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The Free Vibration Equations, Natural Frequencies and Modal Characteristics of Closed Circular Cylindrical Shells

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The Free Vibration Equations, Natural Frequencies and* Modal Characteristics of Closed Circular Cylindrical Shells

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ABSTRACT

The relative accuracies of some of the representative classical theories of thin shells are examined by way of the order-of-magnitude comparison of terms involved in the governing equations for the free vibrations of a circular cylindrical shell. Due consideration is made of the boundary conditions. It will be shown in the end that all the well-known classical theories yield valid solutions accurate enough when taking into consideration of the errors inherent in the Kirchhoff-Love hypothesis. A first approximation, consistent with the order-of-magnitude estimate, leads to a system of free vibration and boundary constraining equations. This system is simple in form yet accurate enough for practical purposes. The approximation enables us to gain a good, qualitative estimate of the forms of the characteristic solutions. The accuracy of the Donnell-type approximation is also investigated. A detailed account of the historical developments of the subject has been given in order to justify our motivation in dealing with this seemingly, most thoroughly investigated topic.

概　　要

周方向に閉じた薄肉弾性円筒シェルの自由振動問題に関して，種々の古典シェル理論の精度を系統的なオーダ評価によって比較し，これらの理論はいずれもKirchhoff - Loveの仮定に含まれる基本誤差内の精度で一致する特性解を与えることを明らかにする。また，Kirchhoff - Love の仮定のもとで振動方程式および境界条件式に対する最も簡単な近似式を導き，特性解の一般的な諸性質を明らかにするとともに，数値計算によって近似式の精度を確認する。さらに Donnell 型の近似式について精度を調べ，その適用範囲を明らかにする。

本論文の解析はほゞ次の順序で進める。まず理論によって異った値をとる単位オーダのパラメタ（theory indicators）を導入して，理論ごとに異なる構成方程式を包括的に一組の式で表示する。運動方程式をたわみ変位関数wのみで表わし，自由振動方程式（八階偏微分方程式）を導く。端末境界条件を与える諸量の関係式をwのみで表わす。Kirchhoff - Love の仮定に含まれる基本誤差のオーダ評価に基づいて自由振動方程式および端末境界条件式の合理的な一次近似式を導く。その結果，これらの近似式には理論の相異を表わすパラメタがまったく含まれないことを示す。また，同様のオーダ評価から特性解の概形を推定し，自由振動においても静的変形と同様に全体変形モードと縁領域変形モードが存在することを明らかにする。次に，Donnell 型の近似式の精度と適用範囲を調べる。数値計算によって解析結果の確認を行う。

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NOTATIONS

Parameters and constants

h	wall-thickness of shell
R	radius of circular cylindrical shell
$2L$	length of cylindrical shell
2ℓ	nondimensional form of 2L; $2\ell = 2L/R$
ρ	mass density
f	frequency of vibration measured in Hz
μ	time factor; $\mu^2 = (1-\nu^2)\rho R^2/E$
k	geometric parameter; $4k^4 = 12(1-\nu^2)(R/h)^2$
δ	geometric parameter; $\delta = h^2/12R^2$
ω	frequency parameter; $\omega^2 = (1-\nu^2)(2\pi Rf)^2\rho/E$
Ω	$\Omega = \omega^2$
E	Young's modulus
ν	Poisson's ratio
K	extensional rigidity; $K = Eh/(1-\nu^2)$
D	bending rigidity; $D = Eh^3/12(1-\nu^2)$
G	shear modulus; $G = E/2(1+\nu)$

Coordinates and differential operators

x	axial coordinate
y	nondimensional form of x; $y = x/R$
θ	circumferential coordinate
t	the time
T	nondimensional form of t; $T = t/\mu$
$(\ )'$	differential operator; $(\ )' = \partial(\ )/\partial y$
$(\ )^{\cdot}$	differential operator; $(\ )^{\cdot} = \partial(\ )/\partial\theta$
$(\ )^*$	differential operator; $(\ )^* = \partial(\ )/\partial T$
$\nabla^2(\ )$	differential operator; $\nabla^2(\ ) = (\ )'' + (\ )^{\cdot\cdot}$
$L(\ )$	differential operator defined in Eq. (19)
$L_0(\ )$	first-approximation form of L( ) defined in Eq. (43)

Vibration and stress

n	circumferential wave number
λ	representative wave length
$\lambda_x, \lambda_\theta$	axial and circumferential wave lengths, respectively
ω_s	frequency parameter corresponding to the lowest thickness-shear vibration of an infinite plate given in Eq. (34)
ω_u	tentatively set upper bound of frequency parameter; $\omega_u = 0(1)$
$\Omega_1, \Omega_2, \Omega_3$	three roots of $A_4(\Omega) = 0$
u_x, u_θ, w_z	midsurface displacement components
u, v, w	nondimensional form of u_x, u_θ, w_z, respectively
$\epsilon_x, \epsilon_\theta, \gamma_{x\theta}$	midsurface strain components
κ_x, κ_θ	changes in curvature of midsurface
$\tau,\ \tau_0$	torsions of midsurface (relative and absolute, respectively)
ω_n	rigid-body rotation about the normal to midsurface
β	lateral rotation of the generator; $\beta = w'$
$N_x, N_\theta, N_{x\theta}, N_{\theta x}$	stress resultants
Q_x, Q_θ	lateral shear resultants
$M_x, M_\theta, M_{x\theta}, M_{\theta x}$	stress couples
$S_x, T_{x\theta}$	lateral and tangential components of the equivalent edge shear, respectively
N, M, S, Q	nondimensional form of N_x, M_x, S_x and $T_{x\theta}$, respectively
$W_i, U_i, V_i, B_i, N_i, M_i, S_i, Q_i$	coefficients of eigenfunction expansions of w, u, v, β, N, M, S and Q, respectively
$\widetilde{W}_i$	real coefficients deduced from W_i in Eqs. (91)
E_i, D_i	theory indicators
p_i	eigenvalues
P	$P = (p_i)^2$
P_m	a value of P at which F(P) becomes minimum
$\xi_1,\ \eta_1$	real and imaginary parts, respectively, of p_i (i = 1, 2, 3, 4)
$\xi_2,\ \eta_2$	real and imaginary parts, respectively, of p_i (i = 5, 6, 7, 8)
$\widetilde{\xi}_i, \widetilde{\eta}_i, p_0, q_0$	real quantities of order of magnitude unity introduced in Eqs. (79)

Coefficients of equations

G_i, H_i, K_i	coefficients in Eqs.

	(12)-(14)
L_i	coefficients in Eq. (19)
f_i, g_i	coefficients in Eqs. (18)
$a_{ij}, b_{ij}, c_{ij}, d_{ij}$	coefficients in Eq. (20)
n_i, m_i, s_i, q_i	coefficients in Eqs. (21)
$\overset{0}{a}_{ij}, \overset{0}{b}_{ij}, \overset{0}{c}_{ij}, \overset{0}{d}_{ij}, \overset{0}{L}_i, \overset{0}{n}_i, \overset{0}{m}_i, \overset{0}{s}_i, \overset{0}{q}_i$	leading terms in the polynomial expressions of a_{ij}, b_{ij}, c_{ij}, d_{ij}, L_i, n_i, m_i, s_i and q_i, respectively
a_{ij}	modified form of $\overset{0}{a}_{ij}$ as given in Eqs. (40)
A_i	coefficients in Eq. (51)
$\mathcal{L}_i$	common denominators in the coefficients given in Eqs. (87)
$\mathcal{W}_i, \mathcal{U}_i, \mathcal{B}_i, \mathcal{V}_i, \mathcal{S}_i, n_i, m_i, \mathcal{O}_i$	coefficients of the boundary constraining equations, Eqs. (92)

Miscellaneous

$F(P)$	function defining the auxiliary equation as $F(P) = 0$ in Eq. (51)
Im( )	to indicate the imaginary part of ( )
$[\;]_f, [\;]_p, [\;]_r$	to indicate the flexural, in-plane, and rotatory inertial components, respectively
$\{C\}$	coefficients matrix of the boundary constraining equations
$\{\widetilde{W}_i\}$	column matrix of the coefficients $\widetilde{W}_i$ given in Eq. (95)
$\{\widetilde{W}_i\}^T$	to indicate the transpose of $\{\widetilde{W}_i\}$
i	unit of imaginary number; $i = (-1)^{1/2}$

1. INTRODUCTION

The present paper is concerned with the theoretical analysis of the free vibrations of a thin elastic circular cylindrical shell. The shell is assumed to be circumferentially closed and to poses two axial circular edges formed by parallel plane cross sections normal to the axis of the shell. This is a classical topic of mechanics of shells. It may be no exaggeration to state that the theory of thin shells finds here its very origin. Let us first explore the historical background of the developments.

Historical Development

It was about one hundred years ago that the analysis on the subject was first undertaken by Lord Rayleigh, born John William Strutt. In his famous treatise on "The Theory of Sound" [1] published in 1877, he has developed a formula for calculating the natural frequencies of an infinitely long circular cylindrical shell assuming that the midsurface of the shell is inextensional and the mode of vibration cylindrical. As anticipated from the assumption of cylindrical mode, his result is applicable for the in-plane vibrations of a circular ring. Indeed, he has shown that his result is identical in form to that derived by R. Hoppe for a ring in a memoir published in Crelle, Bd. 63, 1871. In 1881, Lord Rayleigh presented a paper [2], in which he has developed an inextensional theory of vibrations for shells of revolution and applied it for shells of spherical, cylindrical and conical shape. His primary interest in the paper seems to have been in the calculation of the natural frequencies of a spherical shell, particularly of a hemi-spherical one, as a model for a church bell, because a substantial part of the paper is covered by the analysis on that subject.

A. E. H. Love has developed a general theory of thin shells in a paper [3] presented in 1888 (received January 19, read February 9, 1888). In this paper, Love critisizes the inexactness of Lord Rayleigh's inextensional theory in that it leads to expressions for the displacements which cannot satisfy the boundary conditions at a free edge. He argues that the equations imposing the condition of inextensibility are in the most general case a system of the third order, while the boundary conditions are four in number, so that these equations are not in general of a sufficiently high order to admit of solutions which shall satisfy the boundary conditions at a free edge. He has explicitly shown that this is the case for a spherical shell. In the development of the general theory, he has shown that the strain energy expression consists of two terms; one is proportional to the thickness h depending on the membrane stretching, and the other proportional to h^3 depending on the flexural bending. Needless to say, the inextensional theory of Lord Rayleigh takes no account of the term depending on the membrane stretching. Love dissents from Lord Rayleigh's reasoning for the omission of that term depending on the membrane stretching. He argues that the term proportional to h^3 is small in comparison with the term proportional to h, and that the former instead of the latter should be omitted in the limiting case of $h \to 0$. He has thus formulated an extensional theory and applied it for spherical and cylindrical shells.

Lord Rayleigh immediately responded to Love's criticism in a paper [4] presented in the same year (received December 1, read December 13, 1888). His argument focuses on the justification of his omission of the strain energy depending on the membrane stretching. He states that it is a general mechanical principle that, if displacements be produced in a system by forces, the resulting deformation is determined by the condition that the potential energy of deformation shall be as small as possible, and that the large potential energy which would accompany any stretching of the mid-surface is the very reason why such stretching will not occur. He has also given an explanation to the inexactness of his inextensional theory with regard to the violation of the boundary conditions at free edges through an example of a long circular cylindrical shell subjected to a pair of concentrated normal forces at the extremities of one diameter of the central section. As the thickness reduces, the deformation assumes more and more the character of pure bending such that every normal cross section deforms into an identical configuration. If the thickness remains small but finite, a point will at last be attained when the energy can be made least by a sensible local stretching of the mid-surface such as will dispense with the uniform bending otherwise necessary over so great a length. Thus, he has indicated the possibility of the existence of the so-called edge-zone solutions.

The second edition of "The Theory of Sound" was published in 1894. All the corrections of importance and new matter added to the first edition are clearly indicated. A new chapter is interpolated, devoted to shells. Although calculations of the natural frequencies for spherical and cylindrical shells are presented with the aid of both the inextensional and extensional theories, he seems to have remained an advocate of the inextensional theory. He notes that any extension that may occur must be limited to a region of infinitely small area and affects neither the types nor the frequencies of vibration. The first edition of Love's famous treatise on "The Mathematical Theory of Elasticity" [5] was published in 1892 (volume 1) and 1893 (volume 2). The fourth edition was published in 1927, whose American printing in 1959 is the one available to the present authors at the present time. Some of the important new additions and revisions in the third and fourth editions are indicated. But, as Love notes that the first edition was almost entirely re-written in the second edition published in 1906, it is impossible for us to see from the book available how the material has evolved from the original volumes. In any event, Love has presented for the first time a general method of solution of the free vibration problems of circular cylindrical shells. But he has made no attempt to solve any specific problem except for those extreme cases of purely extensional and inextensional vibrations. The fact that the calculations by the inextensional theory were incorporated in the new editions seems to imply a change in his attitude toward Lord Rayleigh's insistence on the inextensional theory. He seems to have contented himself by noting that the extensional strain, which is necessary in order to secure the satisfaction of the boundary conditions, is practically confined to so narrow a region near the edge that its effect in altering the total amount of the potential energy, and therefore the periods of vibration, is negligible and the greater part of the shell vibrates according to Lord Rayleigh's type.

In 1934, W. Flügge published a book "Statik und Dynamik der Schalen" [6], a highly recognized classic devoted to mechanics of shells. He has developed a general theory of shells, which is nowadays most commonly known as Flügge's theory, and applied it to the free vibrations of a circular cylindrical shell. He has suggested a general method of solution. The method may be summarized as follows: Give the general solutions for the displacement components u, v and w in the form

$$u = U e^{py} \cos n\theta \sin \omega T$$
$$v = V e^{py} \sin n\theta \sin \omega T$$
$$w = W e^{py} \cos n\theta \sin \omega T$$

where U, V and W are the indeterminate coefficients, y and θ the axial and circumferential coordinates, respectively, T the time, n the circumferential wave number, p the characteristic value which determines the axial characteristic mode and ω the natural frequency. The auxiliary equation deduced from the equations of motion is an algebraic equation of the fourth degree in p^2 and n^2 and of the third degree in ω^2. Thus, if n and ω are assumed, eight values of p may be determined. Once p, n and ω, as well as the thickness-to-radius ratio and the material constants, are known, the characteristic equation resulting from eight boundary conditions may be solved for the length of the shell 2ℓ. It will yield infinite number of discrete solutions for ℓ. The minimum of them determines the length of the cylinder which vibrates at ω in a mode characterized by 2n of the axial nodal lines with no circumferential node. As he states: "Die zahlenmässige Durchführung dieser Rechnung ist sehr mühsam und steht noch aus.", calculation by this general method is a very difficult task and he himself has not attempted it. Instead, he has chosen a special case where the edges of the cylinder are simply supported so that the boundary conditions are given by

$$w = v = N = M = 0 \; ; \; y = \pm\ell$$

N and M being the axial stress resultant and couple. Now, the displacement components u, v and w are expressed in trigonometric functions in y, θ and T, and they satisfy the boundary conditions exactly. He then gave a numerical example and has calculated three roots of ω for the auxiliary equation. Comparing the amplitude ratios of the displacement components for each ω, he has shown that the free vibrations corresponding to the lowest, intermediate and highest values of ω are predominantly in lateral, axial and circumferential motion, respectively. Since the free vibrations predominantly in lateral motion are the most important in engineering practice, and since his numerical result shows the lowest value of ω is very small, he has made an approximation neglecting in the auxiliary equation the cubic and quadratic terms in ω^2, and deduced a simple formula applicable for the lowest natural frequencies.

It has become clear by now that an eigenvalue problem of a shell is generally governed by an eighth order partial differential equation and they require solution of an eighth order characteristic determinant resulting from eight homogeneous boundary conditions. This poses a tremendous difficulty in analysis, and consequently, as noted by Flügge, analyses in those early days couldn't go beyond outlining general methods of solution and applying them for some extremely limited cases. A complete solution with mathematical rigour and generality becomes feasible only after the development of high speed digital computers. It should be emphasized, however, that much of the fundamentals of the shell theory and its solution technique were laid down in those early days.

In the fifties, high speed digital computers began to influence on every item of human activities let alone scientific and technological research. But their developments had not yet reached a stage of maturity being enjoyed by the researchers in the sixties and thereafter. Much of the research effort in the fifties, therefore, was focused on the development of approximate methods of solution of the general theory to cope with the growing technological demands. R.N. Arnold and G.B. Warburton [7, 8] have calculated the natural frequencies for circular cylindrical shells of finite length, which are either simply supported or rigidly clamped at both edges. In analysis, they used the strain-displacement relations of the Love-Timoshenko theory, Timoshenko's version of Love's first approximation, to express the strain energy and the kinetic energy in terms of the displacements and then derived Lagrange's equations assuming the displacement components such as to satisfy the boundary conditions. An equivalent wavelength factor was introduced to make estimation of the natural frequencies for cylinders with solidly built or flanged edges as seen in practice. They also performed an experiment and compared the results with the theoretical predictions. In the experiment, the modal characteristics were determined by detecting the sound intensity variations by a stethoscope. M. L. Baron and H. H. Bleich[9] have calculated the natural frequencies of an infinitely long circular cylindrical shell. The shell is first assumed as membrane without bending rigidity, and the bending effects are incorporated subsequently according to Rayleigh's principle for the eigenvalues. Y-Y. Yu[10] used the approximate equations of the Donnell-type to calculate the natural frequencies of a circular cylindrical shell of finite length which were either simply supported or rigidly clamped at both edges, or simply supported at one edge and rigidly clamped at the other. Assuming that the shell is comparatively long and it vibrates in a mode characterized by a fairly small number of longitudinal waves and a large number of circumferential waves, he has derived a simple formula for calculating the natural frequencies. He has also derived the characteristic equations in simple form, which have turned out to be identical to those of the lateral vibrations of an elastic beam. The Donnell-type of approximation was also used by K. Heki[11]. Although he didn't specifically refer to Donnell' work, it is apparent from the fundamental assumptions in his analysis that the approximation is the same as the one here referred to as the Donnell-type. A detailed investigation on the characteristic values for both the circumferentially closed and open shells has led to the following conclusions: (1) The approximation yields accurate solutions for the natural frequencies unless the frequencies are very high and the shell is extremely long. (2) The edge-zone bending solutions which decay out rapidly as the distance from the edges increases do not have significant influence on the natural frequencies. (3) The frequencies are determined mostly by the global solutions which vary gradually over the entire surface of the shell, and they are significantly influenced by the in-plane boundary conditions. His work seems to have been left relatively unknown without receiving worldwide recognition in spite of the importance of his conclusions. In particular, the importance of his findings concerning the influence of the in-plane boundary conditions should not have been overlooked.

The traditional approximate analysis has been carried over even to the present computer age. V. I. Weingarten[12] has made an analysis on a circular cylindrical shell which is rigidly clamped at one edge and completely free at the other, and performed an experiment on butt welded steel test specimens detecting the resonance characteristics by a microphone. J. D. Watkins and R. R. Clary[13] carried out an experiment on spotwelded stainless steel test specimens with the free-free and the clamped-free edges. J. L. Sewall and E. C. Naumann[14] performed an analytical and experimental study on the free vibrations of unstiffened and stringer-stiffened circular cylindrical shells. The unstiffened test specimens were fabricated from aluminum-alloy sheet. Four sets of boundary conditions were investigated; they were (1) the free-free, (2) the simply supported-simply supported, (3) the clamped-free, and (4) the clamped-clamped edges. In Refs.[12, 13, 14], the experimental results have been compared with those determined analytically in one way of approximation or another proposed by earlier researchers. R. W. Nau and J. G. Simmonds[15] have derived a simple formula for the low natural frequencies of a clamped-clamped cylindrical shell by the asymptotic method.

In the meantime, a great progress was observed in the development of high speed digital computers with regard to their memory capacity, efficiency as well as availability. Researchers in the late fifties and thereafter were able to perform extensive numerical calculations taking account of all possible combinations of boundary conditions and the complicated effects of various factors and parameters. Among the overwhelming volume of literatures, we shall here make reference only to a limited number of those concerned with the classical problem of plain homogeneous circular cylindrical shells. B.L. Smith and E.E. Haft[16] and D.F. Vronay and B.L. Smith[17] have calculated the natural frequencies for a shell with rigidly clamped edges. Though they claim that they have solved exactly the equations derived by Flügge and decoupled by Yu, they have actually worked out with the Donnell-type equations. Their numerical solutions may therefore be regarded as the exact solutions to the Donnell-type equations. A.E. Armenàkas[18] has calculated the natural frequencies of a simply supported circular cylindrical shell by the Flügge and the Donnell-type equations and established the range of validity of these equations by a numerical comparison of the results with those obtained by J.E. Greenspon[19] and D.C. Gazis[20] on the basis of the theory of clasticity. He has derived a simple formula for the lowest natural frequencies. Both Greenspon's and Gazis': papers published in 1958 are also typical examples of the computer-age products. Greenspon has calculated the natural frequencies of the flexural vibrations of a hollow circular cylinder of finite length. He treated the cylinder as a three dimensional body assuming that the thickness-bounding surfaces were traction free and the normal cross sectional surface of the axial edges were to remain circular such that $w = v = 0$ be satisfied. The resultant forces and moments acting on the edges have been evaluated and found to be zero except for the case of $n = 1$. Numerical calculations were performed for $n = 1$ and 2 and the results have been compared with those of the Timoshenko beam theory ($n = 1$) and those obtained by Arnold and Warburton ($n = 2$). Gazis has investigated the plane-strain free vibrations of a thick-walled hollow cylinder of infinite length in the framework of the theory of elasticity. The thickness-bounding surfaces are assumed traction free. He has shown that the extensional and shear modes can exist uncoupled in the axisymmetric vibrations and derived approximate expressions of the frequencies for these modes, which tend to the simple thickness-stretch and thickness-shear modes of an infinite plate as the thickness-to-radius ratio approaches zero. He has also deduced an equation for the frequencies of the classical extensional vibrations in cylindrical modes from the analysis of the unsymmetric vibrations. A transition from the shell vibrations to the Pochhammer vibrations of a solid cylinder has been established on the basis of the results of the analysis and the numerical computations covering the entire range of the thickness-to-radius ratio. C.B. Sharma and D.J. Johns[21] have calculated with the aid of the Flügge theory the natural frequencies of a finite cylindrical shell whose edges are either rigidly clamped and force free or rigidly clamped and ring-stiffened. A comparison of the results with those obtained by the Love-Timoshenko theory indicates there exists no significant difference between the two theories.

