}$, since the period of oscillation depends on the amplitude. By using the solution u thus obtained, we may start an iteration for the next β which deviates from the previous value of β by a small increment.

We shall go into details about the case where the interval $[0, \bar{\tau}]$ is consists of a single element, that is, p=1. Letting the weighting functions be $W_1^1=1$ and $W_1^2=1-\xi$, we may obtain

$$\frac{4}{\pi^2}\gamma\Omega^2 = \frac{1}{2} + \frac{1}{12}\gamma + \beta\left(\frac{43}{140} + \frac{43}{840}\gamma + \frac{1}{105}\gamma^2 + \frac{1}{840}\gamma^3\right), \tag{33.1}$$

$$\frac{4}{\pi^2}\Omega^2 = \frac{7}{20} + \frac{1}{30}\gamma + \beta\left(\frac{109}{440} + \frac{29}{924}\gamma + \frac{1}{220}\gamma^2 + \frac{1}{2310}\gamma^3\right), \tag{33.2}$$

where

$$\gamma = -u^{22}/u^{11}, \text{ and } \beta = \delta\{a(0)\}^2 = \delta(u^{11})^2$$

In this case, we need not use the iterative procedure of Eqs. (32). Instead, elimination of Ω^2 yields

$$4\beta\gamma^4 + 31\beta\gamma^3 + (202\beta+308)\gamma^2 + (1816\beta+2464)\gamma - 2838\beta - 4620 = 0. \tag{34}$$

Eq. (34) can numerically be solved to obtain γ for specified values of β. When $\beta=0$, it is obvious that the exact solution of Ω is unity. In this case, we obtain from Eqs. (33)

$$\gamma^2 + 8\gamma - 15 = 0. \tag{35}$$

Eq. (35) gives 0.9962 to Ω.

Finally, we solve the problem with the aid of Galerkin's approach, that is, by employing the Hermitian polynomials as the weighting function. The numerical results with equally spaced elements for p=1 to 4 are presented in Table I, where Ω_{exact} is calculated from the exact solution to Eq. (11) represented by

$$a = \bar{a}\ \mathrm{cn}\left(\sqrt{1+\beta}\ \tau,\ \sqrt{\frac{\beta}{2(1+\beta)}}\right). \tag{36}$$

In Eq. (36), cn denotes Jacobi's elliptic function.

4. EXTENSIONAL VIBRATION

In the case of extensional vibration, it is necessary to solve Eqs. (9) simultaneously.

4.1 Approximation using harmonic functions

The trial functions to Eqs. (9) shall be taken the same as in Ref. 5:

$$a = \bar{a}\cos\Omega\tau, \tag{37.1}$$

$$b = \bar{b}_0 + \bar{b}_2\cos 2\Omega\tau. \tag{37.2}$$

We choose the weighting functions $\cos\Omega\tau$ for Eq. (9.1) and unity and $\cos 2\Omega\tau$ for Eq. (9.2), i.e.

$W_1^1 = \cos\Omega\tau$, $W_2^1 = 1$, and $W_2^2 = \cos 2\Omega\tau$.

Then, Eq. (4) furnishes

$$1-\Omega^2+12\bar{b}_0+6\bar{b}_2+\frac{9}{4}\bar{a}^2=0, \tag{38.1}$$

$$12\bar{b}_0+\frac{3}{2}\bar{a}^2=0, \tag{38.2}$$

$$-4\varepsilon\Omega^2\bar{b}_2+12\bar{b}_2+\frac{3}{2}\bar{a}^2=0. \tag{38.3}$$

Elimination of b_0 and b_2 from Eqs. (38) yields

$$\Omega^2=\frac{1}{8}\varepsilon(12+4\varepsilon+3\beta-\sqrt{(12+4\varepsilon+3\beta)^2-192\varepsilon}). \tag{39}$$

where $\beta = \varepsilon\bar{a}^2$

It is noted that Eqs. (38) coincide with the corresponding equations of Ref. 5 and Eq. (39) agrees with Evensen's amplitude-frequency relationship as ε becomes small compared with unity. Therefore, we may conclude that MWR employed in this section is equivalent to MHB of Ref. 5. The trial functions represented by Eqs. (37) correspond merely to the one-term solution of the constrained vibration, i.e. Eq. (16). It is very tedious, however, to derive and solve the governing nonlinear equations with respect to the unknown coefficients of the higher-term solution since the number of coefficients involved increases rapidly.

