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A Formulation of the Three-Dimensional Potential Flow Field around a Lifting Wing by Use of the Surface Velocity Components

— an Extension of the Prager-Vandrey-Martensen Procedure to the Three-dimensional Case —

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CONTENTS

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ABSTRACT

A Fredholm integral equation of the second kind for the surface velocity components of the three-dimensional incompressible potential flow around a body is obtained by extending the procedure explored by W. Prager, F. Vandrey and E. Martensen for the two-dimensional case. The formulation is based on the representation of the velocity potential by a doublet distribution over the body surface, and is realized by replacing the original surface boundary condition of the vanishing normal velocity component with the equivalent condition of the quiescent flow in the region inside the body.

The formulation is then generalized to cases where the normal velocity component to the body surface does not necessarily vanish identically so that it can be utilized in the boundary-layer-displacement-model procedure designed to account for viscous effects within the scope of the inviscid flow theory. This generalization is accomplished by combining the above-mentioned doublet distribution with a source distribution of prescribed strengths.

Since our formulation lacks redundancy in variables to take care of the Kutta condition at the trailing edge of a lifting wing, the implication of this condition in our formulation is studied by examining the behaviour of our basic integral equation at the trailing edge of a wing. It is found that the geometry of the trailing vortex sheet and the direction of vortex shedding in the immediate neighbourhood of the trailing edge bear essential relations to the fulfilment of the Kutta condition. The classical Prandtl model of the trailing vortex sheet is, in principle, not adequate for our formulation.

物体表面に沿う流速を未知函数とする積分方程式による三次元ポテンシャル流れの新らしい定式化

概　　要

三次元翼のまわりのポテンシャル流れを計算する方法の一つとして，翼表面に沿う速度成分を未知函数とする積分方程式を導き，その性質を調べた。この方程式は，翼のまわりの流れ場を翼表面に分布する二重湧出しで表現し，流れの境界条件として，翼表面に垂直な速度成分の代りに，それと等価な，翼内部の流れに関する条件を利用して得られるもので，二次元翼の場合に対するPrager，Vandrey，Martensen などによる手法を三次元流れに拡張したものである。

揚力のある翼の場合の翼後縁におけるKuttaの条件の重要性に鑑み，この定式化におけるKuttaの条件の意味を調べ，後流渦面の翼後縁における形状とその上の渦の分布の様式がこの条件の成立に密接に関係していることを明らかにした。

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List of Symbols

A	vector potential due to vortex distribution
B	metric coefficient in the β-direction
C	aerofoil contour
C^+	outer (positive) side of C
C^-	inner (negative) side of C
Dμ	gradient of the doublet strength
i	unit vector in the x-direction
j	unit vector in the y-direction, or vortex vector
k	unit vector in the z-direction
n	unit normal vector
q	velocity vector in the flow field
q_L	velocity at the trailing edge on the lower side of the trailing vortex sheet
q_m	mean velocity on the trailing vortex sheet
q_U	velocity at the trailing edge on the upper side of the trailing vortex sheet
q^+	velocity on the upper side of the trailing vortex sheet
q^-	velocity on the lower side of the trailing vortex sheet
R	non-singular part in $\nabla\varphi_D(P)$ as P approaches T
r_{PQ}	distance between the points P and Q
S_L	wing lower surface
S_U	wing upper surface
S_W	trailing vortex sheet
S^+	positive side of a surface S
S^-	negative side of a surface S
T	point on the trailing edge
t	length along the trailing edge
t	unit tangential vector
V	total velocity along the wing surface
v	disturbance velocity along the wing surface
$\mathbf{V}_m$	mean velocity at the trailing edge
$\mathbf{V}_\infty$	free-stream velocity
α	one of the surface coordinates on S
β	the other of the surface coordinates on S, or 'yaw' angle of the vortex vector at the trailing edge
Γ	circulation
δ_T	trailing-edge angle
ξ, η, ζ	coordinates of the variable point Q of a surface integral
θ_U	angle between the wing upper surface and the trailing vortex sheet
θ_L	angle between the wing lower surface and the trailing vortex sheet
$\boldsymbol{\lambda}$	vortex vector
Λ_t	trailing edge sweep angle
μ	doublet strength
ν	coordinate normal to the surface S, or index of Hölder-continuity
ρ	distance between the two points P and T
σ	source strength
ϕ	total potential
ϕ_∞	free-stream potential
φ	perturbation potential, or angle of approaching path against a reference direction
φ_U	approaching-path angle measured from the x_U-axis
φ_L	approaching-path angle measured from the x_L-axis
φ_W	approaching-path angle measured from the x_W-axis
ψ	perturbed component of stream function
ψ_∞	free-stream component of stream function
$\boldsymbol{\psi}$	vector potential of the flow field

1. INTRODUCTION

A standard technique in the subsonic panel method for a lifting wing is to use a combination of sources and doublets. That is, the sources are distributed over the wing surface to represent the displacement effects of the wing while the doublets are placed on the trailing vortex sheet in order to model the vortex layer downstream of the wing, which is inherent in the lifting flow field. The existing representative codes of the subsonic panel method, e.g. those of Rubbert & Saaris,[1), 2)] Hess,[3), 4)] Loeve et al.[5), 6)] and Kraus & Sacher[7)] are all implemented by using this combinations of the surface singularity distributions.

This combination may be the best one among the possible formulations in the method of surface distribution of singularities as long as one is concerned with the practical performance of a numerical procedure. It may not, however, be particularly convenient one to use when one wishes to analyse a detailed nature of the flow field, e.g. the implication of the Kutta condition at the trailing edge of a wing.