A series of papers[22, 23] published by K. Forsberg, a former student of Professor Flügge at Stanford University, present thus far most extensive and rigorous numerical solutions taking account of all possible combinations of boundary conditions. His solutions are exact in the sense that no approximation has been introduced beyond those underlying the basic equations formulated by Flügge. The numerical calculations were performed in a broad range of the geometric parameters; i.e., the length-to-radius and the radius-to-thickness ratios. In general, there are sixteen possible sets of homogeneous boundary conditions at each edge consisting of appropriate combinations of the following four pairs:

lateral displacement or shear;	$w = 0$ or $S = 0$
axial rotation or moment;	$\beta = 0$ or $M = 0$
axial displacement or normal force;	$u = 0$ or $N = 0$
circumferential displacement or shear;	$v = 0$ or $Q = 0$

Most of the previous investigators had been concerned only with three typical boundary conditions loosely defined as the simply supported, the rigidly clamped and the force free boundary conditions specified by

simply supported; $w = M = N = v = 0$
rigidly clamped; $w = \beta = u = v = 0$
force free; $S = M = N = Q = 0$

Forsberg has investigated all sixteen cases and presented the results for ten representative cases including those specified with different conditions at different edges. One of the most important conclusions of his analysis on the breathing vibrations ($n \geq 2$) is that the effect of the axial constraint is significant even for very long shells and for all values of the radius-to-thickness ratio, so that the minimum frequencies may differ by more than 50 per cent depending on whether $u = 0$ or $N = 0$ at both edges. As to the axisymmetric ($n = 0$) and the beam-like bending ($n = 1$) vibrations, it has been shown that the vibration characteristics are governed primarily by the membrane behavor and they are strongly influenced by those in-plane boundary conditions involving u and v or N and Q.

Since the development of a first complete linear theory of shells by Love in 1888, a great progress has been made in the development of the general theory of thin shells along with the aforementioned developments in the analysis of the free vibrations of circular cylindrical shells. Literatures are already almost superfluous, and a number of different theories have been proposed by many authors. Most of these theories are based on the Kirchhoff-Love hypothesis and they are nowadays commonly known as the classical theories. To name some, they are the Love[3], the Flügge[6], the Novozhilov[24], the Koiter[25], the Sanders[26], the Love-Reissner[27], the Love-Timoshenko[28], and the Vlasov[29] theories. Here, the Love-Reissner and the Love-Timoshenko theories mean, respectively, Reissner's and Timoshenko's versions of Love's first approximation theory. There also exist various simplified theories specifically designed for practical applications for particular shell configurations. Probably, the best known of them is the Donnell[30] theory for circular cylindrical shells.

The accuracies and the ranges of validity of the classical theories have been investigated by many researchers, most thoroughly for static deformations and stresses. It is now well-known that a first approximation theory consistent with the Kirchhoff-Love hypothesis should be such that the strain energy expression consists of two terms; one is due to the membrane stretching and the other to the flexural bending. Most of the well-known classical theories such as those of Refs.[3, 6] and [24] through [29] are actually consistent first approximation theories in this regard. As long as a theory is formulated on the basis of the Kirchhoff-Love hypothesis, its accuracy cannot surpass that of Love's theory even if it is seemingly more accurate because of retention of higher order terms in its equations. An exposition of the state of art as well as additional references on this subject can be found in a previous paper of one of the present authors[31]. Much less attention has been paid to the accuracy and the consistency of the general theories of shells when they are applied for the dynamic problems. As far as the consistency check and the global error estimate of solutions are concerned, it appears more or less straightforward to follow the same approach as in the static theories by adding the kinetic energy terms to the energy expression. A pointwise local error estimate of solutions, however, is an extremely difficult task, indeed almost impossible at the present time, if it is dealt with in the framework of the general theory. It has been done mostly by comparing the numerical values of the solutions obtained by some particular shell theories with those obtained by a seemingly more accurate theory for a particular boundary value problem. Examples of this for the free vibrations of circular cylindrical shells may be seen in Refs.[18, 22] and Ref.[21]; the formers are concerned with the comparison of the Donnell and the Flügge theories, and the latter the Love-Timoshenko and the Flügge theories. The relative accuracies of the theories may also be examined analytically (or qualitatively) by comparing the terms involved in the governing equations or in the characteristic equations. This approach has been employed by P.M. Naghdi and J.G. Berry [32], K. Heki[11], and A.W. Leissa[33]. None of their papers gives detailed considerations on the boundary conditions, except that Heki has pointed out the important effect of the in-plane boundary conditions on the natural frequencies and Leissa has examined the characteristic equations for shells with simply supported edges derived from a number of different theories. The reader is urged to refer to Leissa's report for a detailed account of the state of art and a complete list of literatures on the subject of the vibrations of shells by 1973.

A great effort has also been being made to develop higher order shell theories as well as to establish a rational foundation of the theory of shells upon the theory of elasticity. The previously cited papers of Greenspon and Gazis are typical examples of the work in this direction, and they have provided a rational mean for the comparison of the shell theory with the elasticity theory. To make a complete list of literatures on them is out of the scope of the present paper. We only cite a few of them which deal with dynamics of circular cylindrical shells. They include the papers of P.M. Naghdi and R.M. Cooper[34], T.C. Lin and G.W. Morgan [35], G. Herrmann and I. Mirsky[36], and I. Mirsky and G. Herrmann[37]. A notable feature of these papers is that the transverse shear and the rotatory inertia are incorporated in their approximate shell theories deduced from the theory of elasticity just as those incorporated in the Timoshenko beam theory [38] and in the Mindlin plate theory[39].

Present Paper

The purpose of the present paper is twofold: We shall on one hand seek for a system of consistent first approximation equations as simple in form as possible for the free vibrations of a circular cylindrical shell within the framework of the classical theory based on the Kirchhoff-Love hypothesis. We shall on the other hand try to establish analytically (or qualitatively) the relative accuracies of some of the representative classical theories by comparing terms involved in the governing equations. As far as the second part of the purpose is concerned, the present paper belongs to the class represented by the papers of Naghdi and Berry[32], Heki[11], and Leissa[33]. It should be noted, however, that the emphasis in the present paper is placed on the influence of boundary conditions on the accuracies of solutions, which has not been discussed thoroughly in those earlier papers.

A first approximation is achieved neglecting small terms of order of magnitude of the fundamental errors inherent to the Kirchhoff-Love hypothesis. For this, use is made of the well-established fact that the classical shell theory is valid in the ranges (see for instance F.I. Niordson's report[40]).

$$h/R_m + (h/\lambda_m)^2 \ll 1$$

h being the thickness of the shell, R_m the minimum radius of curvature, and λ_m the minimum wave length of deformation. We can also well anticipate, as noted by A. Kalnins[41], that the free vibration spectrum predicted by the classical shell theory is accurate only for the frequencies well below the minimum frequency of all the thickness-mode vibrations obtained by Gazis[20]. We thus tentatively set an upper limit to the frequencies. Small terms in the equations are neglected by an appropriate order-of-magnitude comparison. We assume that the difference from one theory to another is mainly attributed to the difference in the expressions of their constitutive equations. We shall only be concerned with the vibrations in the so-called breathing mode characterized by $n \geq 2$.

The present paper proceeds as follows: In Section 2, the basic equations including the boundary conditions, the kinematic relations, the equations of motion, and the constitutive equations are presented. A unique feature in this section is that a set of the constitutive equations is written in such a way that it can easily be specified for any one of those formulated in the Flügge[6], the Koiter[25], the Novozhilov[24], the Naghdi[42], and the Love-Reissner[27] theories by introducing special parameters here referred to as the theory indicators. In Section 3, the equations of motion are reduced to a single eighth order differential equation in terms of the lateral displacement and its derivatives, which is here referred to as the free vibration equation. Also, those quantities to be prescribed as boundary conditions at the axial edges are expressed only in terms of the lateral displacement and its derivatives. They are called supplemental equations.

In Section 4, an order-of-magnitude estimate is made for the eigenvalues as well as for the geometric parameters in consistence with the fundamental assumptions underlying the classical theory. On the basis of this order-of-magnitude estimate, in Section 5, the free vibration equation and the supplemental equations are simplified. The order-of-magnitude estimate for the eigenvalues is more advanced in Section 6 for relatively low frequencies. Substitution of an assumed form of the eigenfunction into the free vibration equation yields an algebraic equation to be satisfied by the eigenvalues, which is here referred to as the auxiliary equation in order to distinguish it from the characteristic equations resulting from the boundary conditions. The specific forms of the solutions of the auxiliary equation are determined within certain ranges of the frequency, the circumferential wave number and the thickness-to-radius ratio. They are then written out explicitly for very low frequencies to yield a simple formula for calculating the natural frequencies. In Section 7, the supplemental equations are written in terms of the eigenvalues and the indeterminate coefficients of the eigenfunction. Specifying homogeneous boundary conditions at the axial edges, we obtain four pairs of homogeneous linear equations for the indeterminate coefficients, which are here referred to as the boundary constraining equations. They are simplified by an appropriate order-of-magnitude comparison. As an example, the characteristic determinant is calculated for the simply supported edges. The equation obtained by setting the characteristic determinant equal to zero is here referred to as the characteristic equation. In Section 8, a brief explanation is given on the method of numerical calculations to be performed for the verification of the approximate equations derived in the preceding sections. In Section 9, the accuracy of the Donnell-type approximation is examined.

2. BASIC EQUATIONS

Let us consider a thin-walled circular cylindrical shell made of a linearly-elastic, homogeneous and isotropic material with Young's modulus E, Poisson's ratio ν and the mass density ρ . Let the wall-thickness, the radius and the length of the shell be denoted by h, R and 2L, respectively. Let the coordinates x and θ be chosen on the midsurface, such that the x measures the axial distance along the generator from the center of the cylinder and the θ the circumferential angular extent with $R\theta$ being the arc length. The shell geometries, the coordinate system, and the positive direction of the components of the displacements, the forces and the moments are depicted in Fig. 1.

The infinitesimal displacement components u_x, u_θ and w_z are defined in the directions of the base vectors tangent to the x and θ coordinates and along the outward normal to the midsurface, respectively. Then, the strains ϵ_x, ϵ_θ and $\gamma_{x\theta}$, the changes in curvature κ_x and κ_θ, the relative torsion τ , and the rotation about the normal ω_n are related to the displacement components by

$$\begin{aligned}
\varepsilon_x &= \partial u_x/\partial x \\
\varepsilon_\theta &= (\partial u_\theta/\partial_\theta + w_z)/R \\
\gamma_{x\theta} &= \partial u_\theta/\partial x + (\partial u_x/\partial_\theta)/R \\
\kappa_x &= -\partial^2 w_z/\partial x^2 \\
\kappa_\theta &= (\underline{\partial u_\theta/\partial \theta} - \partial^2 w_z/\partial \theta^2)/R^2 \\
2R\tau &= \underline{3\partial u_\theta/\partial x} - (\partial u_x/\partial\theta)/R - 4\partial^2 w_z/\partial x\partial\theta \\
2\omega_n &= \partial u_\theta/\partial x - (\partial u_x/\partial\theta)/R
\end{aligned} \tag{1}$$

Here, the torsion τ is referred to as the relative torsion in order to distinguish it from the torsion τ_0 defined by

$$\tau_0 = \tau - \omega_n/R \tag{2}$$

which may be specifically referred to as the absolute torsion.

The expressions of the constitutive equations differ slightly from one theory to another. But those based on the Kirchhoff-Love hypothesis can most often be written formally for circular cylindrical shells as

$$\begin{aligned}
N_x &= K(\varepsilon_x + \nu\varepsilon_\theta + \delta E_1 R\kappa_x) \\
N_\theta &= K[(1+\delta E_2)\varepsilon_\theta + \nu\varepsilon_x - \delta E_1 R\kappa_\theta] \\
N_{x\theta} &= Gh[(1+\delta E_3/4)\gamma_{x\theta} + \delta E_4 R\tau/2] \\
N_{\theta x} &= Gh[(1+3\delta E_5/4)\gamma_{x\theta} - \delta E_6 R\tau/2]
\end{aligned} \tag{3}$$

$$M_x = D(\kappa_x + \nu\kappa_\theta + D_1\varepsilon_x/R)$$
$$M_\theta = D(\kappa_\theta + \nu\kappa_x - D_1\varepsilon_\theta/R)$$
$$M_{x\theta} = (Gh^3/12)(\tau - D_4\omega_n/R + D_2\gamma_{x\theta}/2R)$$
$$M_{\theta x} = (Gh^3/12)(\tau - D_4\omega_n/R - D_3\gamma_{x\theta}/2R)$$

Here, N_x, N_θ, $N_{x\theta}$ and $N_{\theta x}$ are the stress resultants, and M_x, M_θ, $M_{x\theta}$ and $M_{\theta x}$ the stress couples. The elastic constants K, D and G are given by

$$K = Eh/(1-\nu^2)$$
$$D = Eh^3/12(1-\nu^2) \tag{4}$$
$$G = E/2(1+\nu)$$

The geometric parameter δ is defined by

$$\delta = h^2/12R^2 \tag{5}$$

The parameters E_1, E_2, E_3, E_4, E_5, E_6, D_1, D_2, D_3 and D_4 are quantities at most of order of magnitude unity and their specific values differ with different theories. If their values are specified as in Table 1, Eqs. (3) become identical, or at least equivalent, to those constitutive equations formulated in the Flügge[6], the Naghdi [42], the Koiter[25], the Novozhilov[24], and the Love-Reissner[27] theories. For brevity, we shall refer to these parameters as the theory indicators.

If the effect of the rotatory inertia about the normal to the midsurface is neglected, the equations of motion for the free vibrations of the circular cylindrical shell are given by

$$R\partial N_x/\partial x + \partial N_{\theta x}/\partial\theta = \rho hR[\partial^2 u_x/\partial t^2]_p$$
$$\partial N_\theta/\partial\theta + R\partial N_{x\theta}/\partial x + \underline{Q_\theta} = \rho hR[\partial^2 u_\theta/\partial t^2]_p$$
$$R\partial Q_x/\partial x + \partial Q_\theta/\partial\theta - N_\theta = \rho hR[\partial^2 w_z/\partial t^2]_f \tag{6}$$
$$R\partial M_x/\partial x + \partial M_{\theta x}/\partial\theta - RQ_x = -\delta\rho hR^3[\partial^3 w_z/\partial x\partial t^2]_r$$
$$\partial M_\theta/\partial\theta + R\partial M_{x\theta}/\partial x - RQ_\theta$$
$$= \delta\rho hR^2[\partial^2 u_\theta/\partial t^2 - \partial^3 w_z/\partial\theta\partial t^2]_r$$

Where t is the time, and Q_x and Q_θ are the lateral shear resultants.

The right-hand members of Eqs. (6) represent the inertial components. They are designated by the square brackets attached with the subscripts f, p and r to indicate the lateral, inplane and rotatory inertial components, respectively. It will be shown in the subsequent analysis that the rotatory inertial components are always negligible within the framework of the first approximation shell theory.

In Eqs. (1) and (6), the terms to be neglected in the Donnell approximation are designated by underline. The accuracy and limitation of the Donnell approximation will be examined in Section 9.

The boundary conditions for free vibrations of the circumferentially closed circular cylindrical shell are prescribed at the axial edges x = L and −L as proper combinations of the following four pairs of the homogeneous equations:

$$\left.\begin{array}{l} w_z = 0 \text{ or } S_x = 0 \\ \beta = 0 \text{ or } M_x = 0 \\ u_x = 0 \text{ or } N_x = 0 \\ u_\theta = 0 \text{ or } T_{x\theta} = 0 \end{array}\right\} \quad \text{at } x = \pm L \tag{7}$$

where β is the axial rotation defined by

$$\beta = \partial w_z/\partial x \tag{8}$$

ans S_x and $T_{x\theta}$ are the equivalent edge shears defined by

$$S_x = Q_x + (\partial M_{x\theta}/\partial\theta)/R$$
$$T_{x\theta} = N_{x\theta} + M_{x\theta}/R \tag{9}$$

The following nondimensional quantities and operators are introduced to present the subsequent analysis in nondimensional form:

$$u = u_x/R, \quad v = u_\theta/R, \quad w = w_z/R,$$
$$N = N_x/K, \quad M = RM_x/D, \quad S = S_x/\delta K, \quad Q = T_{x\theta}/K,$$
$$y = x/R, \quad T = t/\mu, \quad \ell = L/R, \tag{10}$$
$$(\ )' = \partial(\ )/\partial y, \quad (\ )^{\cdot} = \partial(\ )/\partial\theta, \quad (\ )^* = \partial(\ )/\partial T,$$
$$\nabla^2(\ ) = (\ )'' + (\ )^{\cdot\cdot}$$

where

$$\mu^2 = (1-\nu^2)\rho R^2/E \tag{11}$$

3. GOVERNING EQUATIONS AND EIGENFUNCTIONS

The number of the equations of motion is reduced from five to three by eliminating from them the lateral shear resultants Q_x and Q_θ. The three equations are then written in terms of the displacement components with the aid of Eqs. (1) and (3). The result may be written in nondimensional form as

$$G_1u''+G_2u^{\cdot\cdot}+G_3v'^{\cdot}+G_4w'''+G_5w'^{\cdot\cdot}+G_6w' -[u]_p^{**}= 0 \qquad (12)$$

$$H_1v''+H_2v^{\cdot\cdot}+H_3u'^{\cdot}+H_4w''^{\cdot}+H_5w^{\cdot\cdot\cdot}+H_6w^{\cdot} -([1]_p+[\delta]_r)v^{**}+[\delta w^{\cdot}]_r^{**}=0 \qquad (13)$$

$$K_1u'''+K_2u'^{\cdot\cdot}+K_3u'+K_4v''^{\cdot}+K_5v^{\cdot\cdot\cdot}+K_6v^{\cdot} - \nabla^4w - K_7w^{\cdot\cdot} - K_8w - [v^{\cdot}]_r^{**} - [w/\delta]_f^{**} + [\nabla^2 w]_r^{**}=0 \qquad (14)$$

where the coefficients G_i, H_i and K_i are constants involving ν, δ, E_i and D_i, whose explicit expressions are given in Appendix A.

Let us multiply Eqs. (12) and (13) with H_3 and G_3, respectively, and differentiate them simultaneously with respect to y and θ to write them in the form

$$G_1(H_3u'^{\cdot})''+G_2(H_3u'^{\cdot})^{\cdot\cdot}+H_3(G_3v''^{\cdot\cdot}+G_4w''''^{\cdot} +G_5w''^{\cdot\cdot\cdot}+G_6w''^{\cdot}) - [H_3u'^{\cdot}]_p^{**}=0 \qquad (15)$$

$$H_1(G_3v'^{\cdot})''+H_2(G_3v'^{\cdot})^{\cdot\cdot}+G_3(H_3u''^{\cdot\cdot}+H_4w'''^{\cdot\cdot} +H_5w'^{\cdot\cdot\cdot\cdot}+H_6w'^{\cdot\cdot}) - ([1]_p+[\delta]_r)(G_3v'^{\cdot})^{**} +G_3[\delta w'^{\cdot\cdot}]_r^{**}=0 \qquad (16)$$

Also, from Eqs. (12) and (13), we have

$$G_3v'^{\cdot}=-(G_1u''+G_2u^{\cdot\cdot}+G_4w'''+G_5w'^{\cdot\cdot}+G_6w')+[u]_p^{**}$$
$$H_3u'^{\cdot}=-(H_1v''+H_2v^{\cdot\cdot}+H_4w''^{\cdot}+H_5w^{\cdot\cdot\cdot}+H_6w^{\cdot}) +([1]_p+[\delta]_r)v^{**}-[\delta w^{\cdot}]_r^{**} \qquad (17)$$

Substituting $H_3u'^{\cdot}$ and $G_3v'^{\cdot}$ from Eqs. (17), we eliminate u from Eq. (15) and v from Eq. (16). We can thus decouple u and v from Eqs. (15) and (16). The result may be written in the form

$$Lu = f_1w'''''+f_2w'''^{\cdot\cdot}+f_3w'^{\cdot\cdot\cdot\cdot}+f_4w'''+f_5w'^{\cdot\cdot} +f_6w'''^{**}+f_7w'^{\cdot\cdot **}+f_8w'^{**} \qquad (18)$$
$$Lv=g_1w''''^{\cdot}+g_2w''^{\cdot\cdot\cdot}+g_3w^{\cdot\cdot\cdot\cdot\cdot}+g_4w''^{\cdot}+g_5w^{\cdot\cdot\cdot} +g_6w''^{\cdot **}+g_7w^{\cdot\cdot\cdot **}+g_8w^{\cdot **}+g_9w^{\cdot \#\#}$$

where L( ) is a differential operator defined by

$$L(\ )=L_1(\ )''''+L_2(\ )''^{\cdot\cdot}+L_3(\ )^{\cdot\cdot\cdot\cdot} +L_4(\ )''^{**}+L_5(\ )^{\cdot\cdot **}+L_6(\ )^{\#\#} \qquad (19)$$

The coefficients f_i, g_i and L_i in Eqs. (18) and (19) are constants, whose explicit expressions in terms of H_i and G_i are given in Appendix A.