4.2 Finite element approximation

The procedure is essentially the same as in Section 3.2. The trial functions in the k-th element are put in the form of

$$a^{(k)}(\xi) = \boldsymbol{\phi}^{(k)}\boldsymbol{u}^{(k)}, \tag{40.1}$$

$$b^{(k)}(\xi) = \boldsymbol{\phi}^{(k)}\boldsymbol{v}^{(k)}, \tag{40.2}$$

where

$$\boldsymbol{v}^{(k)} = (v^{1k}\ v^{2k}\ v^{1k+1}\ v^{2k+1})^T, \tag{41}$$

and

$$v^{1k} = b^{(k)}(0) = b^{(k-1)}(1), \tag{42.1}$$

$$v^{2k}=\frac{1}{\Delta_k}\frac{db^{(k)}(\xi)}{d\xi}\Big|_{\xi=0}=\frac{1}{\Delta_{k-1}}\frac{db^{(k-1)}(\xi)}{d\xi}\Big|_{\xi=1} \tag{42.2}$$

Instead of Eq. (25), we obtain

$$\sum_{k=1}^{p}\Delta_k\int_0^1 r_m^{(k)}w_m^{(k)}d\xi=0, \quad (m=1,2) \tag{43}$$

where the subscript m with respect to $w^{(k)}$ and $r^{(k)}$ are referred to Eqs. (9.1) and (9.2), respectively, and

$$\Delta_k\int_0^1 r_1^{(k)}w_1^{(k)}d\xi=-\lambda D_1^{(k)}\boldsymbol{u}^{(k)}+[K_1^{(k)}+\varepsilon K_N^{(k)}]\boldsymbol{u}^{(k)}, \tag{44.1}$$

$$\Delta_k\int_0^1 r_2^{(k)}w_2^{(k)}d\xi=-\lambda\varepsilon D_2^{(k)}\boldsymbol{v}^{(k)}+12K_2^{(k)}\boldsymbol{v}^{(k)}-3\boldsymbol{f}_N^{(k)}, \tag{44.2}$$

$$D_m^{(k)}=-\frac{1}{\Delta_k}\int_0^1 w_m^{(k)}\frac{d^2\boldsymbol{\phi}^{(k)}}{d\xi^2}d\xi, \quad (m=1,2) \tag{45}$$

$$K_m^{(k)}=\Delta_k\int_0^1 w_m^{(k)}\boldsymbol{\phi}^{(k)}d\xi, \quad (m=1,2) \tag{46}$$

$$K_N^{(k)}=-\frac{\lambda}{\Delta_k}\int_0^1 w_1^{(k)}\{\frac{d^2\boldsymbol{\phi}^{(k)}}{d\xi^2}\boldsymbol{v}^{(k)}\}\boldsymbol{\phi}^{(k)}d\xi, \tag{47}$$

$$\boldsymbol{f}_N^{(k)}=-\Delta_k\int_0^1 w_2^{(k)}\{\boldsymbol{\phi}^{(k)}\boldsymbol{u}^{(k)}\}^2d\xi. \tag{48}$$

W_1^{21}, W_1^{1p+1}, W_2^{21}, and W_2^{2p+1}, should vanish since the corresponding u^{21}, u^{1p+1}, v^{21}, and v^{2p+1} are constrained because of the conditions, Eqs. (10).

Assembling Eqs. (44) yields

$$-\lambda D_1\boldsymbol{u}+(K_1+\varepsilon K_N)\boldsymbol{u}=0, \tag{49.1}$$

$$-\lambda\varepsilon D_2\boldsymbol{v}+12K_2\boldsymbol{v}=3\boldsymbol{f}_N. \tag{49.2}$$

Equations (49) can be solved by the following iterative procedure similar to Eq. (32):

$$D_1^{-1}(K_1+\varepsilon K_{Nq-1})\boldsymbol{u}_q=\lambda_q\boldsymbol{u}_q, \tag{50.1}$$

$$\boldsymbol{v}_q=3(-\lambda_q\varepsilon D_2+12K_2)^{-1}\boldsymbol{f}_{Nq}. \tag{50.2}$$

where the condition, $u^{11}=\bar{a}$, must be held during the procedure.