Needless to say, a proper implementation of the Kutta condition is of utmost importance when one deals with the potential flow field around a lifting wing. It seems that the panel method, in its status quo, still leaves much to be done in this respect. For instance, calculation results by the existing methods may depend on the distance from the trailing edge to the points at which the Kutta condition is actually applied. With a formulation where a source distribution over the wing surface is employed, the analysis of the Kutta condition is unwieldy because the source strength bears no tangible relations to the flow velocity, in terms of which the Kutta condition is stated.

In contrast, the analysis of this kind sould be much facilitated if the working equation is written in terms of the flow velocity along the wing surface. In fact, this can be realized through a formulation in which doublets, instead of sources, are distributed over the wing surface as well as on the trailing vortex sheet. This exclusive use of doublets leads to an integral equation for the surface velocity, and thus offers a convenient basis to examine the flow features near the trailing edge of a wing or the wing-fuselage juncture, where the conventional source-doublet approach needs to engage the problem of how to dispose of the doublet sheet.

The formulation in terms of the surface flow velocity probably was first attained by W. Prager[8) for the two-dimensional flow around an aerofoil, and by F. Vandrey[9) for the three-dimensional flow around an axisymmetric body at zero incidence. Both authors employed a vortex distribution over the body surface (a vortex distribution is always equivalent to a doublet distribution, see Section 2.2 of the present report). Imagining a fictitious flow field for the region inside the body, and combining this fictitious flow with the real flow outside the body, they found that the vortex strength is equal to the tangential velocity of the flow along the body surface, and that a Fredholm integral equation of the second kind for the vortex strength is obtained by taking advantage of the fact that the fictitious flow inside the body is in fact stationary. The Prager's formulation was reinterpreted and extended to the case of two-dimensional cascades by E. Martensen.[10)]

As for the general three-dimensional case, two equivalent formulations have been established by R. Kress[11)] and E. Grodtkjaer.[12)] The former author addressed the problem of expressing the given vector field in terms of vortex distributions over the boundaries as well as within the region for which the vector field is defined (this problem is called the 'Prager problem'). Applying the solution method for this problem to the case of potential flow around a body, Kress then formulated an integral equation to be satisfied by the vortex distribution over the body surface.

On the other hand, Grodtkjær achieved the formulation by proceeding in a rather straightforward manner: starting from the expression of the derivatives of the velocity potential at a point in the flow field by use of the Green's identity, he managed to express the integrand in terms of the known quantities on the body surface together with the derivatives of the velocity potential themselves. By tending then the point under consideration onto the body surface, integral equations are obtained for the derivatives of the velocity potential along the body surface.

Not having noticed these preceding works, the present author[13)] explored the path laid by Prager, Vandrey and Martensen to arrive at the identical formulation for the general three-dimensional potential flow field. The present report intends to describe how the formulation is achieved since our approach involves features which are rather different from, or not identified in, those of Kress and Grodtkjaer, and hence may merit a record.

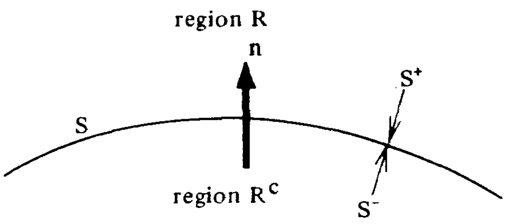
Thus, after presenting in Section 2 preliminary materials for the later discussion, the extension of the Prager-Vandrey-Martensen procedure to the general three-dimensional case is expounded in Section 3 together with a description of the essential features of Kress' and Grodtkjær's approaches. In Section 4, then, some of the other relevant formulations for the potential flow problem are discussed using materials obtained in the preceding sections. Specifically, the basic integral equation for the surface velocity components is extended so as to embrace cases with the boundary condition of not necessarily vanishing normal velocity component. The ensuing section is devoted to a description of a limitation inherent in our formulation which is manifested by considering the process of reducing the wing thickness indefinitely. It is shown that our formulation leads to the basic equation of the classical lifting surface theory in this limiting process. In the final section, the implication of the Kutta condition at the wing trailing edge is explored within the framework of the present formulation by examining the behaviour of our integral equation at the trailing edge of a wing.

2. PRELIMINARIES

2.1. Green's identities

Since the Green's identities will be frequently referred to in the subsequent sections, they are written out here for the sake of convenience together with a few relevant notations.

Let R be a region in three-dimensional space, either bounded or unbounded, whose boundary is a closed surface S. Let R^c be the region complementary to R in the sense that $R + R^c + S$ constitutes the whole space. Let us distinguish between the positive- and the negative sides of S : the positive side, denoted by S^+, is the side of S which faces the region R whereas the negative side, denoted by S^-, is the other side of S, which faces the region R^c, see Sketch 1. Let, further, **n** be the unit normal vector to S whose sense is from S^- to S^+, and $\partial/\partial n$ denote the differentiation in the direction of **n**.