Let the differential operator L( ) be applied to Eq. (14) and the terms in it with Lu and Lv be substituted from Eqs. (18). Then, we can eliminate u and v to have an eighth order differential equation for w. The result may be written in the form

$$a_{81}w''''''''+a_{82}w''''''^{\cdot\cdot}+a_{83}w''''^{\cdot\cdot\cdot\cdot}+a_{84}w''^{\cdot\cdot\cdot\cdot\cdot\cdot}+a_{85}w^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot}$$
$$+a_{61}w''''''+a_{62}w''''^{\cdot\cdot}+a_{63}w''^{\cdot\cdot\cdot\cdot}+a_{64}w^{\cdot\cdot\cdot\cdot\cdot\cdot}+a_{41}w''''$$
$$+a_{42}w''^{\cdot\cdot}+a_{43}w^{\cdot\cdot\cdot\cdot}$$
$$+(b_{61}w''''''+b_{62}w''''^{\cdot\cdot}+b_{63}w''^{\cdot\cdot\cdot\cdot}+b_{64}w^{\cdot\cdot\cdot\cdot\cdot\cdot}$$
$$+b_{41}w''''+b_{42}w''^{\cdot\cdot}+b_{43}w^{\cdot\cdot\cdot\cdot}+b_{21}w''+b_{22}w^{\cdot\cdot})^{**}$$
$$+(c_{41}w''''+c_{42}w''^{\cdot\cdot}+c_{43}w^{\cdot\cdot\cdot\cdot}+c_{21}w''+c_{22}w^{\cdot\cdot}+c_{00}w)^{**}$$
$$+(d_{21}w''+d_{22}w^{\cdot\cdot}+d_{00}w)^{\#\#\#}=0 \qquad (20)$$

where the coefficients a_{ij}, b_{ij}, c_{ij} and d_{ij} are constants, whose explicit expressions in terms of K_i, f_i, g_i and L_i are given in Appendix A.

Let us now derive the differential equations for N, M, S, and Q, similar to Eqs. (18) for u and v, to relate each of these quantities to w, so that the boundary conditions may be given in the form of boundary constraining equations written only in terms of w. The right-hand members of Eqs. (3) for N_x and M_x are written in terms of the displacement components with the aid of Eqs. (1). The differential operator L( ) is applied to these equations and the terms in them with Lu and Lv are substituted from Eqs. (18) to eliminate u and v. As to S and Q, we first eliminate Q_x from Eqs. (9) for S_x by substituting it from the fourth of Eqs. (6). The right hand members of Eqs. (9) can now be expressed in terms of the displacement components with the aid of Eqs. (1) and (3). Again, the differential operator L( ) is applied to these equations and u and v are eliminated substituting Lu and Lv from Eqs. (18). As a result, we have the following nondimensional expressions:

$$
\begin{aligned}
LN = {} & n_1 w^{\prime\prime\prime\prime\cdot\cdot} + n_2 w^{\prime\prime\cdot\cdot\cdot\cdot} + n_3 w^{\cdot\cdot\cdot\cdot\cdot\cdot} + n_4 w^{\prime\prime\cdot\cdot} + n_5 w^{\cdot\cdot\cdot\cdot} \\
& + (n_6 w^{\prime\prime\prime\prime} + n_7 w^{\prime\prime\cdot\cdot} + n_8 w^{\cdot\cdot\cdot\cdot} + n_9 w^{\prime\prime} + n_{10} w^{\cdot\cdot})^{**} \\
& + (n_{11} w^{\prime\prime} + n_{12} w^{\cdot\cdot} + n_{13} w)^{****} \\
LM = {} & m_1 w^{\prime\prime\prime\prime\prime\prime} + m_2 w^{\prime\prime\prime\prime\cdot\cdot} + m_3 w^{\prime\prime\cdot\cdot\cdot\cdot} + m_4 w^{\cdot\cdot\cdot\cdot\cdot\cdot} + m_5 w^{\prime\prime\prime\prime} + m_6 w^{\prime\prime\cdot\cdot} \\
& + m_7 w^{\cdot\cdot\cdot\cdot} + (m_8 w^{\prime\prime\prime\prime} + m_9 w^{\prime\prime\cdot\cdot} + m_{10} w^{\cdot\cdot\cdot\cdot} + m_{11} w^{\prime\prime} \\
& + m_{12} w^{\cdot\cdot})^{**} + (m_{13} w^{\prime\prime} + m_{14} w^{\cdot\cdot})^{****} \qquad (21) \\
LS = {} & s_1 w^{\prime\prime\prime\prime\prime\prime\prime} + s_2 w^{\prime\prime\prime\prime\prime\cdot\cdot} + s_3 w^{\prime\prime\prime\cdot\cdot\cdot\cdot} + s_4 w^{\prime\cdot\cdot\cdot\cdot\cdot\cdot} + s_5 w^{\prime\prime\prime\prime\prime} + s_6 w^{\prime\prime\prime\cdot\cdot} \\
& + s_7 w^{\prime\cdot\cdot\cdot\cdot} + (s_8 w^{\prime\prime\prime\prime\prime} + s_9 w^{\prime\prime\prime\cdot\cdot} + s_{10} w^{\prime\cdot\cdot\cdot\cdot} + s_{11} w^{\prime\prime\prime} \\
& + s_{12} w^{\prime\cdot\cdot})^{**} + (s_{13} w^{\prime\prime\prime} + s_{14} w^{\prime\cdot\cdot})^{****} + s_{15} w^{\prime******} \\
LQ = {} & q_1 w^{\prime\prime\prime\prime\prime\cdot} + q_2 w^{\prime\prime\prime\cdot\cdot\cdot} + q_3 w^{\prime\cdot\cdot\cdot\cdot\cdot} + q_4 w^{\prime\prime\prime\cdot} + q_5 w^{\prime\cdot\cdot\cdot} \\
& + (q_6 w^{\prime\prime\prime\cdot} + q_7 w^{\prime\cdot\cdot\cdot} + q_8 w^{\prime\cdot})^{**} + q_9 w^{\prime\cdot****}
\end{aligned}
$$

where the coefficients n_i, m_i, s_i and q_i are constants, whose explicit expressions in terms of f_i, g_i and L_i are given in Appendix A.

In what follows, we shall refer to Eq. (20) as the free vibration equation and to Eqs. (18) and (21) as the supplemental equations. These equations are exact in the sense that no approximation has been made throughout the entire process of the derivation starting from the basic equations formulated in the preceding section. It should be noted here that the exact expressions of the coefficients of these equations given in Appendix A can be arranged in the form of polynomials in δ if they are written out full in terms of ν, δ, E_i and D_i. This indicates that a good approximation may be achieved for these equations by neglecting the higher order terms in δ in the polynomial expressions of their coefficients if δ is assumed very small in comparison with unity.

It can be shown that the use of the basic equations formulated in the Sanders theory results in the free vibration equation and the supplemental equations which are identical to Eqs. (20), (18) and (21) specialized for the Koiter theory. Thus, in what follows, we shall refer to the theory associated with these equations as the Koiter-Sanders theory.

The homogeneous systems of the free vibration equation and the boundary conditions with due consideration of the supplemental equations constitute eigenvalue problems for w. In these eigenvalue problems, w may be given in the form

$$w = \sum_{i=1}^{8} W_i \, e^{p_i y} \cos n\theta \sin \omega T \qquad (22)$$

Here, W_i are the indeterminate coefficients, n the circumferential wave number, p_i the eigenvalues characterizing the axial wave pattern, and ω the frequency parameter defined by

$$\omega^2 = (1-\nu^2)(2\pi R f)^2 \rho/E \qquad (23)$$

f being the frequency measured in Hz.

The supplemental equations imply that all the dependent variables may also be expressed in the form similar to Eq. (23), such that

$$(u, \beta, N, M, S) = \sum_{i=1}^{8} (U_i, B_i, N_i, M_i, S_i) e^{p_i y} \cos n\theta \sin \omega T \qquad (24)$$

$$(v, Q) = \sum_{i=1}^{8} (V_i, Q_i) \, e^{p_i y} \sin n\theta \sin \omega T$$

where the coefficients U_i, B_i, N_i, M_i, S_i, V_i and Q_i are related with W_i by homogeneous linear relations yet to be determined.

4. ORDER-OF-MAGNITUDE ESTIMATE

It is well-known that the basic equations formulated in the classical shell theories on the basis of the Kirchhoff-Love hypothesis contain errors of order of magnitude

$h/R + (h/\lambda)^2$

λ being a representative wave length of vibration mode (see, for instance, Niordson[40]). Accordingly, as long as we are concerned with the classical shell theories, we may assume

$$h/R \ll 1 \qquad (25)$$

and

$$(h/\lambda_x)^2 \ll 1, \quad (h/\lambda_\theta)^2 \ll 1 \qquad (26)$$

where λ_x and λ_θ are the axial and circumferential wave lengths, respectively.

The assumption of Eq. (25) immediately gives

$$\delta \ll 1 \qquad (27)$$

This implies that the coefficients in the free vibration equation and in the supplemental equations may be given as a first approximation by the leading terms in their polynomial expressions in δ.

Since λ_θ is given by

$$\lambda_\theta = 2\pi R/n \tag{28}$$

we have

$$(\pi^2/3)(h/\lambda_\theta)^2 = \delta n^2 \tag{29}$$

Thus, the second of the assumptions of Eqs. (26) imposes a limitation on n such that

$$\delta n^2 << 1 \tag{30}$$

On the other hand, the first of the assumptions of Eqs. (26) is concerned with the eigenvalues p_i. As shown by the first author of the present paper in his earlier paper[31] on the static problems, four out of eight eigenvalues p_i ($i = 1, 2, 3, 4$) represent the solutions for the global deformations which vary gradually over the entire surface of the shell, whereas the remaining four represent those for the edge-zone deformations which occur in the narrow edge-zones and decay out rapidly as the distance from the edges increases. It is more or less obvious that the global solutions exist in the free vibration problems. The representative axial wave length, λ_x, of the vibration mode is determined by p_i for the global solutions, such that

$$\mathrm{Im}(p_i) = 2\pi R/\lambda_x \tag{31}$$

where Im( ) indicates the imaginary part of p_i. This gives

$$\delta[\mathrm{Im}(p_i)]^2 = (\pi^2/3)(h/\lambda_x)^2 \tag{32}$$

According to the first of the assumptions of Eqs. (26), the right-hand member of Eq. (32) should be a very small quantity in comparison with unity. Since the real and the imaginary parts of p_i are of the same order of magnitude, the following holds for p_i for the global solutions:

$$\delta\,|p_i^2| \ll 1 \tag{33}$$

It has also been shown in the static analysis that p_i for the edge-zone solutions are of order of magnitude of $(R/h)^{1/2}$, so that $\delta\,|p_i^2|$ are of order of magnitude of h/R. Consequently, the order-of-magnitude relation of Eq. (33) holds for all p_i under the assumptions of Eqs. (25) and (26), if the edge-zone solutions similar to those in the static problems exist in the free vibration problems. The existence of the edge-zone solutions in the free vibration problems will be proved in Section 6.

Gazis[20] has shown by an exact three-dimensional analysis on the plane-strain vibrations of hollow elastic cylinders that the simple thickness-shear mode occurs at the lowest frequency of all the thickness-modes, which coincides asymptotically as $h/R \to 0$ with the frequency ω_s of an infinite plate;

$$\omega_s = [(1-\nu)/2]^{1/2}\pi R/h \tag{34}$$

In general, the thickness-modes do not appear by themselves but coupled with other modes. As noted by Kalnins[41], therefore, the free vibration spectrum predicted by the classical shell theories based on the Kirchhoff-Love hypothesis is accurate only for frequencies well below ω_s. An upper bound of the frequency parameter ω may be arbitrarily set for our purpose as ω_u, such that

$$\omega_u = (h/R)\,\omega_s \tag{35}$$

This gives

$$\omega < \omega_u, \quad \omega_u = 0(1) \tag{36}$$

In other words, the range of the frequency parameter we are concerned with is specified by

$$\omega = 0(1) \quad \text{at most} \tag{37}$$

5. FIRST APPROXIMATION

A first approximation to the free vibration equation, Eq. (20), and the supplemental equations, Eqs. (18) and (21), may be established by neglecting higher order terms in the order-of-magnitude comparison consistent with the fundamental assumptions presented in the preceding section.

First, the coefficients of the exact form of these equations given in Appendix A are written out explicitly in terms of ν, δ, E_i and D_i, and they are arranged in the form of polynomials in δ. As a first approximation to these coefficients, only the leading terms in the polynomial expressions may be retained neglecting the higher order terms in δ in accordance with Eq. (27); i.e., $\delta << 1$. The leading terms of these coefficients are designated by the superscript 0 such as a_{ij}^0 for a_{ij}, and their explicit expressions are given in Appendix B.

Second, some of the terms in the exact equations drop as a result of an order-of-magnitude comparison in accordance with Eqs. (30) and (33); i.e., $\delta n^2 \ll 1$ and $\delta\,|\,p_1^2\,| \ll 1$. For instance, $\delta w''''''$ and $\delta w''''^{\cdot\cdot}$ as well as $\delta w''''$ may be omitted in comparison with w''''. Furthermore, Eq. (37) for ω allows us to neglect terms such as $\delta w^{\sharp\sharp}$ in comparison with w^{**}.

As a result of these approximations, the free vibration equation reduces to

$$\delta(\nabla^8 w + a^0_{63} w''^{::} + 2w^{:::} + a^0_{42} w''^{\cdot\cdot} + w^{::}) + (1-\nu^2) w''''$$
$$+\{[\nabla^4 w]_f - [(3+2\nu)w'' + w^{\cdot\cdot}]_p\}^{**}$$
$$-\{[\frac{3-\nu}{1-\nu}\nabla^2 w]_{p,f} - [\frac{2}{1-\nu}w]_p\}^{\sharp\sharp}$$
$$+[\frac{2}{1-\nu}w]^{\sharp\sharp\sharp}_{p,f} = 0 \tag{38}$$

Without loss of generality, terms of order of magnitude of the errors involved in the present approximation may be added to or subtracted from the resulting approximate equations. Thus, it is admissible to write Eq. (38) in the form

$$\nabla^4(\nabla^2+1)^2 w + 4k^4 w'''' + \tilde{a}_{63} w''^{::} + \tilde{a}_{42} w''^{\cdot\cdot}$$
$$+\{[\nabla^4 w]_f - [(3+2\nu)w'' + w^{\cdot\cdot}]_p\}^{**}/\delta$$
$$-\{[\frac{3-\nu}{1-\nu}\nabla^2 w]_{p,f} - [\frac{2}{1-\nu}w]_p\}^{\sharp\sharp}/\delta$$
$$+[\frac{2}{1-\nu}w]^{\sharp\sharp\sharp}_{p,f}/\delta = 0 \tag{39}$$

where

$$\tilde{a}_{63} = a^0_{63} - 6$$
$$\tilde{a}_{42} = a^0_{42} - 2 \tag{40}$$

and $4k^4$ is a geometric parameter defined by

$$4k^4 = 12(1-\nu^2)(R/h)^2$$
$$= (1-\nu^2)/\delta \tag{41}$$

The supplemental equations reduce to

$$L_0 u = -\nu w''' + w'^{\cdot\cdot} + [\frac{2\nu}{1-\nu}w']^{**}_p$$
$$L_0 v = -(2+\nu)w''^{\cdot} - w^{\cdot\cdot\cdot} + [\frac{2}{1-\nu}w^{\cdot}]^{**}_p$$
$$L_0 N = (1-\nu^2)w''^{\cdot\cdot} + \delta(n^0_3 w^{:::} + n^0_5 w^{::}) - [\nu\nabla^2 w]^{**}_p + [\frac{2\nu}{1-\nu}w]^{\sharp\sharp}_p$$
$$L_0 M = -\nabla^4(w'' + \nu w^{\cdot\cdot}) + m^0_5 w'''' + m^0_6 w''^{\cdot\cdot} - \nu w^{::}$$
$$+[\frac{3-\nu}{1-\nu}\nabla^2(w'' + \nu w^{\cdot\cdot}) + m^0_{11} w'' + \frac{2}{1-\nu}w^{\cdot\cdot}]^{**}_p \tag{42}$$
$$-\frac{2}{1-\nu}[w'' + \nu w^{\cdot\cdot}]^{**}_p$$
$$L_0 S = -\nabla^4[w''' + (2-\nu)w'^{\cdot\cdot}] + s^0_5 w''''' + s^0_6 w'''^{\cdot\cdot} + s^0_7 w'^{::}$$
$$+\{[\nabla^4 w']_r + [\frac{3-\nu}{1-\nu}\nabla^4 w' + (3-\nu)\nabla^2 w'^{\cdot\cdot} + s^0_{11} w'''$$
$$+s^0_{12} w'^{\cdot\cdot}]_p\}^{**} - \{[\frac{3-\nu}{1-\nu}\nabla^2 w']_{p,r} + \frac{1}{1-\nu}[2\nabla^2 w'$$
$$-(1-\nu)w'^{\cdot\cdot}]_p\}^{\sharp\sharp} + [\frac{2}{1-\nu}w']^{\sharp\sharp\sharp}_{p,r}$$
$$L_0 Q = -(1-\nu^2)w'''^{\cdot} + \delta(q^0_3 w'^{::} + q^0_5 w'^{\cdot\cdot\cdot})$$
$$+[(1+\nu)w'^{\cdot}]^{**}_p$$

where

$$L_0(\;) = \nabla^4(\;) - [\frac{3-\nu}{1-\nu}\nabla^2(\;)]^{**}_p + [\frac{2}{1-\nu}(\;)]^{\sharp\sharp}_p \tag{43}$$

The supplemental equations, Eqs. (42), can be simplified further if an appropriate use is made of the free vibration equation, Eq. (38) or (39). Take for instance the first of Eqs. (42) and apply it with $\delta\nabla^8(\;)$ to have

$$\delta\nabla^8 L_0 u = -\nu(\delta\nabla^8 w)''' + (\delta\nabla^8 w)'^{\cdot\cdot}$$
$$+\underline{\frac{2\nu}{1-\nu}\delta[\nabla^8 w']^{**}_p} \tag{44}$$

Then, $\delta\nabla^8 w$ in the first and second terms of the right-hand members of Eq. (44) is substituted from Eq. (38). The result is

$$\delta\nabla^8 L_0 u = \nu[\delta(a^0_{63} w'''''^{::} + 2w'''^{:::} + a^0_{42} w'''''^{\cdot\cdot} + w'''^{::})$$
$$+(1-\nu^2)w'''''''] - [\delta(a^0_{63} w'''^{:::} + 2w'^{::::}$$
$$+a^0_{42} w'''^{::} + w'^{:::}) + (1-\nu^2)w''''^{\cdot\cdot}]$$
$$+\nu\{[\nabla^4 w''']_f - [(3+2\nu)w''''' + w'''^{\cdot\cdot}]_p\}^{**}$$
$$-\{[\nabla^4 w'^{\cdot\cdot}]_f - [(3+2\nu)w'''^{\cdot\cdot} + w'^{::}]_p\}^{**}$$
$$-\nu\{[\frac{3-\nu}{1-\nu}\nabla^2 w''']_{p,f} - [\frac{2}{1-\nu}w''']_p\}^{\sharp\sharp}$$
$$+\{[\frac{3-\nu}{1-\nu}\nabla^2 w'^{\cdot\cdot}]_{p,f} - [\frac{2}{1-\nu}w'^{\cdot\cdot}]_p\}^{\sharp\sharp}$$
$$+[\frac{2\nu}{1-\nu}w''']^{\sharp\sharp\sharp}_{p,f} - [\frac{2}{1-\nu}w'^{\cdot\cdot}]^{\sharp\sharp\sharp}_{p,f}$$
$$+\underline{\frac{2\nu}{1-\nu}\delta[\nabla^8 w']^{**}_p} \tag{45}$$

The term with $\delta[\nabla^8 w']^{**}_p$ is negligible in comparison with the terms with $[\nabla^4 w''']^{**}_f$ and $[\nabla^4 w'^{\cdot\cdot}]^{**}_f$ in the right-hand members of Eq. (45). Thus, the term in the right-hand members of Eq. (44) designated by underline is negligible, and the dynamic term due to the inplane inertia in the supplemental equation for u drops. In a similar manner, some of the terms in the right-hand members of Eqs. (42) may be neglected. As a result, the supplemental equations are now simplified to

$$
\begin{aligned}
L_0u &= -\nu w''' + w'^{\cdot\cdot} \\
L_0v &= -(2+\nu)w''^{\cdot} - w^{\cdot\cdot\cdot} \\
L_0N &= (1-\nu^2)w''^{\cdot\cdot} + \delta(n_3^0 w^{\vdots\vdots\vdots} + n_5^0 w^{::}) - [\nu\nabla^2 w]_p^{**} \\
&\quad + \left[\frac{2\nu}{1-\nu}w\right]_p^{\ddagger\ddagger} \\
L_0M &= -\nabla^4(w'' + \nu w^{\cdot\cdot}) + m_6^0 w''^{\cdot\cdot} - \nu w^{::} \\
L_0S &= -\nabla^4[w''' + (2-\nu)w'^{\cdot\cdot}] + s_6^0 w'''^{\cdot\cdot} + s_7^0 w'^{::} \\
L_0Q &= -(1-\nu^2)w'''^{\cdot} + \delta(q_3^0 w'^{\vdots\vdots\vdots} + q_5^0 w'^{\cdot\cdot\cdot}) \\
&\quad + [(1+\nu)w'^{\cdot}]_p^{**}
\end{aligned}
\tag{46}
$$

It should be noted here that none of Eqs. (38), (39) and (46) contains the terms designated by $[\quad]_r$. This indicates that the rotatory inertia can always be neglected within the accuracy of the first approximation shell theory.