5. NUMERICAL RESULTS AND DISCUSSIONS

Constrained Vibration As shown in Table I, the one trigonometric function provides a good approximation for small β since the non-linear effect is comparatively small. Needless to say, the two-term assumption gives a better solution over the wide range of β. On the other hand, the Galerkin type finite element procedure is proved to be a good approximation in the range of the calculation even when p=1. The accuracy of the solution increases rapidly with increase in the number of elements.

Extensional Vibration In this case converged solutions of the finite element approach are deemed to be exact since no exact analytic solution is available. The weighting functions for the finite element method are chosen to be the same as the trial functions. The results on p=1, 2 and 10 for β=0.1 and $\varepsilon/12=10^{-2}$ are shown in Table II. The convergence is very rapid; even the two-element solution gives the same frequency as that obtained by the ten-element solution. In Table III, the results calculated from Eq. (39) are compared with the converged solutions which are obtained by the finite element method (p=10). The agreement between both results is excellent especially when ε is small. The frequencies predicted in Ref. 5 by Evensen are also presented. It is obvious that Eq. (39) reduces to the relationship by Evensen when ε becomes infinitesimal.

6. CONCLUDING REMARKS

The method of weighted residuals in the time domain is successfully applied to nonlinear differential equations which govern the periodic oscillation. The accuracy of the method, as well as its convenience, is assured by the example calculations on the nonlinear vibrations of an infinitely long cylinder. The Galerkin type of finite element procedure shows a rapid convergence and gives an excellent approximation.

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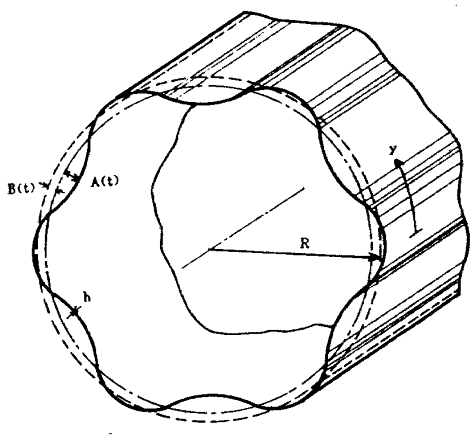

Figure. An infinitely long cylinder.

Table I. Error parameter, $\left(\frac{\Omega}{\Omega_{exact}}\right)^2 - 1$, for the constrained vibration of Eq. (11).

Approximation		Harmonic functions		FEM–Hermitian polynomials					Ω_{exact}
Weighting function		same		1, $1-\xi$	same				
Terms or elements		one term*	two terms**	p=1***	p=1	p=2	p=3	p=4	
β	0	0	0	-0.0075	-0.0004	-0.0000	-0.0000	-0.0000	1
	0.5	0.0031	0.0001	-0.0042	-0.0014	-0.0003	-0.0001	-0.0000	1.3707
	1.0	0.0078	0.0003	-0.0006	-0.0012	-0.0005	-0.0001	-0.0001	1.7365
	1.5	0.0119	0.0005	0.0023	-0.0008	-0.0007	-0.0002	-0.0001	2.1000
	2.0	0.0154	0.0008	0.0048	-0.0002	-0.0009	-0.0002	-0.0001	2.4621

*) Eq.(17), **) Eqs.(13), ***) Eqs.(33).

Table II. Convergence of a nondimensional frequency for the extensional vibration, depending on the number of elements.

$\left(\frac{1}{12}\varepsilon = 10^{-2},\quad \beta = 0.1\right)$

p	Ω^2
1	0.9745
2	0.9747
10	0.9747

Table III. Nondimensional frequency Ω^2 of the extensional vibration of Eqs. (9). (FEM in parentheses)

β \ $\frac{1}{12}\varepsilon$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	Evensen
0	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000
0.1	0.9610 (0.9612)	0.9746 (0.9747)	0.9755 (0.9756)	0.9756 (0.9757)	0.9756
0.2	0.9264 (0.9272)	0.9506 (0.9509)	0.9522 (0.9525)	0.9524 (0.9526)	0.9524
0.3	0.8954 (0.8973)	0.9277 (0.9283)	0.9300 (0.9306)	0.9302 (0.9308)	0.9302
0.4	0.8672 (0.8712)	0.9060 (0.9070)	0.9088 (0.9097)	0.9091 (0.9100)	0.9091

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