Sketch 1.
Notations about Regions and Surfaces

Now, let $\phi(P)$ and $\psi(P)$ be any functions of a point P(x, y, z) in space with sufficiently analytic character (for a precise description, see, e.g. Refs. 14 or 15) in R and on S. When R is bounded, the following identities (Green's identities) hold;

First Identity:

$$\int_R (\psi\nabla^2\phi + \nabla\psi \cdot \nabla\phi)\, dV = -\int_{S^+} \psi \frac{\partial\phi}{\partial n}\, dS, \tag{2.1}$$

Second Identity:

$$\int_R (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \int_{S^+} (\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}) dS, \tag{2.2}$$

Third Identity:

$$\phi(P) = -\frac{1}{4\pi}\int_R \nabla^2\phi \frac{1}{r_{PQ}}\, dV_Q + \frac{1}{4\pi}\int_{S^+} [\phi\frac{\partial}{\partial n}(\frac{1}{r_{PQ}}) - \frac{\partial\phi}{\partial n}\frac{1}{r_{PQ}}]\, dS_Q, \tag{2.3}$$

where P is a point in R, r_{PQ} is the distance from P to the integration point Q in R or on S as the case may be, and $\nabla\phi$ indicates the gradient of ϕ:

$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}).$$

The Green's identities hold true, too, when R is extending to infinity if ϕ and ψ are regular at infinity, i.e. if $r\phi$, $r\psi$, $r^2\nabla\phi$ and $r^2\nabla\psi$ are bounded when $r = \sqrt{x^2 + y^2 + z^2}$ goes to infinity.

In regards to Third Identity, one notes that the integrand in the surface integral on the right-hand side refers to the side of S which faces R, i.e. to S^+. When the integrand is given the values associated with the opposite side S^- (and the differentiation $\partial/\partial n$ is interpreted in the opposite direction to that understood with (2.3), i.e. $\partial/\partial n$ is taken in the direction from S^+ to S^-), then the left-hand side must be put equal to zero as long as P remains in R. To the same thing, the left-hand side of (2.3) must be replaced with zero when P is chosen to be in R^c:

$$0 = -\frac{1}{4\pi}\int_R \nabla^2\phi \frac{1}{r_{PQ}} dV_Q + \frac{1}{4\pi}\int_{S^+}[\phi \frac{\partial}{\partial n}(\frac{1}{r_{PQ}}) - \frac{\partial\phi}{\partial n}\frac{1}{r_{PQ}}]dS_Q. \tag{2.4}$$

2.2. Derivatives of the Potential due to a Surface Distribution of Sources and Doublets

The potential flow field around a three-dimentional body is represented by a combination of sources and doublets distributed over the relevant boundaries. The strengths of sources and/or doublets are determined from the boundary conditions. In applying the boundary conditions, expressions of the derivatives of the potential are needed at points on the boundaries upon which sources or doublets are distributed. The material given below concerning to these expressions is an abridged reiteration of a part of the analysis reported in Ref. 16.

First, the potential of source distribution is considered. The potential φ_S due to a source distribution over a surface S is defined by

$$\varphi_S(P) = \frac{1}{4\pi}\int_S \sigma(Q)\frac{1}{r_{PQ}} dS_Q \tag{2.5}$$

The derivatives of φ_S at a point Q_0 on S are required in order to write down the Neumann-type boundary condition on S. Invoking the concept of the Cauchy integral, they are given[14), 15)] as follows:

$$[\nabla_P\varphi_S(P)]_{P\to Q_0} = \mp\frac{1}{2}\sigma(Q_0)\mathbf{n} + \frac{1}{4\pi}\oint_S \sigma(Q)[\nabla_P(\frac{1}{r_{PQ}})]_{P=Q_0} dS_Q \tag{2.6}$$

where ∇_P indicates the gradient operator at the point P and the symbol $\oint$ denotes the Cauchy integral (a definition of Cauchy integral has been given at the beginning of Appendix I). The double sign preceding the first term on the right-hand side is due to the dependence of the results on the path along which the point Q_0 is approached from outside of S: the upper sign is taken when Q_0 is approached from the region into which the normal **n** is directed, i.e. the region facing the positive side of S, and the lower sign is taken when Q_0 is approached from the region facing the negative side of S.

Next, the potential φ_D due to a doublet distribution over a surface S is given as

$$\varphi_D(P) = \frac{1}{4\pi}\int_S \mu(Q)\frac{\partial}{\partial n_Q}(\frac{1}{r_{PQ}})dS_Q = \frac{1}{4\pi}\int_S \mu(Q)\mathbf{n}_Q\cdot\nabla_Q(\frac{1}{r_{PQ}})dS_Q. \tag{2.7}$$

Differentiation of the right-hand side of (2.7) under the integral sign to obtain $\nabla\varphi_D$ is permissible as long as the point P is away from S. When P approaches a point Q_0 lying on S, the integrand thus differentiated exhibits a singularity of order of r^{-3}, which is not integrable even in the sense of the Cauchy integral. This difficulty is circumvented by integration by parts which is carried out in two steps as shown below.

In the first step, a rearrangement of terms appearing in the integrand for $\nabla\varphi_D$ by use of the fact that $\nabla^2(1/r)$ identically vanishes leads to the relation[16)]

$$\nabla\varphi_D = \nabla \times \psi \tag{2.8}$$

with

$$\psi(P) = -\frac{1}{4\pi}\int_S \mu(Q)\mathbf{n}\times\nabla_Q(\frac{1}{r_{PQ}})dS_Q. \tag{2.9}$$

This relation states the fact that the velocity field due to a doublet distribution over the surface S is expressible either by the scalar potential φ_D or the vector potential ψ (cf. Ref.17, p.47). Since the kernel function $\mathbf{n} \times \nabla_Q(r_{PQ}^{-1})$ in ψ represents an operation confined within the surface S, the differential operator ∇_Q can be transferred from r_{PQ}^{-1} to $\mu(Q)$ by integration by part.