The coefficients a_{ij}^0, $\tilde{a}_{ij}$, n_i^0, m_i^0, s_i^0, and q_i^0 in Eqs. (38), (39) and (46) are quantities of order of magnitude unity, whose specific values differ with different theories as indicated in Appendix B. It is readily recognized that these coefficients are present only accompanied with the static terms; those terms with spatial derivatives only. Since the static equations are to be deduced from the dynamic equations as a limiting case of $\omega \to 0$, the static terms involved in Eqs. (38), (39) and (46) must coincide with those in the static equations derived through a purely static analysis. It has been shown in the first author's earlier paper[31] that those terms with these coefficients are always negligible in the static equations. Instead of repeating the same tedious work for a rigorous proof as that presented in the earlier paper, the following reasonings may be provided to ensure the omission of these terms: The terms with these coefficients as well as the terms with $[w]_p^{**}$ and $[w'^{\cdot}]_p^{**}$ in Eqs. (46) will become important only in a special case where p_i take very small values while n takes relatively large values. The state of stress and deformation in this case is nearly the same as that of the unsymmetric cylindrical mode, or in other words, the unsymmetric in-plane mode of a ring, which is nearly independent of y. Thus, both N and Q are nearly identically equal to zero. Also, the terms with a_{ij}^0, $\tilde{a}_{ij}^0$, m_i^0, s_i^0 are negligible in comparison with other terms involving higher order derivatives with respect to θ in the respective equations.

Finally, the free vibration equation and the supplemental equations are simplified to yield the following system of equations: Free vibration equation;

$$
\begin{aligned}
&\nabla^4(\nabla^2+1)^2 w + 4k^4 w'''' \\
&+\left\{[\nabla^4 w]_f - [(3+2\nu)w'' + w^{\cdot\cdot}]_p\right\}^{**}/\delta \\
&-\left\{\left[\frac{3-\nu}{1-\nu}\nabla^2 w\right]_{p,f} - \left[\frac{2}{1-\nu}w\right]_p\right\}^{\ddagger\ddagger}/\delta \\
&+\left[\frac{2}{1-\nu}w\right]_{p,f}^{\ddagger\ddagger\ddagger}/\delta = 0
\end{aligned}
\tag{47}
$$

Supplemental equations;

$$
\begin{aligned}
L_0u &= -\nu w''' + w'^{\cdot\cdot} \\
L_0v &= -(2+\nu)w''^{\cdot} - w^{\cdot\cdot\cdot} \\
L_0N &= (1-\nu^2)w''^{\cdot\cdot} - [\nu\nabla^2 w]_p^{**} \\
L_0M &= -\nabla^4(w'' + \nu w^{\cdot\cdot}) - \nu w^{::} \\
L_0S &= -\nabla^4[w''' + (2-\nu)w'^{\cdot\cdot}] \\
L_0Q &= -(1-v^2)w'''^{\cdot}
\end{aligned}
\tag{48}
$$

The system of Eqs. (47) and (48) constitute a first approximation to the governing equations for the free vibration of the circumferentially closed circular cylindrical shells. It may now be stated, since none of those coefficients which are affected by the differences of the theories is present in the governing equations, that the use of any of those classical shell theories will yield valid solutions accurate enough within the errors inherent to the Kirchhoff-Love hypothesis.

6. SOLUTION FORMS OF p_i

Substitution of w from Eq. (22) to Eq. (47) yields the auxiliary equation for the present eigenvalue problems. If we write for simplicity

$$P = p_i^2 \tag{49}$$

and

$$\Omega = \omega^2 \tag{50}$$

the auxiliary equation reads

$$F(P) = A_0 P^4 + A_1 P^3 + A_2 P^2 + A_3 P + A_4 = 0 \tag{51}$$

where

$$\begin{aligned}
A_0 &= \delta \\
A_1 &= -2\delta(2n^2-1) \\
A_2 &= -\Omega+1-\nu^2+\delta(6n^4-6n^2+1) \\
A_3 &= -\frac{3-\nu}{1-\nu}\Omega^2+(2n^2+3+2\nu)\Omega-2\delta n^2(2n^4-3n^2+1) \\
A_4 &= -\frac{2}{1-\nu}\Omega^3+\left(\frac{3-\nu}{1-\nu}n^2+\frac{2}{1-\nu}\right)\Omega^2 \\
&\quad -n^2(n^2+1)\Omega+\delta n^4(n^4-2n^2+1)
\end{aligned} \tag{52}$$

Let us examine the functional behavior of F(P) to get a good estimate of the solution forms of p_i. The second derivative of F(P) is calculated as

$$d^2F/dP^2 = 12A_0[(P+A_1/4A_0)^2 + (A_2-3A_1^2/8A_0)/6A_0] \tag{53}$$

After the substitution of A_0, A_1 and A_2 from Eqs. (52), the second term in the square brackets on the right-hand side of Eq. (53) reads

$$(A_2-3A_1^2/8A_0)/6A_0 = (1-\nu^2-\delta/2-\Omega)/6\delta \tag{54}$$

Thus, within the accuracy of the present approximation, we have

$$d^2F/dP^2 > 0 \tag{55}$$

for

$$\Omega < 1-\nu^2 \tag{56}$$

The upper bound of Ω is very close to

$$\Omega = 1 \tag{57}$$

which corresponds to the natural frequency of the circular cylindrical shell in an axisymmetric cylindrical mode, or of a ring in an axisymmetric inplane mode. As a matter of fact, if we set $(\ )' = 0$ and $(\ )^{\cdot} = 0$ in Eq. (47), we have

$$[w]_p^{**} + [w]_{p,f}^{***} = 0 \tag{58}$$

This readily gives $\omega^2 = 1$ as a nontrivial solution.

Since the vibration in the axisymmetric cylindrical mode induces high inplane stretching, it occurs only at a very high frequency. For most of the problems of practical importance, therefore, we may restrict our attention to the frequency spectrum specified by Eq. (56).

The functional behavior of F(P) is such that it monotonically decreases as P increases from $-\infty$ to a point where it takes its minimal value and then it increases as P exceeds that point. If $A_4 < 0$, F(P) = 0 has one positive and one negative real roots. If $A_4 > 0$, on the other hand, it has either no real root or only two negative or positive real roots depending on the signs of the minimal value of $F(P_m)$ and of P_m at which F(P) becomes minimal. The sign of P_m depends on that of A_3.

Let us now examine the behavior of A_4 as a function of Ω. Let the three roots of $A_4(\Omega) = 0$ be designated by Ω_1, Ω_2 and Ω_3. Then, the following relations hold:

$$\begin{aligned}
\Omega_1+\Omega_2+\Omega_3 &= (3-\nu)n^2/2+1 \\
\Omega_1\Omega_2+\Omega_2\Omega_3+\Omega_3\Omega_1 &= (1-\nu)n^2(n^2+1)/2 \\
\Omega_1\Omega_2\Omega_3 &= (1-\nu)\delta n^2(n^2-1)^2/2
\end{aligned} \tag{59}$$

These indicate that one of the roots, say Ω_1, must be of order of magnitude δn^4 and the remaining ones, Ω_2 and Ω_3, of order of magnitude n^2. Thus, the first two of Eqs. (59) reduce to

$$\begin{aligned}
\Omega_2+\Omega_3 &= (3-\nu)n^2/2+1 \\
\Omega_2\Omega_3 &= (1-\nu)n^2(n^2+1)/2
\end{aligned} \tag{60}$$

The roots Ω_2 and Ω_3 are readily obtained from Eqs. (60). These are subsequently used to determine Ω_1 from the third of Eqs. (59). The result is

$$
\begin{aligned}
\Omega_1 &= \delta n^2 (n^2-1)^2/(n^2+1) \\
\Omega_2 &= (1-\nu)\, n^2/2 \\
\Omega_3 &= n^2+1
\end{aligned}
\qquad (61)
$$

For n = 0, Eqs. (61) give

$$\Omega_1 = \Omega_2 = 0, \qquad \Omega_3 = 1 \qquad (62)$$

The value of $A_4(\Omega)$ is positive within the range of the frequency parameter specified in Eq. (56). We now have

$$A_4 > 0 \qquad \text{for } n = 0 \qquad (63)$$

For n = 1, Eqs. (61) give

$$\Omega_1 = 0, \quad \Omega_2 = (1-\nu)/2, \quad \Omega_3 = 2 \qquad (64)$$

Since

$$(1-\nu)/2 < 1-\nu^2 \qquad (65)$$

we have

$$\left.\begin{aligned} &A_4 > 0\,;\ (1-\nu)/2 < \Omega < 1-\nu^2 \\ &A_4 < 0\,;\ 0 < \Omega < (1-\nu)/2 \end{aligned}\right\} \text{ for } n = 1 \qquad (66)$$

As noted before, in the cases where $A_4 > 0$, there exist either no real root or only two negative or positive real roots of P for F(P) = 0. This means that all the roots of p_i are either real, purely imaginary or complex. This form of solutions is completely different from that in the cases of $n \geq 2$ where it will be proved subsequently that $A_4 < 0$ holds. These two different cases, therefore, has to be dealt with separately. In the present paper, we shall restrict our attention to the cases of $n \geq 2$ leaving detailed analyses on the cases of n = 0 and 1 to separate papers to follow.

For $n \geq 2$, we have

$$1-\nu^2 < \Omega_2 < \Omega_3 \qquad (67)$$

so that

$$\left.\begin{aligned} &A_4 < 0\,;\ \Omega_1 < \Omega < 1-\nu^2 \\ &A_4 > 0\,;\ 0 < \Omega < \Omega_1 \end{aligned}\right\} \text{ for } n \geq 2 \qquad (68)$$

It can be shown easily that Ω_1 corresponds to the lowest natural frequency of the flexural vibration of the cylindrical shell in the cylindrical mode, or of the flexural in-plane vibration of a ring. As a matter of fact, if we set $(\ )' \equiv 0$ in Eq. (47), the free-vibration equation for the cylindrical mode reads

$$
\begin{aligned}
&[(\ )^{\cdot\cdot}+1]^2 w^{\cdot\cdot\cdot\cdot} + \{[w^{\cdot\cdot\cdot\cdot}]_f - [w^{\cdot\cdot}]_p\}^{**}/\delta \\
&-\{[\tfrac{3-\nu}{1-\nu} w^{\cdot\cdot}]_{p,f} - [\tfrac{2}{1-\nu} w]_p\}^{\#}/\delta \\
&+ [\tfrac{2}{1-\nu} w]_{p,f}^{\#\#}/\delta = 0
\end{aligned}
\qquad (69)
$$

For the lowest natural frequency, we may assume $\omega^2 \ll 1$ to reduce Eq. (69) to the form

$$[(\ )^{\cdot\cdot}+1]^2 w^{\cdot\cdot\cdot\cdot} + \{[w^{\cdot\cdot\cdot\cdot}]_f - [w^{\cdot\cdot}]_p\}^{**}/\delta = 0 \qquad (70)$$

This immediately gives

$$\omega^2 = \delta n^2 (n^2-1)^2/(n^2+1) \qquad (71)$$

which coincides with Ω_1 given in Eqs. (61).

The flexural vibration in the cylindrical mode induces little inplane stretching. It is therefore well anticipated that Ω_1 is the lowest possible of all the natural frequencies of the circular cylindrical shell for $n \geq 2$. We may now state, therefore, that the entire range of the frequency parameter we are concerned with is covered by

$$\delta n^2 (n^2-1)^2/(n^2+1) < \Omega < 1-\nu^2 \qquad (72)$$

and that we always have

$$A_4 < 0 \qquad \text{for} \quad n \geq 2 \qquad (73)$$

It now becomes clear that the auxiliary equation, Eq. (51), has one positive and one negative real roots and a pair of complex conjugate roots of P. Thus, the eight eigenvalues p_i take the form

$$
\begin{aligned}
&p_1, p_2 = \pm \xi_1, \qquad p_3, p_4 = \pm i\, \eta_1 \\
&p_5, p_6, p_7, p_8 = \pm (\xi_2 \pm i\, \eta_2)
\end{aligned}
\qquad (74)
$$

where ξ_1, η_1, ξ_2 and η_2 are all real, and is unit of imaginary number; $i = (-1)^{1/2}$.

If the use is made of Eqs. (74), the auxiliary equation, Eq. (51), can be written in the form

$$
\begin{aligned}
&\delta(P-\xi_1^2)(P+\eta_1^2)[P-(\xi_2^2-\eta_2^2)+i\,2\xi_2\eta_2] \\
&[P-(\xi_2^2-\eta_2^2)-i\,2\xi_2\eta_2] = 0
\end{aligned}
\qquad (75)
$$

Rewriting the left-hand side of Eq. (75) in the form of a quartic equation in P and comparing its coefficients with those of Eq. (51), we have

$$
\begin{aligned}
&(\xi_1^2-\eta_1^2)+2(\xi_2^2-\eta_2^2)=2(2n^2-1)\\
&\delta[2(\xi_1^2-\eta_1^2)(\xi_2^2-\eta_2^2)-\xi_1^2\eta_1^2+(\xi_2^2+\eta_2^2)^2]\\
&\quad=1-\nu^2-\Omega+\delta(6n^4-6n^2+1)\\
&\delta[2\xi_1^2\eta_1^2(\xi_2^2-\eta_2^2)-(\xi_1^2-\eta_1^2)(\xi_2^2+\eta_2^2)^2]\\
&\quad=-\frac{3-\nu}{1-\nu}\Omega^2+(2n^2+3+2\nu)\Omega-\delta 2n^2(2n^4-3n^2+1)\\
&\delta\xi_1^2\eta_1^2(\xi_2^2+\eta_2^2)^2=\frac{2}{1-\nu}\Omega^3-\left(\frac{3-\nu}{1-\nu}n^2+\frac{2}{1-\nu}\right)\Omega^2\\
&\qquad+n^2(n^2+1)\Omega-\delta n^4(n^4-2n^2+1)
\end{aligned}
\tag{76}
$$

Let us temporarily assume that the frequency is so small that

$$\Omega \ll 1 \tag{78}$$

Then, an examination of Eqs. (76) reveals that ξ_1^2, η_1^2, ξ_2^2 and η_2^2 are given in the form

$$
\begin{aligned}
\xi_1^2&=\delta^{1/2}n^4q_0^2+\delta n^3\tilde{\xi}_1\\
\eta_1^2&=\delta^{1/2}n^4q_0^2+\delta n^3\tilde{\eta}_1\\
\xi_2^2&=\delta^{-1/2}p_0^2+\tilde{\xi}_2\\
\eta_2^2&=\delta^{-1/2}p_0^2+\tilde{\eta}_2
\end{aligned}
\tag{79}
$$

where q_0, p_0, $\tilde{\xi}_1$, $\tilde{\eta}_1$, $\tilde{\xi}_2$ and $\tilde{\eta}_2$ are quantities of order of magnitude unity. These indicate that ξ_1 and η_1 are of order of magnitide n^2/k, whereas ξ_2 and η_2 of order of magnitude k, so that

$$
|p_i|\begin{cases}=0(n^2/k)\ ; & i=1,2,3,4\\ =0(k)\ ; & i=5,6,7,8\end{cases}
\tag{80}
$$

Since k is proportional to $(R/h)^{1/2}$ which takes large values for thin shells, Eqs. (80) indicate that the eigenvalues p_1, p_2, p_3 and p_4 represent the global solutions, whereas p_5, p_6, p_7 and p_8 the edge-zone solutions. Thus, the existence of the edge-zone solutions similar to those in the static deformation problems has been proved.

It is of practical interest to investigate an extreme case where the natural frequencies assume low values in the vicinity of the lower bound of the frequency spectrum specified in Eqs. (72) and the thickness-to-radius ratio takes very small values, such that

$$
\begin{aligned}
&\Omega=0(\delta n^4)\\
&\delta^{1/2}n^2 \ll 1
\end{aligned}
\tag{81}
$$

Accordingly, we have from Eqs. (79)

$$
\begin{aligned}
\xi_1^2&=\eta_1^2=\delta^{1/2}n^4q_0^2\\
\xi_2^2&=\eta_2^2=\delta^{-1/2}p_0^2
\end{aligned}
\tag{82}
$$

Also, from the second of Eqs. (76), we have

$$p_0^2=(1-\nu^2)/4 \tag{83}$$

These lead to

$$\xi_2=\eta_2=k \tag{84}$$

and

$$p_5,\ p_6,\ p_7,\ p_8=\pm k(1\pm i) \tag{85}$$

which coincide with those obtained in the static analysis in Ref.[31].

The values of q_0 remains as yet undetermined. If we use the notation ξ_1 instead of q_0, the fourth of Eqs. (76) leads to a simple formula for calculating Ω;

$$\Omega=[(1-\nu^2)\xi_1^4+\delta n^4(n^2-1)^2]/n^2(n^2+1) \tag{86}$$

Once ξ_1 is known, Ω is calculated easily with the aid of Eq. (86). This simple formula is identical in form to that derived by Nau and Simmonds[15] by the asymptotic method.

7. BOUNDARY CONDITIONS

The coefficients U_i, B_i, N_i, M_i, S_i, V_i and Q_i in Eqs. (24) are expressed only in terms of W_i with the aid of the supplemental equations, Eqs. (48), and the equation defining β, Eq. (8). As a result, we have

$$
\begin{aligned}
U_i&=-p_i(\nu p_i^2+n^2)W_i/\mathscr{L}_i\\
B_i&=p_iW_i\\
N_i&=-[(1-\nu^2)p_i^2n^2+\nu(p_i^2-n^2)\omega^2]W_i/\mathscr{L}_i\\
M_i&=-[(p_i^2-n^2)^2(p_i^2-\nu n^2)+\nu n^4]W_i/\mathscr{L}_i\\
S_i&=-p_i(p_i^2-n^2)^2[p_i^2-(2-\nu)n^2]W_i/\mathscr{L}_i\\
V_i&=n[(2+\nu)p_i^2-n^2]W_i/\mathscr{L}_i\\
Q_i&=(1-\nu^2)np_i^3W_i/\mathscr{L}_i
\end{aligned}
\tag{87}
$$

where

$$\mathscr{L}_i=(p_i^2-n^2)^2+\frac{3-\nu}{1-\nu}(p_i^2-n^2)\omega^2+\frac{2}{1-\nu}\omega^4 \tag{88}$$

Assuming that the homogeneous boundary conditions hold independent of θ and T, we have the following four pairs of the boundary constraining equations at $y = +\ell$:

$$\begin{cases} w=0;\ \sum_{i=1}^{8}\mathscr{L}_i e^{+p_i\ell}W_i=0 \\ S=0;\ \sum_{i=1}^{8}p_i(p_i^2-n^2)^2[p_i^2-(2-\nu)n^2]e^{+p_i\ell}W_i=0 \end{cases} \tag{89a}$$

$$\begin{cases} \beta=0;\ \sum_{i=1}^{8}\mathscr{L}_i p_i\ e^{+p_i\ell}W_i=0 \\ M=0;\ \sum_{i=1}^{8}[(p_i^2-n^2)^2(p_i^2-\nu n^2)+\nu n^4]e^{+p_i\ell}W_i=0 \end{cases} \tag{89b}$$

$$\begin{cases} u=0;\ \sum_{i=1}^{8}p_i(\nu p_i^2+n^2)e^{+p_i\ell}W_i=0 \\ N=0;\ \sum_{i=1}^{8}[(1-\nu^2)p_i^2n^2+\nu(p_i^2-n^2)\omega^2] \\ \quad e^{+p_i\ell}W_i=0 \end{cases} \tag{89c}$$

$$\begin{cases} v=0;\ \sum_{i=1}^{8}[(2+\nu)p_i^2-n^2]e^{+p_i\ell}W_i=0 \\ Q=0;\ \sum_{i=1}^{8}p_i^3\ e^{+p_i\ell}W_i=0 \end{cases} \tag{89d}$$

A set of additional four pairs of the boundary constraining equations at $y = -\ell$ is obtained by replacing $+\ell$ in Eqs. (89) by $-\ell$.

Here, the denominators $\mathscr{L}_i$ in Eqs. (87) have been multiplied on all the corresponding i-th members in the boundary constraining equations, so that the members in $w = 0$ and $\beta = 0$ contain $\mathscr{L}_i$ as multipliers, whereas the denominators in the remaining equations drop. This manipulation is admissible because we are concerned with the solutions of the homogeneous linear systems with eight equations with eight unknown W_i.

It may be more convenient to write Eqs. (89) in terms of real variables instead of the complex variables p_i and W_i. If use is made of Eqs. (74), the right-hand members of Eq. (22) can be written only in terms of the real variables ξ_1, η_1, ξ_2 and η_2, such that

$$\begin{aligned} w=(&\widetilde{W}_1e^{\xi_1y}+\widetilde{W}_2e^{-\xi_1y}+\widetilde{W}_3\cos\eta_1y+\widetilde{W}_4\sin\eta_1y \\ &+\widetilde{W}_5e^{\xi_2y}\cos\eta_2y+\widetilde{W}_6e^{\xi_2y}\sin\eta_2y \\ &+\widetilde{W}_7e^{-\xi_2y}\cos\eta_2y+\widetilde{W}_8e^{-\xi_2y}\sin\eta_2y)\cos n\theta\ \sin\omega T \end{aligned} \tag{90}$$

where the coefficients $\widetilde{W}_i$ are real and related with W_i by

$$\begin{gathered} \widetilde{W}_1=W_1,\ \widetilde{W}_2=W_2,\ \widetilde{W}_3=W_3+W_4,\ \widetilde{W}_4=i(W_3-W_4) \\ \widetilde{W}_5=W_5+W_6,\ \widetilde{W}_6=i(W_5-W_6),\ \widetilde{W}_7=W_7+W_8, \\ \widetilde{W}_8=-i(W_7-W_8) \end{gathered} \tag{91}$$

Similarly, the boundary constraining equations, Eqs. (89), can be rewritten in terms of the real variables, such that

$$\begin{cases} w=0;\quad \sum_{i=1}^{8}\mathscr{W}_i\widetilde{W}_i=0 \\ S=0;\quad \sum_{i=1}^{8}\mathscr{S}_i\widetilde{W}_i=0 \end{cases} \tag{92a}$$

$$\begin{cases} \beta=0;\quad \sum_{i=1}^{8}\mathscr{B}_i\widetilde{W}_i=0 \\ M=0;\quad \sum_{i=1}^{8}\mathscr{M}_i\widetilde{W}_i=0 \end{cases} \tag{92b}$$

$$\begin{cases} u=0;\quad \sum_{i=1}^{8}\mathscr{U}_i\widetilde{W}_i=0 \\ N=0;\quad \sum_{i=1}^{8}\mathscr{N}_i\widetilde{W}_i=0 \end{cases} \tag{92c}$$

$$\begin{cases} v=0;\quad \sum_{i=1}^{8}\mathscr{V}_i\widetilde{W}_i=0 \\ Q=0;\quad \sum_{i=1}^{8}\mathscr{O}_i\widetilde{W}_i=0 \end{cases} \tag{92d}$$

The explicit expressions of the real coefficients $\mathscr{W}_i, \mathscr{S}_i, \mathscr{B}_i, \mathscr{M}_i, \mathscr{U}_i, \mathscr{N}_i, \mathscr{V}_i$ and $\mathscr{O}_i$ are given in Appendix C. These expressions have been determined by writing the left-hand members of Eqs. (89) in terms of ξ_1, η_1, ξ_2, η_2 and $\widetilde{W}_i$ and rearranging the resulting equations in such a way that the multiplying factors upon $e^{\pm\xi_2\ell}$ in the coefficients associated with the edge-zone solutions become of order of magnitude unity in the case where Ω takes very small values in comparison with unity. Since it has been shown in Eqs. (74) that ξ_1 and η_1 are of order of magnitude (n^2/k), whereas ξ_2 and

η_2 of order of magnitude k, in that particular case, some of those coefficients associated with the global solutions are very small divided by large quantities of higher order in k, so that the boundary constraining equations containing these small coefficients may play only an insignificant role in the calculations of the characteristic equations. It is also due to these small coefficients that we encounter difficulties in numerical computations resulting in numerical overflow.

These will become more evident if we consider such an extreme case as specified by Eqs. (81). An approximation consistent with Eqs. (81) with due consideration of Eqs. (82) and (84) drastically simplifies the expressions of the coefficients of the boundary constraining equations. The result is given in Appendix D. This makes it much easier to calculate the characteristic determinants. Let us examine, for instance, a case where the boundary conditions are prescribed at both edges by

$$w = M = N = v = 0\ ;\quad y = \pm\ell \tag{93}$$

The boundary constraining equations read in matrix form

$$\{C\}\{\widetilde{W}_i\} = 0 \tag{94}$$

where

$$\{\widetilde{W}_i\} = \{e^{\pm\xi_1\ell}\widetilde{W}_1,\ e^{\mp\xi_1\ell}\widetilde{W}_2,\ \widetilde{W}_3,\ \widetilde{W}_4,\ e^{\pm k\ell}\widetilde{W}_5,\ e^{\pm k\ell}\widetilde{W}_6,\ e^{\mp k\ell}\widetilde{W}_7,\ e^{\mp k\ell}\widetilde{W}_8\}^T \tag{95}$$

$$\{C\} = \begin{bmatrix} -\dfrac{n^4}{4k^4}(1,1,\ \cos\xi_1\ell,\ \pm\sin\xi_1\ell), \\ \qquad (\cos k\ell,\ \pm\sin k\ell,\ \cos k\ell,\ \pm\sin k\ell) \\ \dfrac{\nu n^6}{8k^6}(1,1,\ \cos\xi_1\ell,\ \pm\sin\xi_1\ell), \\ \qquad (\mp\sin k\ell,\ \cos k\ell,\ \pm\sin k\ell,\ -\cos k\ell) \\ \dfrac{n^2}{2k^2}\left(\dfrac{\xi_1}{n}\right)^2(1,1,\ -\cos\xi_1\ell,\ \mp\sin\xi_1\ell), \\ \qquad (\mp\sin k\ell,\ \cos k\ell,\ \pm\sin k\ell,\ -\cos k\ell) \\ -\dfrac{n^2}{2(2+\nu)k^2}(1,1,\ \cos\xi_1\ell,\pm\sin\xi_1\ell), \\ \qquad (\mp\sin k\ell,\ \cos k\ell,\ \pm\sin k\ell,\ -\cos k\ell) \end{bmatrix} \tag{96}$$

Here, $\{\widetilde{W}_i\}^T$ indicates transpose of $\{\widetilde{W}_i\}$. The + and − signs should be assigned depending on the edges $+\ell$ and $-\ell$, so that each row of the matrix $\{C\}$ actually consists of two for the $+\ell$ and $-\ell$ edges. The matrices on the third and fourth rows on the fifth through eighth columns of $\{C\}$ cancel out by subtracting from them the corresponding matrices on the second row. Since (ξ_1/n) is of order of magnitude (n/k), the smallest factor involved in $\{C\}$ is $n^6/8k^6$ on the first four matrices on the second row. These small matrices are negligible in comparison with the corresponding ones on the third and fourth rows when the formers are subtracted from the latters. Consequently, within the accuracy of the present approximation, the matrix $\{C\}$ can be reduced to

$$\{C\} = \begin{bmatrix} -\dfrac{n^4}{4k^4}(1,1,\ \cos\xi_1\ell,\ \pm\sin\xi_1\ell), \\ \qquad (\cos k\ell,\ \pm\sin k\ell,\ \cos k\ell,\ \pm\sin k\ell) \\ \dfrac{\nu n^6}{8k^6}(1,1,\ \cos\xi_1\ell,\ \pm\sin\xi_1\ell), \\ \qquad (\mp\sin k\ell,\ \cos k\ell,\ \pm\sin k\ell,\ -\cos k\ell) \\ \dfrac{n^2}{2k^2}\left(\dfrac{\xi_1}{n}\right)^2(1,1,-\cos\xi_1\ell,\ \mp\sin\xi_1\ell),(0,0,0,0) \\ -\dfrac{n^2}{2(2+\nu)k^2}(1,1,\cos\xi_1\ell,\ \pm\sin\xi_1\ell),(0,0,0,0) \end{bmatrix} \tag{97}$$

It can be proved easily that the determinant of the coefficients matrix of the matrix equation, Eq. (94), is given only by the non-zero matrices on the third and fourth rows of $\{C\}$. Thus, the characteristic equation is given by

$$\begin{vmatrix} e^{\xi_1\ell}, & e^{-\xi_1\ell}, & -\cos\xi_1\ell, & -\sin\xi_1\ell \\ e^{-\xi_1\ell}, & e^{\xi_1\ell}, & -\cos\xi_1\ell, & \sin\xi_1\ell \\ e^{\xi_1\ell}, & e^{-\xi_1\ell}, & \cos\xi_1\ell, & \sin\xi_1\ell \\ e^{-\xi_1\ell}, & e^{\xi_1\ell}, & \cos\xi_1\ell, & -\sin\xi_1\ell \end{vmatrix} = 0 \tag{98}$$

A simple calculation yields

$$\sin 2\xi_1\ell = 0 \tag{99}$$

which gives

$$\xi_1 = m\pi/2\ell \tag{100}$$

The natural frequencies can easily be calculated with the aid of Eq. (86) substituting for ξ_1 from Eq. (100). It is well anticipated from the assumptions underlying these equations that the frequencies thus calculated are accurate only for small values of m and large values of ℓ and n. For a given vale of ℓ, therefore, m must be as small as possible, that is m = 1, to get a good approximation. The characteristic equation (99) is identical in form to that of a beam simply supported at both edges. This indicates that the axial modal characteristics of a circular cylindrical shell may be determined by supposing a beam along the generator.

8. NUMERICAL CALCULATIONS

Numerical calculations are performed to compare numerically the solutions of the exact equations of various theories and those of the approximate equations and to check the results of the preceding analysis.

Numerical calculations of the solutions of the eigenvalue problems constituted by the characteristic equation and the auxiliary equation may be performed by a number of different ways including the use of a library computer code. Here, we shall present a brief account of our method of calculations which makes extensive use of the developments in the preceding sections. Among other things, the eigenvalues p_i are given in the form of Eqs. (87), so that calculations are performed on real variables only. Thus, the difficulty arising from dealing with complex variables is avoided. For simplicity, the numerical procedures are demonstrated for the system of the approximate equations derived in Section 5. The same procedures can be applied to the exact equations of Section 4. Throughout the present calculation, the value of ν is assumed as

$$\nu = 0.3 \tag{101}$$

The calculations proceed as follows: A set of the values of the geometric parameters k and ℓ is given. An integer value of n is assigned. The value of ω is assumed intuitively as an initial guess. The coefficients A_i of the auxiliary equation are calculated with the aid of Eqs. (52).

Then, the auxiliary equation, Eq. (51), is solved for p by Ferrari's method. The eigenvalues p_i are determined taking the square roots of p. It turns out that the p_i thus determined always fall into the form given by Eqs. (74), confirming our theoretical prediction. The values of ξ_1 , η_1 , ξ_2 and η_2 are thus determined. The values of the coefficients of the boundary constraining equations, $\mathscr{W}_i, \mathscr{S}_i, \mathscr{B}_i, \mathscr{M}_i, \mathscr{U}_i, \mathscr{N}_i, \mathscr{V}_i$ and $\mathscr{O}_i$, are calculated by the formulae presented in Appendix C.

We now specify the boundary conditions. For a specified set of the boundary conditions, the determinant of the coefficients matrix of the boundary constraining equations is calculated. If the determinant is zero within a prescribed error, the initial values of ω, ξ_1 , η_1 , ξ_2 and η_2 are registered as the solutions of that eigenvalue problem. If the determinant is not small enough to be regarded zero, we make an appropriate modification on the initial value of ω and follow the same procedure to calculate the determinant using the modified value of ω as a new initial value. The iterative procedure is repeated until the determinant becomes zero with the prescribed accuracy. Thus, the solutions for ω, ξ_1 , η_1 , ξ_2 and η_2 are determined.

The calculations are performed for each integer value of n ranging from 1 to 10 and for various values of ℓ and k. Here, the cases of n = 1 are included in the numerical calculations for the sake of future reference to see if the validity of the form of p_i given in Eqs. (74) is violated as the theoretical prediction indicates its possibility.

Essentially the same procedure is used to determine the solutions for the exact systems of the equations. Of course, the expressions of the coefficients of the auxiliary equations and the boundary constraining equations are much more involved than those of the first approximation equations. But the calculations of their numerical values can be carried out without difficulty by the digital computer if use is made of the formulae presented in Appendix A. In these calculations, however, we must first specify the values of the theory indicators E_i and D_i. Here, they are specified as in Table 1, so that the exact solutions of the Flügge, the Koiter-Sanders, the Novozhilov, and the Love-

Reissner theories are obtained.

Some of the results of the numerical calculations of the natural frequencies ω are presented in Tables 2a through 4c. Tables 2, 3 and 4 are the results for shells with the clamped-clamped, the simply supported-simply supported, and the clamped-free edges, respectively. In these tables, the boundary conditions are designated for simplicity by C1, S1 and FR, such that

C1: clamped edge; $w = \beta = u = v = 0$

S1: simply supported edge;
$w = M = N = v = 0$

FR: free edge; $S = M = N = Q = 0$

Three values have been selected for k; i.e., k = 5, 10 and 50, to cover the entire range of h/R of practical importance. A simple calculation with $\nu = 0.3$ gives

$k = 5$; $h/R = 0.06609$, $R/h = 15.1307$

$k = 10$; $h/R = 0.01653$, $R/h = 60.5228$

$k = 50$; $h/R = 0.00066$, $R/h = 1513.07$

The results are presented for n = 2, 4 and 6, though the calculations were actually performed for integer values ranging from n = 1 through 10.

In these tables, Flügge, Naghdi, K-S, Novozhilov, and L-R indicates the solutions of the exact systems of equations for the Flügge, the Naghdi, the Koiter-Sanders, the Novozhilov, and the Love-Reissner theories, respectively. No significant difference is observed among these solutions. Thus, the numerical results confirm the theoretical predictions about the accuracies of these classical theories.

The results designated by "1st. approx." theory are the solutions of the system of the first approximation equations, Eqs. (47) and (48), derived in the present paper. They are fairly in good agreement with the exact solutions of the classical theories. A general trend in the first approximation solutions is that the agreement with the exact solutions becomes closer as k, n and ℓ increase. The agreement is best in the case of S1-S1 and worst in C1-FR. But, even in the case of D1-FR, the errors relative to the exact solutions of the Flügge theory are less than 4 per cent or the value of h/R, except for the case of $2\ell = 2.0$ and $n = 2$ where the relative errors as much as 15 per cent are observed. It may be stated, therefore, that the present approximation does not yield accurate solutions for short shells less than $2\ell = 2.0$ with free edges vibrating with n = 2. Otherwise, it will provide valid solutions accurate enough for practical purpose.

The solutions of the system of equations of the Donnell-type approximation, Eqs. (107) and (108), are designated as "Donnell" in the tables. They generally confirm the theoretical predictions in that the accuracy is increasingly lowered as n decreases and ℓ increases and as the boundary conditions are prescribed more and more by the physical quantities. Thus, the agreement of the solutions with the exact solutions of the classical theories is best in the case of C1-C1 and worst in C1-FR.

9. DONNELL-TYPE APPROXIMATION

In the Donnell-type approximation, the kinematic relations and the equations of motion are given by Eqs. (1) and (6) neglecting the terms in them designated by underline, and the constitutive equations are specified to those of the Love-Reissner theory. Starting out from these approximate basic equations and following the order-of-magnitude comparison procedures set in Section 5, we derive the following system of equations: Free fibration equation;

$$\nabla^8 w + 4k^4 w'''' + \left\{[\nabla^4 w]_f - [(3+2\nu)w'' + w^{\cdot\cdot}]_p\right\}^{**}/\delta$$
$$-\left\{\left[\frac{3-\nu}{1-\nu}\nabla^2 w\right]_{p,f} - \left[\frac{2}{1-\nu}w\right]_p\right\}^{\#\#}/\delta$$
$$+\left[\frac{2}{1-\nu}w\right]_{p,f}^{\#\#\#}/\delta = 0 \qquad (102)$$

Supplemental equations;

$$L_0 u = -\nu w''' + w'^{\cdot\cdot}$$
$$L_0 u = -(2+\nu) w''^{\cdot} - w^{\cdot\cdot\cdot}$$
$$L_0 N = (1-\nu^2) w''^{\cdot\cdot}$$
$$L_0 M = -\nabla^4 (w'' + \nu w^{\cdot\cdot}) \qquad (103)$$
$$L_0 S = -\nabla^4 [w''' + (2-\nu) w'^{\cdot\cdot}]$$
$$L_0 Q = -(1-\nu^2) w'''^{\cdot}$$

Obviously, the omission of the term with $\partial u_\theta/\partial x$ in τ of Eqs. (1) is a consequence of the omission of the term with $\partial u_\theta/\partial\theta$ in κ_θ. It also becomes clear from the process of derivation of Eqs. (102) and (103) that the omission of Q_θ in Eqs. (6) is consistent with the order-of-magnitude arguments for the omission of $\partial u_\theta/\partial\theta$ in κ_θ. Therefore, if one assumes that the term with $\partial u_\theta/\partial\theta$ is negligible in κ_θ, all the other terms designated by underline in Eqs. (1) and (6) become negligible as a logical consequence of the assumption.

The kinematic relation for κ_θ given in Eqs. (1) is rewritten in nondimensional form as

$$R\kappa_\theta = \underline{v^{\cdot}} - w^{\cdot\cdot} \tag{104}$$

Let us apply $L_0(\ )$ to Eq. (104) and substitute $L_0 v$ from Eqs. (103). Then, we have

$$RL_0\kappa_\theta = \underline{-(2+\nu)w''^{\cdot\cdot} - w^{::}} - \nabla^4 w^{\cdot\cdot} - \frac{3-\nu}{1-\nu}\nabla^2 w^{\cdot\cdot **} - \frac{2}{1-\nu}w^{\cdot\cdot\#} \tag{105}$$

The terms designated by underline in the right-hand members of Eq. (105) are negligible if either $|p_i^2|$ for the global solutions or n^2 is much greater than unity. In other words, the Donnell-type approximation is valid for the vibrations characterized by a large number of waves. It may suffice for our purpose to state that the Donnell-type approximation is a consistent approximation under the assumption that

$$n^2 \gg 1 \tag{106}$$

Accordingly, Eqs. (102) and (103) can be simplified further to yield the free vibration equation and the supplemental equations consistent with Donnell's approximation.

Free vibration equation:

$$\nabla^8 w + 4k^4 w'''' + [\nabla^4 w]_f^{**}/\delta = 0 \tag{107}$$

Supplemental equations:

$$\nabla^4 u = -\nu w''' + w'^{\cdot\cdot}$$

$$\nabla^4 v = -(2+\nu)w''^{\cdot} - w^{\cdot\cdot\cdot}$$

$$\nabla^4 N = (1-\nu^2)w''^{\cdot\cdot}$$

$$\nabla^4 M = -\nabla^4(w'' + \nu w^{\cdot\cdot})$$

$$\nabla^4 S = -\nabla^4[w''' + (2-\nu)w'^{\cdot\cdot}]$$

$$\nabla^4 Q = -(1-\nu^2)w'''^{\cdot} \tag{108}$$

Equation (107) may be regarded as Donnell's equation for free vibrations. The dynamic term in this equation is designated by $[\ \ ]_f$ and is only of the second order in timewise derivative. This indicates that Donnell's equation is valid for flexural vibrations characterized by a large number of waves.

If we write out the auxiliary equation, it takes the same form as Eq. (51), wherein A_i now read

$$\begin{aligned} A_0 &= \delta \\ A_1 &= -\delta 4n^2 \\ A_2 &= 1-\nu^2-\Omega+\delta 6n^4 \\ A_3 &= 2n^2(\Omega-\delta 2n^4) \\ A_4 &= -n^4(\Omega-\delta n^4) \end{aligned} \tag{109}$$

Following the same arguments given in Section 6, we find that $A_4 < 0$ and that p_i take the form of Eqs. (74) if Ω is in the range specified by

$$\delta n^4 < \Omega < 1-\nu^2 \tag{110}$$

Comparing Eq. (110) with Eq. (72), we see that Donnell's equation fails to provide valid solutions in the form of Eqs. (74) if Ω is as low as

$$\delta n^2(n^2-1)^2/(n^2+1) < \Omega < \delta n^4 \tag{111}$$

This indicates that the validity of Donnell's equation becomes deteriorated as the vibration mode comes closer to an unsymmetric cylindrical mode.

As long as Ω remains in the range of Eq. (110), p_i take the form of Eqs. (74) and all the subsequent development through Eqs. (86) in Section 6 hold for the solutions of Donnell's equation, provided that the terms of order of magnitude of $1/n^2$ be neglected in comparison with unity. In particular, if we restrict our attention to the case where Ω takes a very small value, such that Eqs. (81) hold; $\Omega = 0(\delta n^4)$ and $\delta^{1/2}n^2 \ll 1$, we have from Eq. (86)

$$\Omega = \frac{1}{n^4}[(1-\nu^2)\xi_1^4 + \delta n^8] \tag{112}$$

This formula is accurate only when n takes very large values, and its errors increase as n takes on smaller values.