The second step is to perform this integration by part after rewriting the integrand of ψ in terms of a surface coordinate system on S. The result is[16)]

$$\psi(P) = \frac{1}{4\pi}\int_S \frac{\lambda(Q)}{r_{PQ}}\, dS_Q - \frac{1}{4\pi}\oint_{\partial S} \frac{\mu(Q)}{r_{PQ}}\, \mathbf{t}(Q)\, ds_Q \qquad (2.10)$$

where λ is explained shortly later, ∂S is the boundary curve of S if any, and $\mathbf{t}(Q)$ is the unit tangent vector to ∂S at the point Q on it pointing to the direction of integration along ∂S. The second integral in (2.10) refers to the line integral along ∂S, and the symbol $\oint$ denotes that the sense of the integral is in the direction such that $\mathbf{n} \times \mathbf{t}$ always falls on S. In order to give the expression of λ, let us define the gradient of the doublet strength μ within the surface S, which is denoted as $\mathbf{D}\mu$. Expressing by (α, β) an orthogonal surface coordinate system on S, a three-dimensional coordinate system (α, β, ν) is established by supplementing to (α, β) the third coordinate ν which designates a distance along the normal to S. Then $\mathbf{D}\mu$ is defined in this system as the vector

$$\mathbf{D}\mu = \left(\frac{1}{A}\frac{\partial\mu}{\partial\alpha}, \frac{1}{B}\frac{\partial\mu}{\partial\beta}, 0\right) \qquad (2.11)$$

where A and B are the metric coefficients in respective coordinate directions. Now λ is given as

$$\lambda = \mathbf{n} \times \mathbf{D}\mu. \qquad (2.12)$$

When a velocity field is expressed as $\nabla \times \psi$ with ψ given by (2.10), then such velocity field is said to have been induced by the surface vortex distribution of strength λ over S and the line vortex distribution of strength μ along ∂S. Reiteraing the equivalence between the two expressions (2.9) and (2.10) for ψ, one can states that the velocity field due to the doublet distribution of strength μ over S is identical with that due to the surface vortex distribution of strength λ over S plus the line vortex distribution of strength μ along ∂S, where λ is related to μ via the equations (2.11) and (2.12). It is noted here that by definition λ is aligned with the curves μ = const on S. This equivalence rule, though independently reached by the present author,[16)] has been known in the field of electrodynamics as the Theorem of Ampere (cf. Ref. 17, p.54).

Now one sees from (2.8) and (2.10) that $\nabla\varphi_D$ can be expressed in a form in which the integrand possesses a singularity of order of r^{-2}, which is Cauchy-integrable. Restricting ourselves to cases where S is a closed surface whereby the line integral on the right-hand side of (2.10) is absent, the operation analogous to that which was used in deriving (2.6) from (2.5) is applied to ψ of (2.10) and leads to the following formula for $\nabla\varphi_D$ at a point Q_0 on S:

$$[\nabla_P \varphi_D(P)]_{P\to Q_0} = \pm\frac{1}{2}\mathbf{D}\mu + \frac{1}{4\pi}\oint_S (\mathbf{D}\mu \times \mathbf{n})_Q \times [\nabla_P(\frac{1}{r_{PQ}})]_{P\to Q_0}\, dS_Q \qquad (2.13)$$

where the double sign preceding the first term on the right-hand side designates the same convention as used with the expression (2.6), viz. the upper sign (+) is meant when P approaches Q_0 from the region facing the positive side of S, while the lower sign (−) is taken when P approaches Q_0 remaining in the region facing the negative side of S. It should be noticed that the formulae (2.6) and (2.13) presuppose certain regularity conditions on the geometry of S and the density $\sigma(Q)$ or $\mathbf{D}\mu(Q)$.

For instance, S with continuous curvature and the density with Hölder-continuity (a function $\alpha(Q)$ is said to be Hölder-continuous at Q if there exist two positive constants C and ν such that

$$|\alpha(P) - \alpha(Q)| < C\, r_{PQ}^{\nu}$$

for P sufficiently close to Q, where r_{PQ} is a distance between P and Q) are sufficient as these conditions, cf. Refs. 14 and 15 for a more detailed discussion.

3. EXTENSION OF THE PRAGER-VANDREY-MARTENSEN PROCEDURE TO THE GENERAL THREE-DIMENSIONAL CASE

The potential flow of incompressible fluid around a body is described by the Laplace equation for the velocity potential ϕ with the boundary condition of zero normal derivative on the body surface. Restricting ourselves to the case where the body is immersed in an oncoming potential flow characterized by the potential ϕ_∞, the disturbance potential φ due to the presence of the body defined as $\varphi = \phi - \phi_\infty$ can be expressed by a distribution of sources and doublets over the boundary surface S:

$$\varphi(P) = \frac{1}{4\pi}\int_S [\mu(Q)\frac{\partial}{\partial n}(\frac{1}{r_{PQ}}) - \sigma(Q)\frac{1}{r_{PQ}}]\, dS_Q. \qquad (3.1)$$

Since the right-hand side of this expression always satisfies the Laplace equation, the problem is reduced to the determination of the doublet strength μ and the source strength σ so that φ satisfies the prescribed boundary condition on S. The description of this boundary condition by use of the expression (3.1) leads to an integral equation for μ and σ.

Since the boundary condition on the body surface provides only one equation for the two variables μ and σ, the formulation thus far is redundant unless one specifies another condition to be satisfied by μ and/or σ. This additional condition can be freely chosen so that the formulation fits in with specific purposes of the analysis in hand. Thus, quite a few formulations of the potential flow problem already exist, and a discussion about them will be given in the next section.

Our formulation assumes the doublet distribution only. As is seen from the expression (2.13) for the surface derivatives of the potential due to a doublet distribution, a straightforward description of the boundary condition $\partial(\varphi + \phi_\infty)/\partial n = 0$ on S leads to an integral equation of the first kind for the gradient of the doublet strength $D\mu$. On the other hand, the approach explored by Prager[8] admits a formulation that leads to an integral equation of the second kind for $D\mu$. Moreover, $D\mu$ is identified in the course of the analysis with the tangential component of the flow velocity along the body surface.

In the present Section, this approach is described for the general three-dimensional situation together with the two precedent analyses, viz. those by Kress[11] and Grodtkjær,[12] who reached the identical formulation through reasonings somewhat different from ours.

3.1. Prager's Approach

According to Green's third identity (2.3), a harmonic function φ in the region bounded by a closed surface S is expressed as

$$\varphi(P) = \frac{1}{4\pi}\int_{S^+} [\varphi\frac{\partial}{\partial n}(\frac{1}{r_{PQ}}) - \frac{\partial\varphi}{\partial n}\frac{1}{r_{PQ}}]\, dS_Q \qquad (3.2)$$

for P(x, y, z) lying in the region R which faces the positive side of S. When the region R includes the point at infinity, then φ is required to satisfy the condition that both $r\varphi$ and $r^2\nabla\varphi$ are bounded when $r = x^2 + y^2 + z^2$ goes to infinity. It is known that the disturbance velocity potential of the flow around a body submerged in a uniform oncoming stream can be found in the class of harmonic functions satisfying this regularity condition at infinity.

Now, W. Prager[8] employed a stream function in lieu of the velocity potential to express the flow around a two-dimensional aerofoil. Let the stream function be given by a sum of ψ_∞ and ψ, the former representing the oncoming flow while the latter being associated with the disturbance due to the presence of the aerofoil. The boundary condition to be satisfied by ψ is

$$\psi + \psi_\infty = 0 \tag{3.3}$$

along the aerofoil contour C, or to be more precise, along the side of the contour which faces the flow field outside the aerofoil. This side of C is denoted as C^+. The other side of C, i.e. the side touching with the region inside the aerofoil is denoted by C^-.

Applying Green's third identity to ψ, one obtains

$$\psi(P) = \frac{1}{2\pi}\int_{C^+}[\psi\frac{\partial}{\partial n_Q}\log(\frac{1}{r_{PQ}}) - \frac{\partial\psi}{\partial n_Q}\log(\frac{1}{r_{PQ}})]\,ds_Q \tag{3.4}$$

for P lying in the flow field. On the other hand, applying Green's third identity in the form of (2.4) to the oncoming flow stream function ψ_∞ which is harmonic throughout the whole space results in

$$0 = \frac{1}{2\pi}\int_{C^-}[-\psi_\infty(Q)\frac{\partial}{\partial n_Q}\log(\frac{1}{r_{PQ}}) + \frac{\partial\psi_\infty}{\partial n_Q}\log(\frac{1}{r_{PQ}})]\,ds_Q \tag{3.5}$$

Note that the differentiation $\partial/\partial n$ is in the direction of the normal **n** to C whose sense is from C^- to C^+. Since ψ_∞ together with its derivatives is continuous across C, the contour integral in (3.5) can be reinterpreted as the one along C^+.

By definition, the tangential velocity component V_t of the flow along the aerofoil surface is related to ψ and ψ_∞ via

$$V_t = -\frac{\partial}{\partial n}(\psi + \psi_\infty) \text{ along } C^+. \tag{3.6}$$

Combining the equations (3.4) and (3.5), and using the relations (3.3) and (3.6), one obtains the following expression for ψ:

$$\psi(P) = \frac{1}{2\pi}\int_{C^+} V_t(Q)\log(\frac{1}{r_{PQ}})\,ds_Q \tag{3.7}$$

which expresses the fact that the stream function of the two-dimensional flow around an aerofoil can be expressed by the vortex distribution over the aerofoil contour C whose strength is equal to the tangential velocity component V_t along C.

How is this vortex strength determined? An equation for V_t is obtained by substituting (3.7) into the boundary condition (3.3). This yields an integral equation of the first kind. Prager, however, gave an integral equation of the second kind by inserting (3.7) instead into (3.6). Since, analogous to (2.6), one possesses the expression

$$\frac{\partial\psi}{\partial n}(Q_0) = \mp\frac{1}{2}V_t(Q_0) + \frac{1}{2\pi}\int_C V_t(Q)K(Q_0,Q)\,ds_Q \tag{3.8}$$

for a point Q_0 lying on C where

$$K(Q_0,Q) = [\frac{\partial}{\partial n_P}\log(\frac{1}{r_{PQ}})]_{P=Q_0},$$

the condition (3.6) leads to the following equation for V_t:

$$V_t(Q_0) + \frac{1}{\pi}\int_C V_t(Q)K(Q_0,Q)\,ds_Q = -2\frac{\partial\psi_\infty}{\partial n}. \tag{3.9}$$

The question whether ψ expressed by (3.7) with V_t given by the solution of (3.9) satisfies the boundary condition (3.3) sets up an essential step in our formulation, and will be addressed shortly.

3.2. Martensen's Reasoning

Prager's stream-function method cannot be readily extended to the general three-dimensional case. However, reinterpretation of the Prager's approach due to E. Martensen[10] offers a basis on which an extension to the three-dimensional case is made most easily.