In this particular case, the boundary constraining equations are identical in form to Eqs. (92) with the coefficients given in Appendix D. Suppose we have a correct value of ω for a particular boundary value problem. We then solve Eq. (112) for ξ_1. The value of ξ_1 thus determined is used to calculate the determinant of the coefficients matrix for that particular boundary conditions. The result may not be zero. The magnitude of the residual depends on the errors involved in ξ_1 as well as the boundary conditions. Comparing the small factors n/k and ξ_1/n in the coefficients matrix, we may anticipate it most likely, as in the static boundary value problems, that the errors in the determinants is the greatest when the boundary conditions are prescribed entirely by the physical quantities and the smallest when they are prescribed entirely by the geometric quantities. This is a reasonable guess because we have shown that the validity of Donnell's equation is deteriorated for unsymmetric cylindrical modes which occur only when the both edges are free. However, since we are dealing with the homogeneous boundary conditions in contrast with the inhomogeneous ones in the static boundary value problems, the factor ξ_1/n may drop out as a common multiplier and may not have a significant influence on the calculations of the determinants. At the present time, therefore, we may only state that the solutions in Donnell's approximation become increasingly inaccurate as n takes on smaller values and ℓ larger values. The influence of ℓ is due to the fact that the coefficients matrices of the boundary constraining equations contain terms with $\exp(\xi_1 \ell)$.

10. CONCLUSION

Some of the important conclusions of the present analysis may be summarized as follows:

1. The classical shell theories including the Flügge, the Koiter-Sanders, the Novozhilov, and the Love-Reissner theories provide valid solutions for the free vibration problems characterized by $n \geq 2$ of circumferentially closed circular cylindrical shells. The solutions are accurate within the errors inherent to the Kirchhoff-Love hypothesis.

2. A first approximation consistent to the order-of-magnitude estimate of the errors involved in the Kirchhoff-Love hypothesis yields the following system of equations:
Free vibration equation;

$$\begin{aligned}
&\nabla^4(\nabla^2+1)^2 w + 4k^4 w'''' \\
&+ \left\{[\nabla^4 w]_f - [(3+2\nu)w'' + w^{\cdot\cdot}]_p\right\}^{**}/\delta \\
&- \left\{[\tfrac{3-\nu}{1-\nu}\nabla^2 w]_{p,f} - [\tfrac{2}{1-\nu} w]_p\right\}^{\overset{**}{**}}/\delta \\
&+ [\tfrac{2}{1-\nu} w]_{p,f}^{\overset{***}{***}}/\delta = 0
\end{aligned}$$

Supplemental equations;

$$\begin{aligned}
L_0 u &= -\nu w''' + w'^{\cdot\cdot} \\
L_0 v &= -(2+\nu) w''^{\cdot} - w^{\cdot\cdot\cdot} \\
L_0 N &= (1-\nu^2) w''^{\cdot\cdot} - [\nu \nabla^2 w]_p^{**} \\
L_0 M &= -\nabla^4 (w'' + \nu w^{\cdot\cdot}) - \nu w^{::} \\
L_0 S &= -\nabla^4 [w''' + (2-\nu) w'^{\cdot\cdot}] \\
L_0 Q &= -(1-\nu^2) w'''^{\cdot}
\end{aligned}$$

with

$$L_0(\ ) = \nabla^4(\ ) - [\tfrac{3-\nu}{1-\nu}\nabla^2(\ )]_p^{**} + [\tfrac{2}{1-\nu}(\ )]_p^{\overset{**}{**}}$$

3. Among the three inertial components, the lateral component plays the most important role and is unavoidable in the analysis in general within the framework of the clasical shell theory. The inplane inertial component gives a significant contribution to the free vibration characteristics when the vibration occurs either at a very high frequency or with a small number of the circumferential waves. On the other hand, the rotatory inertial component have no significant influence on the vibration characteristics, and that it can be disregarded entirely in the analysis.

4. We write the eigenfunction for w such that

$$w = \sum_{i=1}^{8} W_i \, e^{p_i y} \cos n\theta \sin \omega T$$

Then, the eigenvalues p_i are given in the form

$$p_1, p_2 = \pm \xi_1, \quad p_3, p_4 = \pm i \eta_1$$

$$p_5, p_6, p_7, p_8 = \pm(\xi_2 \pm i \eta_2)$$

provided that the frequency parameter ω remains in the range specified by

$$\delta n^2(n^2-1)^2/(n^2+1)<\omega^2<1-\nu^2$$

The upper bound of the specified frequency spectrum is very close to

$$\omega^2=1$$

which corresponds to the natural frequency of the circular cylindrical shell in the axisymmetric cylindrical mode. The lower bound

$$\omega^2=\delta n^2(n^2-1)^2/(n^2+1)$$

corresponds to the lowest natural frequencies in unsymmetric cylindrical modes.

5. For $\omega^2 \ll 1$, the leading terms in ξ_1 and η_1 are inversely proportional to k, whereas those in ξ_2 and η_2 proportional to it, indicating that the formers represent the global solutions while the latters the edge-zone solutions. Thus, the existence of the edge-zone solutions in the free vibration problems has been proved just as in the static boundary value problems.

In cases where $\omega^2 = 0\ (\delta n^4)$ and $\delta^{1/2}n^2 \ll 1$, a drastic simplification is achieved for the governing equations and the solutions are obtained in a simple form, such that

$$\xi_2=\eta_2=k$$

and

$$\omega^2=[(1-\nu^2)\xi_1^4+\delta n^4(n^2-1)^2]/n^2(n^2+1)$$

where ξ_1 is to be determined from the characteristic equation.

6. The Donnell-type approximation is a consistent first approximation within the errors inherent to the Kirchhoff-Love hypothesis for vibrations with large number of waves; namely, for

$$n^2 \gg 1$$

The Donnell-type approximation leads to the following system of equations:

Free vibration equation (Donnell's equations);

$$\nabla^8 w+4k^4w''''+[\nabla^4 w]_f^{**}/\delta=0$$

Supplemental equations;

$$\nabla^4 u=-\nu w'''+w'^{\cdot\cdot}$$

$$\nabla^4 v=-(2+\nu)w''^{\cdot}-w^{\cdot\cdot\cdot}$$

$$\nabla^4 N=(1-\nu^2)w''^{\cdot\cdot}$$

$$\nabla^4 M=-\nabla^4(w''+\nu w^{\cdot\cdot})$$

$$\nabla^4 S=-\nabla^4[w'''+(2-\nu)w'^{\cdot\cdot}]$$

$$\nabla^4 Q=-(1-\nu^2)w'''^{\cdot}$$

7. Donnell's equation is valid for flexural vibrations with large number of waves in which only the lateral inertial component plays an important role, and it gives only one natural frequency for one particular vibration mode. The validity of Donnell's equation deteriorates as the frequency goes down to the range

$$\delta n^2(n^2-1)^2/(n^2+1)<\omega^2<\delta n^4$$

indicating that Donnell's equation fails to provide accurate solutions, even for those flexural vibration modes characterized by large number of waves, if they are nearly in unsymmetric cylindrical modes.

8. If the Donnell-type approximation is applied for the problems characterized by small number of waves, the errors in the solutions increase as n decreases, the length-to-radius ratio increases, and the boundary conditions are prescribed more and more by the physical quantitites.

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APPENDIX A. EXACT EXPRESSIONS OF THE COEFFICIENTS

G_i :

$$G_1 = 1$$
$$G_2 = (1-\nu)[1+\delta(3E_5+E_6)/4]/2$$
$$G_3 = (1+\nu)/2+3(1-\nu)\delta(E_5-E_6)/8 \qquad (A.1)$$
$$G_4 = -\delta E_1$$
$$G_5 = \delta(1-\nu)E_6/2$$
$$G_6 = \nu$$

H_i :

$$H_1 = (1-\nu)\{1+\delta[(E_3+3E_4)/2+3+D_2-D_4]/2\}/2$$
$$H_2 = 1+\delta(1+E_2-E_1-D_1)$$
$$H_3 = (1+\nu)/2+\delta(1-\nu)[(E_3-E_4)/2+D_2+D_4-1]/4$$
$$H_4 = -\delta[1+(1-\nu)E_4/2] \qquad (A.2)$$
$$H_5 = -\delta(1-E_1)$$
$$H_6 = 1+\delta(E_2-D_1)$$

K_i :

$$K_1 = D_1$$
$$K_2 = (1-\nu)[-1+D_4+(D_2-D_3)/2]/2$$
$$K_3 = -\nu/\delta$$
$$K_4 = (3-\nu)/2+(1-\nu)[(D_2-D_3)/2-D_4]/2$$
$$K_5 = 1-D_1 \qquad (A.3)$$
$$K_6 = -1/\delta-(E_2-E_1)$$
$$K_7 = D_1+E_1$$
$$K_8 = 1/\delta+E_2$$

f_i :

$$f_1 = -H_1G_4$$
$$f_2 = H_4G_3-H_1G_5-H_2G_4$$
$$f_3 = H_5G_3-H_2G_5$$
$$f_4 = -H_1G_6 \qquad (A.4)$$
$$f_5 = H_6G_3-H_2G_6$$
$$f_6 = ([1]_p+[\delta]_r)G_4$$
$$f_7 = [G_5]_p+\delta[G_3+G_5]_r$$
$$f_8 = ([1]_p+[\delta]_r)G_6$$

g_i :

$$g_1 = G_4H_3-G_1H_4$$
$$g_2 = G_5H_3-G_1H_5-G_2H_4$$
$$g_3 = -G_2H_5$$
$$g_4 = G_6H_3-G_1H_6$$
$$g_5 = -G_2H_6 \qquad (A.5)$$
$$g_6 = [H_4]_p-\delta[G_1]_r$$
$$g_7 = [H_5]_p-\delta[G_2]_r$$
$$g_8 = [H_6]_p$$
$$g_9 = [\delta]_{p,r}$$

L_i :

$$L_1 = H_1G_1$$
$$L_2 = H_1G_2+H_2G_1-H_3G_3$$
$$L_3 = H_2G_2 \qquad (A.6)$$
$$L_4 = -[H_1+G_1]_p-\delta[G_1]_r$$
$$L_5 = -[H_2+G_2]_p-\delta[G_2]_r$$
$$L_6 = [1]_p+[\delta]_{p,r}$$

a_{ij} :

$$a_{81} = K_1f_1-L_1$$
$$a_{82} = K_1f_2+K_2f_1+K_4g_1-L_2-2L_1$$
$$a_{83} = K_1f_3+K_2f_2+K_4g_2+K_5g_1-L_3-2L_2-L_1$$
$$a_{84} = K_2f_3+K_4g_3+K_5g_2-2L_3-L_2$$
$$a_{85} = K_5g_3-L_3$$
$$a_{61} = K_1f_4+K_3f_1 \qquad (A.7)$$
$$a_{62} = K_1f_5+K_2f_4+K_3f_2+K_4g_4+K_6g_1-K_7L_1$$
$$a_{63} = K_2f_5+K_3f_3+K_4g_5+K_5g_4+K_6g_2-K_7L_2$$
$$a_{64} = K_5g_5+K_6g_3-K_7L_3$$
$$a_{41} = K_3f_4-K_8L_1$$
$$a_{42} = K_3f_5+K_6g_4-K_8L_2$$
$$a_{43} = K_6g_5-K_8L_3$$

b_{ij} :

$$
\begin{aligned}
b_{61} &= K_1f_6 - L_4 + [L_1]_r \\
b_{62} &= K_1f_7 + K_2f_6 + K_4g_6 - L_5 - 2L_4 + [L_2 + L_1 - g_1]_r \\
b_{63} &= K_2f_7 + K_4g_7 + K_5g_6 - 2L_5 - L_4 + [L_3 + L_2 - g_2]_r \\
b_{64} &= K_5g_7 - L_5 + [L_3 - g_3]_r \\
b_{41} &= K_1f_8 + K_3f_6 - [L_1/\delta]_f \\
b_{42} &= K_2f_8 + K_3f_7 + K_4g_8 + K_6g_6 - K_7L_4 \\
&\quad - [L_2/\delta]_f - [g_4]_r \\
b_{43} &= K_5g_8 + K_6g_7 - K_7L_5 - [L_3/\delta]_f - [g_5]_r \\
b_{21} &= K_3f_8 - K_8L_4 \\
b_{22} &= K_6g_8 - K_8L_5
\end{aligned}
\tag{A.8}
$$

c_{ij} :

$$
\begin{aligned}
c_{41} &= -L_6 + [L_4]_r \\
c_{42} &= K_4g_9 - 2L_6 + [L_5 + L_4 - g_6]_r \\
c_{43} &= K_5g_9 - L_6 + [L_5 - g_7]_r \\
c_{21} &= -[L_4/\delta]_f \\
c_{22} &= K_6g_9 - K_7L_6 - [L_5/\delta]_f - [g_8]_r \\
c_{00} &= -K_8L_6
\end{aligned}
\tag{A.9}
$$

d_{ij} :

$$
\begin{aligned}
d_{21} &= [L_6]_r \\
d_{22} &= [L_6 - g_9]_r \\
d_{00} &= -[L_6/\delta]_f
\end{aligned}
\tag{A.10}
$$

n_i :

$$
\begin{aligned}
n_1 &= f_2 + \nu g_1 - \delta E_1L_2 \\
n_2 &= f_3 + \nu g_2 - \delta E_1L_3 \\
n_3 &= \nu g_3 \\
n_4 &= f_5 + \nu g_4 + \nu L_2 \\
n_5 &= \nu g_5 + \nu L_3 \\
n_6 &= f_6 - \delta E_1L_4 \\
n_7 &= f_7 + \nu g_6 - \delta E_1L_5 \\
n_8 &= \nu g_7 \\
n_9 &= f_8 + \nu L_4 \\
n_{10} &= \nu g_8 + \nu L_5 \\
n_{11} &= -\delta E_1L_6 \\
n_{12} &= \nu g_9 \\
n_{13} &= \nu L_6
\end{aligned}
\tag{A.11}
$$

m_i :

$$
\begin{aligned}
m_1 &= D_1f_1 - L_1 \\
m_2 &= D_1f_2 + \nu g_1 - L_2 - \nu L_1 \\
m_3 &= D_1f_3 + \nu g_2 - L_3 - \nu L_2 \\
m_4 &= \nu g_3 - \nu L_3 \\
m_5 &= D_1f_4 \\
m_6 &= D_1f_5 + \nu g_4 \\
m_7 &= \nu g_5 \\
m_8 &= D_1f_6 - L_4 \\
m_9 &= D_1f_7 + \nu g_6 - L_5 - \nu L_4 \\
m_{10} &= \nu g_7 - \nu L_5 \\
m_{11} &= D_1f_8 \\
m_{12} &= \nu g_8 \\
m_{13} &= -L_6 \\
m_{14} &= \nu g_9 - \nu L_6
\end{aligned}
\tag{A.12}
$$

s_i :

$$
\begin{aligned}
s_1 &= D_1f_1 - L_1 \\
s_2 &= D_1f_2 + K_sf_1 + K_tg_1 - L_2 - (2-\nu)L_1 \\
s_3 &= D_1f_3 + K_sf_2 + K_tg_2 - L_3 - (2-\nu)L_2 \\
s_4 &= K_sf_3 + K_tg_3 - (2-\nu)L_3 \\
s_5 &= D_1f_4 \\
s_6 &= D_1f_5 + K_sf_4 + K_tg_4 \\
s_7 &= K_sf_5 + K_tg_5 \\
s_8 &= D_1f_6 - L_4 + [L_1]_r \\
s_9 &= D_1f_7 + K_sf_6 + K_tg_6 - L_5 - (2-\nu)L_4 + [L_2]_r \\
s_{10} &= K_sf_7 + K_tg_7 - (2-\nu)L_5 + [L_3]_r \\
s_{11} &= D_1f_8 \\
s_{12} &= K_sf_8 + K_tg_8
\end{aligned}
\tag{A.13}
$$

$$s_{13} = -L_6 + [L_4]_r$$
$$s_{14} = K_t g_9 - (2-\nu)L_6 + [L_5]_r$$
$$s_{15} = [L_6]_r$$

where

$$K_s = -(1-\nu)/2 + (1-\nu)(D_2 - D_3 + 2D_4)/4$$
$$K_t = (3-\nu)/2 + (1-\nu)(D_2 - D_3 - 2D_4)/4$$

q_i :

$$q_1 = K_u f_1 + K_v g_1 + \delta K_w L_1$$
$$q_2 = K_u f_2 + K_v g_2 + \delta K_w L_2$$
$$q_3 = K_u f_3 + K_v g_3 + \delta K_w L_3$$
$$q_4 = K_u f_4 + K_v g_4$$
$$q_5 = K_u f_5 + K_v g_5 \qquad \text{(A.14)}$$
$$q_6 = K_u f_6 + K_v g_6 + \delta K_w L_4$$
$$q_7 = K_u f_7 + K_v g_7 + \delta K_w L_5$$
$$q_8 = K_u f_8 + K_v g_8$$
$$q_9 = K_v g_9 + \delta K_w L_6$$

where

$$K_u = (1-\nu)\{1 + \delta[-1 + D_2 + D_4 + (E_3 - E_4)/2]/2\}/2$$
$$K_v = (1-\nu)\{1 + \delta[3 + D_2 - D_4 + (E_3 + 3E_4)/2]/2\}/2$$
$$K_w = -(1-\nu)(2 + E_4)/2$$

APPENDIX B. LEADING TERMS OF THE COEFFICIENTS

$a_{ij} \simeq -\dfrac{1-\nu}{2} a^0_{ij}$:

$$a^0_{81} = a^0_{85} = 1, \quad a^0_{82} = a^0_{84} = 4, \quad a^0_{83} = 6$$
$$a^0_{61} = \nu(E_1 + D_1)$$
$$a^0_{62} = 5 + \nu + (1-\nu)[(2+\nu)E_4 - \nu E_6 + (1+\nu)(D_2 - D_3) - 2D_4]/2$$
$$a^0_{63} = 7 + \nu - \nu(E_1 + D_1) + (1-\nu)(E_4 + E_6 - 2D_4)/2$$
$$a^0_{64} = 2 \qquad \text{(B.1)}$$
$$a^0_{41} = (1-\nu^2)/\delta$$
$$a^0_{42} = 3 + \nu - \nu(E_1 + D_1) + (1-\nu)(E_4 + E_6 - D_4)/2$$
$$a^0_{43} = 1$$

$b_{ij} \simeq -\dfrac{1-\nu}{2} b^0_{ij}$:

$$b^0_{61} = b^0_{64} = -\left\{\left[\frac{3-\nu}{1-\nu}\right]_p + [1]_r\right\}$$
$$b^0_{62} = b^0_{63} = -3\left\{\left[\frac{3-\nu}{1-\nu}\right]_p + [1]_r\right\}$$
$$b^0_{41} = b^0_{43} = [1/\delta]_f, \quad b^0_{42} = [2/\delta]_f \qquad \text{(B.2)}$$
$$b^0_{21} = -[(3+2\nu)/\delta]_p, \quad b^0_{22} = -[1/\delta]_p$$

$c_{ij} \simeq -\dfrac{1-\nu}{2} c^0_{ij}$:

$$c^0_{41} = c^0_{43} = \left[\frac{2}{1-\nu}\right]_p + \left[\frac{3-\nu}{1-\nu}\right]_{p,r}$$
$$c^0_{42} = \left[\frac{4}{1-\nu}\right]_p + \left[\frac{2(3-\nu)}{1-\nu}\right]_{p,r} \qquad \text{(B.3)}$$
$$c^0_{21} = c^0_{22} = -\left[\frac{3-\nu}{\delta(1-\nu)}\right]_{f,p}$$
$$c^0_{00} = \left[\frac{2}{\delta(1-\nu)}\right]_p$$

$d_{ij} \simeq -\dfrac{1-\nu}{2} d^0_{ij}$:

$$d^0_{21} = d^0_{22} = -\left[\frac{2}{1-\nu}\right]_{p,r} \qquad \text{(B.4)}$$
$$d^0_{00} = \left[\frac{2}{\delta(1-\nu)}\right]_{f,p}$$

$L_i \simeq \dfrac{1-\nu}{2} L^0_i$:

$$L^0_1 = L^0_3 = 1, \quad L^0_2 = 2 \qquad \text{(B.5)}$$
$$L^0_4 = L^0_5 = -\left[\frac{3-\nu}{1-\nu}\right]_p, \quad L^0_6 = \left[\frac{2}{1-\nu}\right]_p$$