Assume that the stream function (3.7) employing as V_t the solution of (3.9) satisfies the boundary conditions (3.3). Martensen asks the key question : what kind of flow does this stream function describe for the region inside of C? Since a harmonic function of single layer distribution over a boundary exemplified by ψ given as (3.7) is continuous across the boundary (cf., e.g. Reg.15, p.259), the condition (3.3) holds on C^- whenever it holds on C^+. According to the maximum principle for a harmonic function (cf., e.g. Reg. 15, p. 255), then, $\psi + \psi_\infty$ vanishes not only along C^- but also throughout the whole region inside of C. Thus, the answer to the question posed above is: combined with ψ_∞, the stream function ψ given by (3.7) represents a stationary flow in the region inside of C provided it satisfies the boundary condition (3.3). As a consequence of this fact, the normal derivative $\partial(\psi + \psi_\infty)/\partial n$ evaluated on C^- vanishes identically because the derivatives of ψ are continuous in the region up to C^-(cf., e.g. Ref.14, p.165) :

$$\frac{\partial \psi}{\partial n} + \frac{\partial \psi_\infty}{\partial n} = 0 \quad \text{along } C^- \qquad (3.10)$$

Then, in view of the property of $\partial\psi/\partial n$ on C^+ and C^- which is expressed by (3.8) (the minus sign for C^+ and the plus sign for C^- are meant in the double sign preceding $V_t/2$ in (3.8)), one sees that the condition (3.10) is equivalent to the condition (3.6) (because $(\partial\psi/\partial n)_{C^-} = V_t + (\partial\psi/\partial n)_{C^+}$) and hence to (3.9).

Now, the proof that the stream function ψ of (3.7) with V_t given as the solution of (3.9) satisfies the boundary condition (3.3) is carried out by proceeding in the reverse direction.
By combining (3.9) and (3.8), firstly one sees that ψ satisfies the condition (3.10). Secondly one observes that a specific form of Green's first identity applicable to a harmonic function ϕ :

$$\int_R (\nabla\phi)^2 \, dV = \int_{S^+} \phi \frac{\partial \phi}{\partial n} \, dS \qquad (3.11)$$

implies that ϕ is constant throughout R when $\partial\phi/\partial n$ identically vanishes on S. Then applying this theorem to $\psi + \psi_\infty$, one realizes that $\psi + \psi_\infty$ is constant throughout the region inside of C, which leads to the condition (3.3) by virtue of the continuity of ψ up to and across C.

3.3 Extension to the Three-Dimensional Case

Martensen's contribution is to have pointed out that the condition (3.3) and (3.10) are equivalent with each other, and that the equation (3.9) is an alias of the condition (3.10). That the stream function given by (3.7) possesses such properties as these hinges on the fact that the boundary condition (3.3) is continuous across the boundary C when ψ is given as (3.7).

Once this point is noted, extension of Prager's formulation to the general three-dimensional case is readily carried out in terms of a velocity potential as follows. Consider the potential flow around a body whose surface is denoted as S. Using a velocity potential ϕ to describe the flow, the boundary condition on S is given as

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } S^+ \qquad (3.12)$$

where S^+ denotes the outer (positive) side of S which faces the flow field. Let us assume that the potential ϕ is expressed in such a manner that the normal derivative $\partial\phi/\partial n$ on S is continuous across S. Then the condition $\partial\phi/\partial n = 0$ also holds on S^-, where S^- is the inner side of S. Since the region inside the body is a bounded region, Green's first identity (3.11) is applicable to ϕ, assuming ϕ is also harmonic in this region. The result is the fact that ϕ is constant throughout this region, and, specifically, a tangential derivative $\partial\phi/\partial t$ along S^- vanishes identically provided the derivatives of ϕ are continuous in this region up to S^- :

$$\frac{\partial\phi}{\partial t} = 0 \quad \text{on} \quad S^-. \qquad (3.13)$$

That is, the condition (3.13) is derived from the condition (3.12) on the assumptions that

A.1. $\partial\phi/\partial n$ is continuous across S,

A.2. ϕ is also harmonic in the region inside of S, and

A.3. the derivatives of ϕ are continuous in the region inside of S up to S^-.

Conversely, it is shown readily that the condition (3.12) is a consequence of the condition (3.13) when the conditions A.1 through A.3 are satisfied. Because: (1) the condition (3.13) implies that ϕ is constant over S^-, (2) the maximum principle of a harmonic function then indicates that ϕ is constant throughout the region inside of S, (3), in particular, $\partial\phi/\partial n = 0$ on S^- provided the derivatives of ϕ are continuous in the region up to S^-, and (4) the condition (3.12) is thus arrived at since $\partial\phi/\partial n$ is assumed to be continuous across the surface S.

Now suppose that the potential ϕ is given by a sum of the free-stream potential ϕ_∞ and the disturbance potential φ_D which is expressed by a doublet distribution over S as was already given by (2.7):

$$\phi = \varphi_D + \phi_\infty \qquad (3.14)$$

Referring to the expression (2.13) for the surface values of the derivatives of φ_D, the condition (3.12) is then expressed as

$$\frac{1}{4\pi}\mathbf{n}(Q_0)\cdot\oint_S(\mathbf{D}\mu\times\mathbf{n})_Q\times\nabla_{Q_0}\left(\frac{1}{r_{Q_0Q}}\right)dS_Q + \frac{\partial\phi_\infty}{\partial n}(Q_0) = 0 \qquad (3.15)$$

for Q_0 lying on S, whereas the condition (3.13) reads

$$-\frac{1}{2}\mathbf{D}\mu\cdot\mathbf{t} + \frac{1}{4\pi}\mathbf{t}\cdot\oint_S(\mathbf{D}\mu\times\mathbf{n})_Q \times\nabla_{Q_0}\left(\frac{1}{r_{Q_0Q}}\right)dS_Q + \frac{\partial\phi_\infty}{\partial t} = 0 \qquad (3.16)$$

where t designates the unit vector aligned with the direction of a tangential differentiation $\partial/\partial t$ at Q_0. The former is an integral equation of the first kind for $\mathbf{D}\mu$ while the latter is the one of the second kind. Since the potential ϕ given by (3.14) satisfies the conditions A.1 through A.3, these two integral equations are equivalent to each other.

The tangential component of the flow velocity along the surface S is given by

$$V_t = \mathbf{t}\cdot\nabla(\varphi_D + \phi_\infty) \quad \text{on} \quad S^+.$$

Utilizing the expression (2.13), one obtains the following expression for V_t:

$$V_t(Q_0) = \frac{1}{2}\mathbf{D}\mu\cdot\mathbf{t} + \frac{1}{4\pi}\mathbf{t}\cdot\oint_S(\mathbf{D}\mu\times\mathbf{n})_Q \times\nabla_{Q_0}\left(\frac{1}{r_{Q_0Q}}\right)dS_Q + \frac{\partial\phi_\infty}{\partial t} \qquad (3.17)$$

which is combined with (3.16) to yield

$$V_t = \mathbf{D}\mu\cdot\mathbf{t}.$$

That is, the gradient $\mathbf{D}\mu$ is identical with the flow velocity along the surface $\mathbf{V}$:

$$\mathbf{D}\mu = \mathbf{V}. \qquad (3.18)$$

In other words, the equations (3.15) and (3.16) have been integral equations by which the flow velocity $\mathbf{V}$ along the body surface S can be determined.

Thus, unifying (3.15) and (3.16), the following integral equation for $\mathbf{V}$ is obtained:

$$\mathbf{V}(P) - \frac{1}{2\pi}\oint_S(\mathbf{V}\times\mathbf{n})_Q\times\nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q = 2\mathbf{V}_\infty \qquad (3.19)$$

where P denotes a point on S and $\mathbf{V}_\infty$ is the oncoming flow velocity.

It is emphasized here that the derivation of the equation (3.19) postulates the boundary condition (3.12), viz. the condition of vanishing normal component of flow velocity on S. It is desirable to obtain an equation for **V** of the type of (3.19) which is also valid for cases where non-vanishing normal component of flow velocity is prescribed on S. An obvious example of such cases is the formulation in which the viscous effects are accounted for within the realm of inviscid flow theory by expressing them as displacement effects upon the external flow. According to Lighthill[18], the displacement effect in this situation is equivalently represented as an additional normal velocity component at the outer edge of the boundary layer.

In fact, an extension of (3.19) to cases of non-vanishing normal velocity component can be realized and the resulting equation takes the form

$$\mathbf{V}(P) - \frac{1}{2\pi}\oint_S (\mathbf{V}\times\mathbf{n})_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q = 2\mathbf{V}_\infty - \frac{1}{2\pi}\oint_S (\mathbf{V}\cdot\mathbf{n})\nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q \tag{3.19a}$$

How this equation is derived will be explained in Section 4.2.

3.4. Grodtkjær's Approach[12]

Grodtkjær derived the equation which is identical with our (3.16) starting from the Green's third identity for the derivative $\partial\varphi/\partial t = \mathbf{t}\cdot\nabla\varphi$ of a harmonic function φ, which is itself a harmonic function:

$$\frac{\partial\varphi}{\partial t} = \frac{1}{4\pi}\int_{S^+}\left[\frac{\partial\varphi}{\partial t}\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial}{\partial n}\left(\frac{\partial\varphi}{\partial t}\right)\frac{1}{r}\right] dS$$

where **t** is a unit vector in a fixed direction. Let φ be identified with the perturbation velocity potential of the flow around a body whose surface is denoted by S. Naturally φ satisfies the regularity condition at infinity. Denoting the total potential and the free-stream potential by ϕ and ϕ_∞ respectively, substitution of $\varphi = \phi - \phi_\infty$ in the above expression and use of the property

$$0 = \frac{1}{4\pi}\int_S\left[-\frac{\partial\phi_\infty}{\partial t}\frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \frac{\partial}{\partial n}\left(\frac{\partial\phi_\infty}{\partial t}\right)\frac{1}{r}\right] dS,$$

which is a three-dimensional analogue of (3.5) with ψ_∞ replaced by $\partial\phi_\infty/\partial t$, leads to the following expression for $\partial\phi/\partial t$:

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi_\infty}{\partial t} + \frac{1}{4\pi}\int_{S^+}\left[\frac{\partial\phi}{\partial t}\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial t}\right)\frac{1}{r}\right] dS. \tag{3.20}$$

Grodtkjær then transforms the term $\partial/\partial n\,(\partial\phi/\partial t)$ in the integrand in (3.20) by use of the fact that $\nabla^2\phi = 0$ throughout the flow field so that it is expressed in terms of the tangential derivatives of ϕ on S. The result is

$$\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial t}\right) = -\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\mathbf{t}\cdot\mathbf{n}\frac{B}{A}\frac{\partial\phi}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\mathbf{t}\cdot\mathbf{n}\frac{A}{B}\frac{\partial\phi}{\partial\beta}\right)\right]$$

where (α, β) is an orthogonal coordinate system on S with A and B being the respective metric coefficients.