$f_i \simeq \dfrac{1-\nu}{2} \tilde{f}^0_i$:

$$\tilde{f}^0_1 = \delta f^0_1, \quad \tilde{f}^0_2 = \delta f^0_2, \quad \tilde{f}^0_3 = \delta f^0_3$$
$$\tilde{f}^0_4 = f^0_4, \quad \tilde{f}^0_5 = f^0_5, \quad \tilde{f}^0_6 = \delta f^0_6$$
$$\tilde{f}^0_7 = \delta f^0_7, \quad \tilde{f}^0_8 = f^0_8$$

where

$$f^0_1 = E_1$$
$$f^0_2 = -\frac{1+\nu}{1-\nu} + \frac{2}{1-\nu} E_1 - \frac{1+\nu}{2} E_4 - \frac{1-\nu}{2} E_6$$

$$f_3^0 = -\frac{1+\nu}{1-\nu} + \frac{1+\nu}{1-\nu} E_1 - E_6 \tag{B.6}$$

$$f_4^0 = -\nu, \quad f_5^0 = 1$$

$$f_6^0 = -\left[\frac{2}{1-\nu} E_1\right]_p$$

$$f_7^0 = \left[\frac{1+\nu}{1-\nu}\right]_r + [E_6]_p, \quad f_8^0 = \left[\frac{2\nu}{1-\nu}\right]_p$$

$$g_i \simeq \frac{1-\nu}{2} \tilde{g}_i^0 :$$

$$\tilde{g}_1^0 = \delta g_1^0, \quad \tilde{g}_2^0 = \delta g_2^0, \quad \tilde{g}_3^0 = \delta g_3^0$$

$$\tilde{g}_4^0 = g_4^0, \quad \tilde{g}_5^0 = g_5^0, \quad \tilde{g}_6^0 = \delta g_6^0$$

$$\tilde{g}_7^0 = \delta g_7^0, \quad \tilde{g}_8^0 = g_8^0, \quad \tilde{g}_9^0 = \delta g_9^0$$

where

$$g_1^0 = \frac{2}{1-\nu} - \frac{1+\nu}{1-\nu} E_1 + E_4$$

$$g_2^0 = \frac{3-\nu}{1-\nu} - \frac{2}{1-\nu} E_1 + \frac{1-\nu}{2} E_4 + \frac{1+\nu}{2} E_6$$

$$g_3^0 = 1 - E_1$$

$$g_4^0 = -(2+\nu), \quad g_5^0 = -1 \tag{B.7}$$

$$g_6^0 = -\left[\frac{2}{1-\nu}\right]_r - \left[\frac{2}{1-\nu}\right]_p - [E_4]_p$$

$$g_7^0 = -[1]_r - \frac{2}{1-\nu}[1 - E_1]_p$$

$$g_8^0 = \left[\frac{2}{1-\nu}\right]_p, \quad g_9^0 = \left[\frac{2}{1-\nu}\right]_{p,r}$$

$$n_i \simeq \frac{1-\nu}{2} \tilde{n}_i^0 :$$

$$\tilde{n}_1^0 = \delta n_1^0, \ \tilde{n}_2^0 = \delta n_2^0, \quad \tilde{n}_3^0 = \delta n_3^0, \quad \tilde{n}_4^0 = n_4^0$$

$$\tilde{n}_5^0 = \delta n_5^0, \quad \tilde{n}_6^0 = n_6^0, \quad \tilde{n}_7^0 = \delta n_7^0, \quad \tilde{n}_8^0 = \delta n_8^0$$

$$\tilde{n}_9^0 = n_9^0, \quad \tilde{n}_{10}^0 = n_{10}^0, \quad \tilde{n}_{11}^0 = \delta n_{11}^0, \ \tilde{n}_{12}^0 = \delta n_{12}^0$$

$$\tilde{n}_{13}^0 = n_{13}^0$$

where

$$n_1^0 = -1 + \nu E_1 - \frac{1-\nu}{2}(E_4 + E_6)$$

$$n_2^0 = \frac{1-\nu}{2}[-2 + \nu E_4 - (2+\nu) E_6]$$

$$n_3^0 = n_5^0 = \nu(1-E_1), \quad n_4^0 = 1-\nu^2, \quad n_6^0 = [E_1]_p$$

$$n_7^0 = [1]_r - \left[\frac{2\nu}{1-\nu} - \frac{3-\nu}{1-\nu} E_1 + \nu E_4 - E_6\right]_p \tag{B.8}$$

$$n_8^0 = -[\nu]_r - \frac{2\nu}{1-\nu}[1 - E_1]_p$$

$$n_9^0 = n_{10}^0 = -[\nu]_p, \quad n_{11}^0 = -\left[\frac{2}{1-\nu} E_1\right]_p$$

$$n_{12}^0 = \left[\frac{2\nu}{1-\nu}\right]_{p,r}, \quad n_{13}^0 = \left[\frac{2\nu}{1-\nu}\right]_p$$

$$m_i \simeq \frac{1-\nu}{2} m_i^0 :$$

$$m_1^0 = -1, \quad m_2^0 = -(2+\nu), \quad m_3^0 = -(1+2\nu)$$

$$m_4^0 = m_7^0 = -\nu, \quad m_5^0 = -\nu D_1, \quad m_6^0 = D_1 - \nu(2+\nu)$$

$$m_8^0 = \left[\frac{3-\nu}{1-\nu}\right]_p, \quad m_9^0 = \left[\frac{(3-\nu)(1+\nu)}{1-\nu}\right]_p$$

$$m_{10}^0 = \left[\frac{\nu(3-\nu)}{1-\nu}\right]_p, \quad m_{11}^0 = \left[\frac{2\nu}{1-\nu} D_1\right]_p \tag{B.9}$$

$$m_{12}^0 = \left[\frac{2\nu}{1-\nu}\right]_p, \quad m_{13}^0 = -\left[\frac{2}{1-\nu}\right]_p$$

$$m_{14}^0 = -\left[\frac{2\nu}{1-\nu}\right]_p$$

$$s_i \simeq \frac{1-\nu}{2} s_i^0 :$$

$$s_1^0 = -1, \quad s_2^0 = -(4-\nu), \quad s_3^0 = -(5-2\nu)$$

$$s_4^0 = -(2-\nu), \quad s_5^0 = -\nu D_1$$

$$s_6^0 = -3 + D_1 - \frac{1-\nu^2}{2}(D_2 - D_3) + (1-\nu) D_4$$

$$s_7^0 = -2 + \nu + (1-\nu) D_4$$

$$s_8^0 = \left[\frac{3-\nu}{1-\nu}\right]_p + [1]_r$$

$$s_9^0 = \left[\frac{(3-\nu)^2}{1-\nu}\right]_p + [2]_r$$

$$s_{10}^0 = \left[\frac{(3-\nu)(2-\nu)}{1-\nu}\right]_p + [1]_r \tag{B.10}$$

$$s_{11}^0 = \left[\frac{2\nu}{1-\nu} D_1\right]_p$$

$$s_{12}^0 = \left[\frac{3-2\nu+\nu^2}{1-\nu} + \frac{1+\nu}{2}(D_2 - D_3) - (1-\nu) D_4\right]_p$$

$$s_{13}^0 = -\left[\frac{2}{1-\nu}\right]_p - \left[\frac{3-\nu}{1-\nu}\right]_{p,r}$$

$$s_{14}^0 = -\left[\frac{2(2-\nu)}{1-\nu}\right]_p - \left[\frac{3-\nu}{1-\nu}\right]_{p,r}$$

$$s_{15}^0 = \left[\frac{2}{1-\nu}\right]_{p,r}$$

$$q_i \simeq \frac{1-\nu}{2} \tilde{q}_i^0 :$$

$$\tilde{q}_1^0 = \delta q_1^0, \quad \tilde{q}_2^0 = \delta q_2^0, \quad \tilde{q}_3^0 = \delta q_3^0$$

$$\tilde{q}_4^0 = q_4^0, \quad \tilde{q}_5^0 = \delta q_5^0, \quad \tilde{q}_6^0 = \delta q_6^0$$

$$\tilde{q}_7^0 = \delta q_7^0, \quad \tilde{q}_8^0 = q_8^0, \quad \tilde{q}_9^0 = \delta q_9^0$$

where

$$q_1^0 = \nu(1-E_1)$$

$$q_2^0 = -1+\nu-\frac{(1-\nu)(2+\nu)}{2}E_4+\frac{\nu(1-\nu)}{2}E_6$$

$$q_3^0 = -1+\nu E_1-\frac{1-\nu}{2}(E_4+E_6)$$

$$q_4^0 = -(1-\nu^2)$$

$$q_5^0 = -1+\nu E_1-\frac{1-\nu}{2}(E_4+E_6-D_4) \qquad \text{(B.11)}$$

$$q_6^0 = -[1]_r+[2-\nu-E_1+E_4]_p$$

$$q_7^0 = [\nu]_r+[2-\nu+E_1+\frac{3-\nu}{2}E_4+\frac{1-\nu}{2}E_6]_p$$

$$q_8^0 = [1+\nu]_p$$

$$q_9^0 = [1]_{p,r}-[2+E_4]_p$$

APPENDIX C. COEFFICIENTS OF BOUNDARY CONSTRAINING EQUATIONS (FIRST APPROXIMATION; $y=+\ell$)

$\mathscr{W}_i$:

$$(\mathscr{W}_1,\mathscr{W}_2) = -\frac{1}{4\xi_2^2\eta_2^2}(\xi_1^2-n^2+\omega^2)(\xi_1^2-n^2+\frac{2}{1-\nu}\omega^2)(e^{\xi_1\ell},\ e^{-\xi_1\ell})$$

$$(\mathscr{W}_3,\mathscr{W}_4) = -\frac{1}{4\xi_2^2\eta_2^2}(\eta_1^2+n^2-\omega^2)(\eta_1^2+n^2-\frac{2}{1-\nu}\omega^2)(\cos\eta_1\ell,\ \sin\eta_1\ell)$$

$$(\mathscr{W}_5,\mathscr{W}_6) = e^{\xi_2\ell}[(A_w\cos\eta_2\ell-B_w\sin\eta_2\ell),\ (A_w\sin\eta_2\ell+B_w\cos\eta_2\ell)] \qquad \text{(C.1)}$$

$$(\mathscr{W}_7,\mathscr{W}_8) = e^{-\xi_2\ell}[(A_w\cos\eta_2\ell+B_w\sin\eta_2\ell),\ (A_w\sin\eta_2\ell-B_w\cos\eta_2\ell)]$$

where

$$A_w = 1-\frac{1}{4\xi_2^2\eta_2^2}(\xi_2^2-\eta_2^2-n^2+\omega^2)(\xi_2^2-\eta_2^2-n^2+\frac{2}{1-\nu}\omega^2)$$

$$B_w = -\frac{1}{2\xi_2\eta_2}[2(\xi_2^2-\eta_2^2-n^2)+\frac{3-\nu}{1-\nu}]$$

$\mathscr{S}_i$:

$$(\mathscr{S}_1,\mathscr{S}_2) = \frac{1}{8\xi_2^3\eta_2^4}\xi_1(\xi_1^2-n^2)^2[\xi_1^2-(2-\nu)n^2](e^{\xi_1\ell},\ -e^{-\xi_1\ell})$$

$$(\mathscr{S}_3,\mathscr{S}_4) = \frac{1}{8\xi_2^3\eta_2^4}\eta_1(\eta_1^2+n^2)^2[\eta_1^2+(2-\nu)n^2](\sin\eta_1\ell,\ -\cos\eta_1\ell)$$

$$(\mathscr{S}_5,\mathscr{S}_6) = e^{\xi_2\ell}[(A_s\cos\eta_2\ell-B_s\sin\eta_2\ell),\ (A_s\sin\eta_2\ell+B_s\cos\eta_2\ell)] \qquad \text{(C.2)}$$

$$(\mathscr{S}_7,\mathscr{S}_8) = -e^{-\xi_2\ell}[(A_s\cos\eta_2\ell+B_s\sin\eta_2\ell),\ (A_s\sin\eta_2\ell-B_s\cos\eta_2\ell)]$$

where

$$A_s = [1-\frac{(\xi_2^2-\eta_2^2-n^2)^2}{4\xi_2^2\eta_2^2}][1-\frac{\xi_2^2-\eta_2^2-(2-\nu)n^2}{2\eta_2^2}]-\frac{1}{\eta_2^2}(\xi_2^2-\eta_2^2-n^2)[1+\frac{\xi_2^2-\eta_2^2-(2-\nu)n^2}{2\xi_2^2}]$$

$$B_s = -\frac{\xi_2}{\eta_2}\{[1-\frac{(\xi_2^2-\eta_2^2-n^2)^2}{4\xi_2^2\eta_2^2}][1+\frac{\xi_2^2-\eta_2^2-(2-\nu)n^2}{2\xi_2^2}]+\frac{1}{\xi_2^2}(\xi_2^2-\eta_2^2-n^2)[1-\frac{\xi_2^2-\eta_2^2-(2-\nu)n^2}{2\eta_2^2}]\}$$

$\mathcal{B}_i$:

$$\begin{aligned}
(\mathcal{B}_1, \mathcal{B}_2) &= \frac{1}{4\xi_2^3\eta_2^2}\xi_1(\xi_1^2-n^2+\omega^2)(\xi_1^2-n^2+\frac{2}{1-\nu}\omega^2)(-e^{\xi_1\ell}, e^{-\xi_1\ell}) \\
(\mathcal{B}_3, \mathcal{B}_4) &= \frac{1}{4\xi_2^3\eta_2^2}\eta_1(\eta_1^2+n^2-\omega^2)(\eta_1^2+n^2-\frac{2}{1-\nu}\omega^2)(\sin\eta_1\ell, -\cos\eta_1\ell) \\
(\mathcal{B}_5, \mathcal{B}_6) &= e^{\xi_2\ell}[(A_b\cos\eta_2\ell - B_b\sin\eta_2\ell), (A_b\sin\eta_2\ell + B_b\cos\eta_2\ell)] \\
(\mathcal{B}_7, \mathcal{B}_8) &= -e^{-\xi_2\ell}[(A_b\cos\eta_2\ell + B_b\sin\eta_2\ell), (A_b\sin\eta_2\ell - B_b\cos\eta_2\ell)]
\end{aligned} \tag{C.3}$$

where

$$A_b = 1-\frac{1}{4\xi_2^2\eta_2^2}(\xi_2^2-\eta_2^2-n^2+\omega^2)(\xi_2^2-\eta_2^2-n^2+\frac{2}{1-\nu}\omega^2)+\frac{1}{\xi_2^2}[\xi_2^2-\eta_2^2-n^2+\frac{3-\nu}{2(1-\nu)}\omega^2]$$

$$B_b = \frac{\eta_2}{\xi_2}[1-\frac{1}{4\xi_2^2\eta_2^2}(\xi_2^2-\eta_2^2-n^2+\omega^2)(\xi_2^2-\eta_2^2-n^2+\frac{2}{1-\nu}\omega^2)]-\frac{1}{\eta_2^2}[\xi_2^2-\eta_2^2-n^2+\frac{3-\nu}{2(1-\nu)}\omega^2]$$

m_i :

$$\begin{aligned}
(m_1, m_2) &= -\frac{1}{8\xi_2^3\eta_2^3}[(\xi_1^2-n^2)^2(\xi_1^2-\nu n^2)+\nu n^4](e^{\xi_1\ell}, e^{-\xi_1\ell}) \\
(m_3, m_4) &= \frac{1}{8\xi_2^3\eta_2^3}[(\eta_1^2+n^2)^2(\eta_1^2+\nu n^2)-\nu n^4](\cos\eta_1\ell, \sin\eta_1\ell) \\
(m_5, m_6) &= e^{\xi_2\ell}[(A_m\cos\eta_2\ell - B_m\sin\eta_2\ell), (A_m\sin\eta_2\ell + B_m\cos\eta_2\ell)] \\
(m_7, m_8) &= e^{-\xi_2\ell}[(A_m\cos\eta_2\ell + B_m\sin\eta_2\ell), (A_m\sin\eta_2\ell - B_m\cos\eta_2\ell)]
\end{aligned} \tag{C.4}$$

where

$$A_m = \frac{1}{2\xi_2\eta_2}\{[1-\frac{(\xi_2^2-\eta_2^2-n^2)^2}{4\xi_2^2\eta_2^2}](\xi_2^2-\eta_2^2-\nu n^2)+2(\xi_2^2-\eta_2^2-n^2)-\frac{\nu n^4}{4\xi_2^2\eta_2^2}\}$$

$$B_m = 1-\frac{(\xi_2^2-\eta_2^2-n^2)^2}{4\xi_2^2\eta_2^2}-\frac{1}{2\xi_2^2\eta_2^2}(\xi_2^2-\eta_2^2-n^2)(\xi_2^2-\eta_2^2-\nu n^2)$$

$\mathcal{U}_i$:

$$\begin{aligned}
(\mathcal{U}_1, \mathcal{U}_2) &= \frac{\xi_1}{2\nu\xi_2\eta_2^2}(\nu\xi_1^2+n^2)(-e^{\xi_1\ell}, e^{-\xi_1\ell}) \\
(\mathcal{U}_3, \mathcal{U}_4) &= \frac{\eta_1}{2\nu\xi_2\eta_2^2}(\eta_1^2-n^2)(-\sin\eta_1\ell, \cos\eta_1\ell) \\
(\mathcal{U}_5, \mathcal{U}_6) &= e^{\xi_2\ell}[(A_u\cos\eta_2\ell - B_u\sin\eta_2\ell), (A_u\sin\eta_2\ell + B_u\cos\eta_2\ell)]
\end{aligned} \tag{C.5}$$

$$(\mathcal{U}_7, \mathcal{U}_8) = -e^{-\xi_2\ell}[(A_u\cos\eta_2\ell + B_u\sin\eta_2\ell),\ (A_u\sin\eta_2\ell - B_u\cos\eta_2\ell)]$$

where

$$A_u = 1 - \frac{\nu(\xi_2^2-\eta_2^2)+n^2}{2\nu\eta_2^2}$$

$$B_u = -\frac{\xi_2}{\eta_2}\left[1+\frac{\nu(\xi_2^2-\eta_2^2)+n^2}{2\nu\xi_2^2}\right]$$

n_i :

$$(n_1, n_2) = \frac{1}{2\xi_2\eta_2}\left[\xi_1^2 - \frac{\nu}{1-\nu^2}\frac{(\xi_1^2-n^2)\omega^2}{n^2}\right](e^{\xi_1\ell},\ e^{-\xi_1\ell})$$

$$(n_3, n_4) = -\frac{1}{2\xi_2\eta_2}\left[\eta_1^2+\frac{\nu}{1-\nu^2}\frac{(\eta_1^2+n^2)\omega^2}{n^2}\right](\cos\eta_1\ell,\ \sin\eta_1\ell)$$

$$(n_5, n_6) = e^{\xi_2\ell}[(A_n\cos\eta_2\ell - B_n\sin\eta_2\ell),\ (A_n\sin\eta_2\ell + B_n\cos\eta_2\ell)] \tag{C.6}$$

$$(n_7, n_8) = e^{-\xi_2\ell}[(A_n\cos\eta_2\ell + B_n\sin\eta_2\ell),\ (A_n\sin\eta_2\ell - B_n\cos\eta_2\ell)]$$

where

$$A_n = \frac{1}{2\xi_2\eta_2}\left[\xi_2^2-\eta_2^2-\frac{\nu}{1-\nu^2}\frac{(\xi_2^2-\eta_2^2-n^2)\omega^2}{n^2}\right]$$

$$B_n = 1+\frac{\nu}{1-\nu^2}\frac{\omega^2}{n^2}$$

$\mathcal{V}_i$:

$$(\mathcal{V}_1, \mathcal{V}_2) = \frac{1}{2\xi_2\eta_2}\left(\xi_1^2-\frac{n^2}{2+\nu}\right)(e^{\xi_1\ell},\ e^{-\xi_1\ell})$$

$$(\mathcal{V}_3, \mathcal{V}_4) = -\frac{1}{2\xi_2\eta_2}\left(\eta_1^2+\frac{n^2}{2+\nu}\right)(\cos\eta_1\ell,\ \sin\eta_1\ell)$$

$$(\mathcal{V}_5, \mathcal{V}_6) = e^{\xi_2\ell}[(A_v\cos\eta_2\ell - B_v\sin\eta_2\ell),\ (A_v\sin\eta_2\ell + B_v\cos\eta_2\ell)] \tag{C.7}$$

$$(\mathcal{V}_7, \mathcal{V}_8) = e^{-\xi_2\ell}[(A_v\cos\eta_2\ell + B_v\sin\eta_2\ell),\ (A_v\sin\eta_2\ell - B_v\cos\eta_2\ell)]$$

where

$$A_v = \frac{1}{2\xi_2\eta_2}\left(\xi_2^2-\eta_2^2-\frac{n^2}{2+\nu}\right)$$

$$B_v = 1$$

$\mathcal{O}_i$:

$$
\begin{aligned}
(\mathcal{O}_1, \mathcal{O}_2) &= \frac{\xi_1^3}{2\xi_2^2\eta_2}(e^{\xi_1 \ell}, -e^{-\xi_1 \ell}) \\
(\mathcal{O}_3, \mathcal{O}_4) &= \frac{\eta_1^3}{2\xi_2^2\eta_2}(\sin\eta_1\ell, -\cos\eta_1\ell) \\
(\mathcal{O}_5, \mathcal{O}_6) &= e^{\xi_2\ell}[(A_q\cos\eta_2\ell - B_q\sin\eta_2\ell), (A_q\sin\eta_2\ell + B_q\cos\eta_2\ell)] \\
(\mathcal{O}_7, \mathcal{O}_8) &= -e^{-\xi_2\ell}[(A_q\cos\eta_2\ell + B_q\sin\eta_2\ell), (A_q\sin\eta_2\ell - B_q\cos\eta_2\ell)]
\end{aligned}
\tag{C.8}
$$

where

$$A_q = -\frac{\eta_2}{\xi_2}[1 - \frac{1}{2\xi_2\eta_2}(\xi_2^2 - \eta_2^2)]$$

$$B_q = 1 + \frac{1}{2\xi_2^2}(\xi_2^2 - \eta_2^2)$$

APPENDIX D. COEFFICIENTS OF BOUNDARY CONSTRAINING EQUATIONS (APPROXIMATION FOR LOW FREQUENCIES; $y = +\ell$)

$\mathcal{W}_i$:

$$
\begin{aligned}
(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4) &= -\frac{n^4}{4k^4}(e^{\xi_1\ell}, e^{-\xi_1\ell}, \cos\xi_1\ell, \sin\xi_1\ell) \\
(\mathcal{W}_5, \mathcal{W}_6) &= e^{k\ell}(\cos k\ell, \sin k\ell) \\
(\mathcal{W}_7, \mathcal{W}_8) &= e^{-k\ell}(\cos k\ell, \sin k\ell)
\end{aligned}
\tag{D.1}
$$

$\mathcal{S}_i$:

$$
\begin{aligned}
(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4) &= -\frac{(2-\nu)n^7}{8k^7}(\frac{\xi_1}{n})(e^{\xi_1\ell}, -e^{-\xi_1\ell}, -\sin\xi_1\ell, \cos\xi_1\ell) \\
(\mathcal{S}_5, \mathcal{S}_6) &= e^{k\ell}[(\cos k\ell + \sin k\ell), (\sin k\ell - \cos k\ell)] \\
(\mathcal{S}_7, \mathcal{S}_8) &= -e^{-k\ell}[(\cos k\ell - \sin k\ell), (\sin k\ell + \cos k\ell)]
\end{aligned}
\tag{D.2}
$$

$\mathcal{B}_i$:

$$
\begin{aligned}
(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4) &= -\frac{n^5}{4k^5}(\frac{\xi_1}{n})(e^{\xi_1\ell}, -e^{-\xi_1\ell}, -\sin\xi_1\ell, \cos\xi_1\ell)] \\
(\mathcal{B}_5, \mathcal{B}_6) &= e^{k\ell}[(\cos k\ell - \sin k\ell), (\sin k\ell + \cos k\ell)] \\
(\mathcal{B}_7, \mathcal{B}_8) &= -e^{-k\ell}[(\cos k\ell + \sin k\ell), (\sin k\ell - \cos k\ell)]
\end{aligned}
\tag{D.3}
$$

$\mathcal{m}_i$:

$$(\mathcal{m}_1, \mathcal{m}_2, \mathcal{m}_3, \mathcal{m}_4) = \frac{\nu n^6}{8k^6}(1-\frac{1}{n^2})(e^{\xi_1 \ell}, e^{-\xi_1 \ell}, \cos\xi_1\ell, \sin\xi_1\ell)$$

$$(\mathcal{m}_5, \mathcal{m}_6) = -e^{k\ell}(\sin k\ell, -\cos k\ell) \tag{D.4}$$

$$(\mathcal{m}_7, \mathcal{m}_8) = e^{-k\ell}(\sin k\ell, -\cos k\ell)$$

$\mathcal{U}_i$:

$$(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4) = -\frac{n^3}{2\nu k^3}(\frac{\xi_1}{n})(e^{\xi_1 \ell}, -e^{-\xi_1 \ell}, -\sin\xi_1\ell, \cos\xi_1\ell)$$

$$(\mathcal{U}_5, \mathcal{U}_6) = e^{k\ell}[(\cos k\ell + \sin k\ell), (\sin k\ell - \cos k\ell)] \tag{D.5}$$

$$(\mathcal{U}_7, \mathcal{U}_8) = -e^{-k\ell}[(\cos k\ell - \sin k\ell), (\sin k\ell + \cos k\ell)]$$

$\mathcal{n}_i$:

$$(\mathcal{n}_1, \mathcal{n}_2, \mathcal{n}_3, \mathcal{n}_4) = \frac{n^2}{2k^2}(\frac{\xi_1}{n})^2(e^{\xi_1 \ell}, e^{-\xi_1 \ell}, -\cos\xi_1\ell, -\sin\xi_1\ell)$$

$$(\mathcal{n}_5, \mathcal{n}_6) = -e^{k\ell}(\sin k\ell, -\cos k\ell) \tag{D.6}$$

$$(\mathcal{n}_7, \mathcal{n}_8) = e^{-k\ell}(\sin k\ell, -\cos k\ell)$$

$\mathcal{V}_i$:

$$(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4) = -\frac{n^2}{2(2+\nu)k^2}(e^{\xi_1 \ell}, e^{-\xi_1 \ell}, \cos\xi_1\ell, \sin\xi_1\ell)$$

$$(\mathcal{V}_5, \mathcal{V}_6) = -e^{k\ell}(\sin k\ell, -\cos k\ell) \tag{D.7}$$

$$(\mathcal{V}_7, \mathcal{V}_8) = e^{-k\ell}(\sin k\ell, -\cos k\ell)$$

$\mathcal{O}_i$:

$$(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4) = \frac{n^3}{2k^3}(\frac{\xi_1}{n})^3(e^{\xi_1 \ell}, -e^{-\xi_1 \ell}, \sin\xi_1\ell, -\cos\xi_1\ell)$$

$$(\mathcal{O}_5, \mathcal{O}_6) = -e^{k\ell}[(\cos k\ell + \sin k\ell), (\sin k\ell - \cos k\ell)] \tag{D.8}$$

$$(\mathcal{O}_7, \mathcal{O}_8) = e^{-k\ell}[(\cos k\ell - \sin k\ell), (\sin k\ell + \cos k\ell)]$$

FIGURES AND TABLES

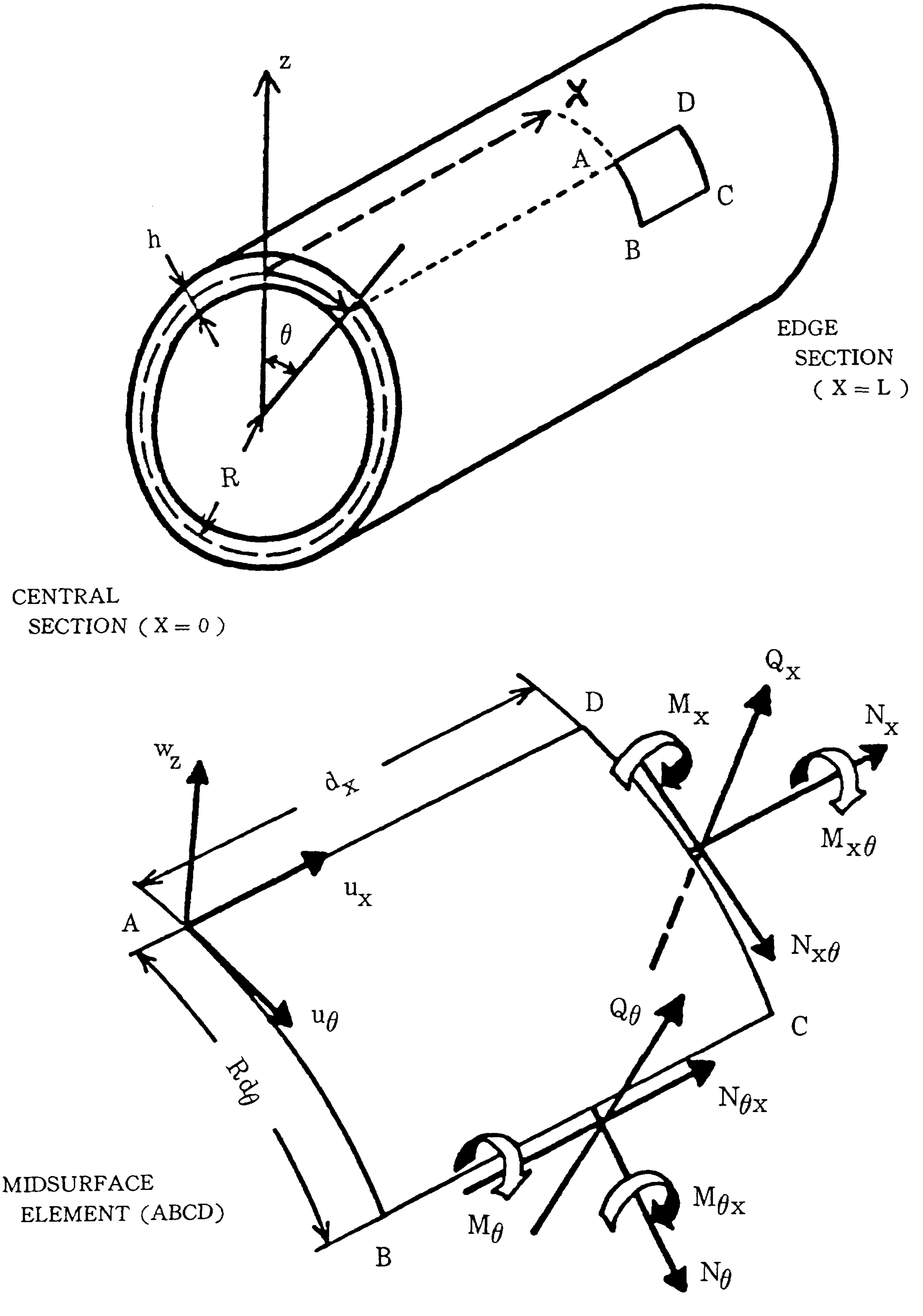

Fig. 1 Shell Geometries, Coordinates, and Positive Direction of Vectors

	E_1	E_2	E_3	E_4	E_5	E_6	D_1	D_2	D_3	D_4
Flügge	1	1	1	1	1	1	1	1	1	0
Naghdi	1	0	1	1	1	1	1	1	1	0
Koiter	0	0	0	1	0	1	0	0	0	0
Novozhilov	0	0	2	2	0	0	0	1	-1	0
Love-Reissner	0	0	0	0	0	0	0	0	0	1

Table 1 Values of Theory Indicators E_i and D_j

k	Theory	n = 2	n = 4	n = 6
5	Flügge	6.4017×10^{-1}	5.5199×10^{-1}	9.8501×10^{-1}
	Naghdi	6.4015	5.5208	9.8531
	K-S	6.3896	5.5162	9.8587
	Novozhilov	6.3955	5.5259	9.8689
	L-R	6.4138	5.5437	9.8824
	1st approx.	6.7894	5.7850	10.5894
	Donnell	6.1352	5.8195	10.3666
10	Flügge	4.4855×10^{-1}	1.9126×10^{-1}	1.3289×10^{-1}
	Naghdi	4.4855	1.9127	1.3291
	K-S	4.4849	1.9123	1.3290
	Novozhilov	4.4852	1.9129	1.3296
	L-R	4.4862	1.9140	1.3308
	1st approx.	4.6682	1.9370	1.3370
	Donnell	4.2408	1.9645	1.3724
50	Flügge	4.2972×10^{-1}	1.5996×10^{-1}	7.1422×10^{-1}
	Naghdi	4.2972	1.5996	7.1422
	K-S	4.2972	1.5996	7.1422
	Novozhilov	4.2972	1.5996	7.1422
	L-R	4.2972	1.5996	7.1423
	1st approx.	4.4623	1.6201	7.1608
	Donnell	4.1227	1.6360	7.2675

Table 2a Calculated Values of ω; Cl-Cl, $2\ell = 2.0$

k	Theory	n		
		2	4	6
	Flügge	1.0342×10^{-1}	1.1969×10^{-1}	4.8724×10^{-1}
	Naghdi	1.0343	1.1978	4.8739
	K-S	1.0349	1.1998	4.8798
5	Novozhilov	1.0359	1.2008	4.8805
	L-R	1.0373	1.2027	4.8797
	1st approx.	1.0670	1.2284	5.1435
	Donnell	1.1948	1.4036	5.2939
	Flügge	9.0892×10^{-2}	2.5576×10^{-2}	3.5921×10^{-2}
	Naghdi	9.0892	2.5581	3.5929
	K-S	9.0896	2.5588	3.5944
10	Novozhilov	9.0902	2.5595	3.5949
	L-R	9.0910	2.5610	3.5965
	1st approx.	9.3735	2.5655	3.6076
	Donnell	10.2538	2.7672	3.8557
	Flügge	8.8994×10^{-2}	1.8808×10^{-2}	5.6977×10^{-3}
	Naghdi	8.8994	1.8808	5.6978
	K-S	8.8994	1.8808	5.6978
50	Novozhilov	8.8994	1.8808	5.6978
	L-R	8.8994	1.8809	5.6978
	1st approx.	9.1985	1.8831	5.6984
	Donnell	10.0340	1.9773	5.8441

Table 2b Calculated Values of ω ; Cl - Cl, $2\ell = 5.0$

k	Theory	n		
		2	4	6
	Flügge	1.9994×10^{-2}	8.3687×10^{-2}	4.4462×10^{-1}
	Naghdi	2.0007	8.3726	4.4466
	K-S	2.0045	8.3847	4.4506
5	Novozhilov	2.0061	8.3858	4.4506
	L-R	2.0095	8.3867	4.4481
	1st approx.	2.0177	8.5569	4.6791
	Donnell	2.6733	10.0420	4.8335
	Flügge	1.6308×10^{-2}	7.1834×10^{-3}	2.8300×10^{-2}
	Naghdi	1.6309	7.1859	2.8303
	K-S	1.6311	7.1902	2.8310
10	Novozhilov	1.6312	7.1909	2.8310
	L-R	1.6315	7.1949	2.8313
	1st approx.	1.6375	7.2045	2.8399
	Donnell	1.9716	8.3406	3.0716
	Flügge	1.5916×10^{-2}	2.0572×10^{-3}	5.2738×10^{-4}
	Naghdi	1.5916	2.0572	5.2738
	K-S	1.5916	2.0572	5.2739
50	Novozhilov	1.5916	2.0572	5.2739
	L-R	1.5916	2.0572	5.2739
	1st approx.	1.5983	2.0573	5.2740
	Donnell	1.9081	2.1793	5.4427

Table 2c Calculated Values of ω ; Cl - Cl, $2\ell = 10.0$

k	Theory	n = 2	n = 4	n = 6
	Flügge	4.8260 × 10^{-1}	3.3610 × 10^{-1}	7.4732 × 10^{-1}
	Naghdi	4.8261	3.3648	7.4808
	K-S	4.8201	3.3705	7.4989
5	Novozhilov	4.8257	3.3773	7.5039
	L-R	4.8388	3.3904	7.5131
	1st approx.	4.8891	3.4922	8.0195
	Donnell	4.4806	3.5710	7.9458
	Flügge	4.2884 × 10^{-1}	1.3842 × 10^{-1}	8.5116 × 10^{-1}
	Naghdi	4.2884	1.3844	8.5158
	K-S	4.2880	1.3846	8.5222
10	Novozhilov	4.2883	1.3850	8.5254
	L-R	4.2892	1.3860	8.5352
	1st approx.	4.2916	1.3878	8.5646
	Donnell	3.9265	1.4023	8.8502
	Flügge	4.2525 × 10^{-1}	1.2523 × 10^{-1}	4.0954 × 10^{-2}
	Naghdi	4.2525	1.2523	4.0954
	K-S	4.2525	1.2523	4.0954
50	Novozhilov	4.2525	1.2523	4.0954
	L-R	4.2525	1.2523	4.0954
	1st approx.	4.2525	1.2523	4.0954
	Donnell	3.8896	1.2580	4.1506

Table 3a Calculated Values of ω; S1 - S1, $2\ell = 2.0$

k	Theory	n = 2	n = 4	n = 6
	Flügge	6.3306 × 10^{-2}	9.9687 × 10^{-2}	4.7341 × 10^{-1}
	Naghdi	6.3360	9.9835	4.7360
	K-S	6.3508	10.0158	4.7428
5	Novozhilov	6.3563	10.0191	4.7429
	L-R	6.3694	10.0354	4.7418
	1st approx.	6.3840	10.2593	4.9979
	Donnell	7.1225	11.7664	5.1358
	Flügge	5.7928 × 10^{-2}	1.2679 × 10^{-2}	3.1079 × 10^{-2}
	Naghdi	5.7931	1.2688	3.1089
	K-S	5.7940	1.2703	3.1109
10	Novozhilov	5.7944	1.2705	3.1110
	L-R	5.7952	1.2720	3.1125
	1st approx.	5.7959	1.2740	3.1227
	Donnell	6.2243	1.4114	3.3599
	Flügge	5.7570 × 10^{-2}	6.8811 × 10^{-3}	1.6083 × 10^{-3}
	Naghdi	5.7570	6.8811	1.6083
	K-S	5.7570	6.8811	1.6083
50	Novozhilov	5.7570	6.8811	1.6083
	L-R	5.7570	6.8812	1.6084
	1st approx.	5.7570	6.8812	1.6084
	Donnell	6.1646	7.2216	1.6516

Table 3b Calculated Values of ω; S1 - S1, $2\ell = 5.0$

k	Theory	n = 2	n = 4	n = 6
	Flügge	8.9301×10^{-3}	8.1255×10^{-2}	4.4303×10^{-1}
	Naghdi	8.9539	8.1299	4.4309
	K-S	9.0023	8.1428	4.4349
5	Novozhilov	9.0071	8.1430	4.4349
	L-R	9.0478	8.1434	4.4323
	1st approx.	9.0752	8.3101	4.6624
	Donnell	13.3884	9.7849	4.8174
	Flügge	5.9586×10^{-3}	5.5467×10^{-3}	2.7807×10^{-2}
	Naghdi	5.9601	5.5493	2.7810
	K-S	5.9630	5.5540	2.7816
10	Novozhilov	5.9633	5.5542	2.7816
	L-R	5.9659	5.5577	2.7819
	1st approx.	5.9670	5.5662	2.7904
	Donnell	7.2492	6.6077	3.0209
	Flügge	5.7607×10^{-3}	5.0368×10^{-4}	1.4846×10^{-4}
	Naghdi	5.7607	5.0368	1.4846
	K-S	5.7607	5.0369	1.4847
50	Novozhilov	5.7607	5.0369	1.4847
	L-R	5.7607	5.0369	1.4848
	1st approx.	5.7607	5.0370	1.4848
	Donnell	6.8406	5.3461	1.5512

Table 3c Calculated Values of ω; Sl - Sl , $2\ell = 10.0$

k	Theory	n = 2	n = 4	n = 6
	Flügge	1.3519×10^{-1}	1.2913×10^{-1}	4.9693×10^{-1}
	Naghdi	1.3525	1.2940	4.9729
	K-S	1.3524	1.2931	4.9750
5	Novozhilov	1.3537	1.2943	4.9758
	L-R	1.3577	1.3045	4.9850
	1st approx.	1.5596	1.3810	5.3252
	Donnell	1.7527	1.5260	5.4105
	Flügge	1.1768×10^{-1}	2.9578×10^{-2}	3.7201×10^{-2}
	Naghdi	1.1768	2.9593	3.7224
	K-S	1.1768	2.9583	3.7214
10	Novozhilov	1.1769	2.9590	3.7219
	L-R	1.1771	2.9654	3.7303
	1st approx	1.3416	3.0249	3.7776
	Donnell	1.4946	3.2598	4.0148
	Flügge	1.1437×10^{-1}	2.2205×10^{-2}	6.2541×10^{-3}
	Naghdi	1.1437	2.2205	6.2541
	K-S	1.1437	2.2205	6.2541
50	Novozhilov	1.1437	2.2205	6.2541
	L-R	1.1437	2.2205	6.2542
	1st approx.	1.3113	2.2528	6.2785
	Donnell	1.4566	2.3831	6.4523

Table 4a Calculated Values of ω; Cl - FR, $2\ell = 2.0$

k	Theory	n = 2	n = 4	n = 6
	Flügge	1.2819×10^{-2}	8.0898×10^{-2}	4.3813×10^{-1}
	Naghdi	1.2851	8.0960	4.3820
	K-S	1.2844	8.0993	4.3848
5	Novozhilov	1.2852	8.0999	4.3848
	L-R	1.2983	8.1160	4.3841
	1st approx.	1.3836	8.3797	4.6266
	Donnell	1.9618	9.2853	4.7007
	Flügge	9.6036×10^{-3}	5.9836×10^{-3}	2.7707×10^{-2}
	Naghdi	9.6056	5.9880	2.7712
	K-S	9.6050	5.9866	2.7711
10	Novozhilov	9.6055	5.9869	2.7711
	L-R	9.6136	6.0021	2.7726
	1st approx.	9.9945	6.0762	2.7896
	Donnell	12.6243	7.1172	3.0136
	Flügge	9.2882×10^{-3}	9.4723×10^{-4}	2.4953×10^{-4}
	Naghdi	9.2882	9.4723	2.4953
	K-S	9.2882	9.4723	2.4953
50	Novozhilov	9.2882	9.4723	2.4953
	L-R	9.2882	9.4726	2.4956
	1st approx.	9.6396	9.5079	2.4983
	Donnell	12.0289	10.1129	2.5931

Table 4b Calculated Values of ω ; Cl - FR, $2\ell = 5.0$

k	Theory	n = 2	n = 4	n = 6
	Flügge	3.5350×10^{-3}	7.7381×10^{-2}	4.5373×10^{-1}
	Naghdi	3.5471	7.7399	4.5385
	K-S	3.5446	7.7438	4.5430
5	Novozhilov	3.5452	7.7439	4.5431
	L-R	3.5889	7.7454	4.5421
	1st approx.	3.7770	7.9383	4.7894
	Donnell	5.5033	8.3226	4.9327
	Flügge	9.7008×10^{-4}	4.3361×10^{-3}	2.7133×10^{-2}
	Naghdi	9.7083	4.3304	2.7134
	K-S	9.7059	4.3303	2.7134
10	Novozhilov	9.7064	4.3302	2.7134
	L-R	9.7345	4.3129	2.7139
	1st approx.	9.9314	4.2350	2.7257
	Donnell	13.9738	5.2632	2.8257
	Flügge	7.9403×10^{-4}	7.2007×10^{-5}	5.6950×10^{-5}
	Naghdi	7.9403	7.2009	5.6952
	K-S	7.9403	7.2009	5.6951
50	Novozhilov	7.9403	7.2009	5.6951
	L-R	7.9403	7.2016	5.6959
	1st approx.	8.0399	7.2105	5.6993
	Donnell	10.0638	7.7732	6.1143

Table 4c Calculated Values of ω ; Cl - FR, $2\ell = 10.0$

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