The gradient operator ∇_Q associated with the running coordinates $Q(\xi, \eta, \zeta)$ of integration on S is expressible in the (α, β, ν)–system introduced in relation to the expression (2.11) as

$$\nabla_Q = \left(\frac{1}{A}\frac{\partial}{\partial\beta}, \frac{1}{B}\frac{\partial}{\partial\beta}, \frac{\partial}{\partial\nu}\right).$$

By virtue of the condition $\partial\phi/\partial n=0$, the following relation holds:

$$\nabla_Q\phi\cdot\nabla_Q\left(\frac{1}{r}\right) = \frac{1}{A^2}\frac{\partial\phi}{\partial\alpha}\frac{\partial}{\partial\alpha}\left(\frac{1}{r}\right) + \frac{1}{B^2}\frac{\partial\phi}{\partial\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{r}\right)$$

Combining this with (3.20), one sees that the following equation holds:

$$\frac{1}{r}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial t}\right)=-\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{\mathbf{t}\cdot\mathbf{n}}{r}\frac{B}{A}\frac{\partial\phi}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{\mathbf{t}\cdot\mathbf{n}}{r}\frac{A}{B}\frac{\partial\phi}{\partial\beta}\right)\right]+(\mathbf{t}\cdot\mathbf{n})\nabla_Q\phi\cdot\nabla_Q\left(\frac{1}{r}\right).$$

Hence the integrand of (3.20) is transformed as

$$\frac{\partial\phi}{\partial t}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial t}\right)\frac{1}{r}=\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{\mathbf{t}\cdot\mathbf{n}}{r}\frac{B}{A}\frac{\partial\phi}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{\mathbf{t}\cdot\mathbf{n}}{r}\frac{A}{B}\frac{\partial\phi}{\partial\beta}\right)\right]+(\mathbf{t}\cdot\nabla_Q\phi)\mathbf{n}\cdot\nabla_Q\left(\frac{1}{r}\right)-(\mathbf{t}\cdot\mathbf{n})\nabla_Q\phi\cdot\nabla_Q\left(\frac{1}{r}\right).$$

Upon integration over the closed surface S, the square-bracketed term on the right-hand side of the above equation identically vanishes. Since

$$(\mathbf{t}\cdot\nabla_Q\phi)\mathbf{n}\cdot\nabla_Q\left(\frac{1}{r}\right)-(\mathbf{t}\cdot\mathbf{n})\nabla_Q\phi\cdot\nabla_Q\left(\frac{1}{r}\right)=\mathbf{t}\cdot\left[(\mathbf{n}\times\nabla_Q\phi)\times\nabla_Q\left(\frac{1}{r}\right)\right]$$

and since $\nabla_Q(r_{PQ}^{-1})=-\nabla_P(r_{PQ}^{-1})$, the expression (3.20) becomes

$$\mathbf{t}\cdot\nabla\phi=\mathbf{t}\cdot\nabla\phi_\infty+\frac{1}{4\pi}\,\mathbf{t}\cdot\int_S(\nabla_Q\phi\times\mathbf{n})\times\nabla_P\left(\frac{1}{r}\right)dS_Q.$$

Now let the point P tend onto the surface S. Understanding that the normal **n** is directed into the flow field, one obtains

$$\int_S(\nabla_Q\phi\times\mathbf{n})\times\nabla_P\left(\frac{1}{r}\right)dS_Q\rightarrow 2\pi\nabla\phi+\oint_S(\nabla_Q\phi\times\mathbf{n})\times\nabla_P\left(\frac{1}{r}\right)dS_Q$$

Thus, Grodtkjær reaches the equation

$$\nabla\phi(P)-\frac{1}{2\pi}\oint_S(\nabla\phi\times\mathbf{n})_Q\times\nabla_P\left(\frac{1}{r}\right)dS_Q=2\nabla\phi_\infty$$

for P on S, where scalar multiplication by **t** has been omitted since **t** is arbitary. This is identical with our equation (3.19). It is noted that although having started from the identity (3.20), the final expression (3.21) is an equation for $\nabla\phi$ since the boundary condition $\partial\phi/\partial n=0$ has been incorporated in the process of its derivation.

3.5 Formulation by Kress[11)]

In the field of electrodynamics it is of practical value to know how to distribute electric current in a region **B** of three-dimensional space and on its boundary S in order to generate a given magnetic field in B. This problem, named as the 'Prager problem',[19)] can be stated in the jargon of fluid mechanics as 'given a solenoidal velocity field **v** in B, find a vortex distribution, consisting of a volume density **J** in B and a surface density j on S, which induces the given field **v** in B'. That is, **J** and j are sought such that the given field **v** is expressed by a vector potential **A** generated by them:

$$\mathbf{v}=\nabla\times\mathbf{A}\qquad(3.22a)$$

with

$$\mathbf{A}=\frac{1}{4\pi}\int_B\mathbf{J}\frac{1}{r}\,dV+\frac{1}{4\pi}\int_S\mathbf{j}\frac{1}{r}\,dS.\qquad(3.22b)$$

Here, j is assumed to be tangential to S, i.e. to satisfy $\mathbf{j}\cdot\mathbf{n}=0$. Also, when B includes the point at infinity, **v** is required to vanish uniformly at point infinity.

The vector potential **A** satisfies the relation

$$\nabla\times(\nabla\times\mathbf{A})=-\nabla^2\mathbf{A}$$

provided **J** and j satisfy the following continuity conditions:

C.1. $\nabla\cdot\mathbf{J}=0$ in B, and

C.2 $\mathbf{D}\cdot\mathbf{j}\equiv\frac{1}{A}\frac{\partial j_\alpha}{\partial\alpha}+\frac{1}{B}\frac{\partial j_\beta}{\partial\beta}=\mathbf{n}\cdot\mathbf{J}$ on S

because these two conditions assure that **A** is solenoidal : $\nabla\cdot\mathbf{A}=0$ in the whole space, where **n** is the unit normal to S directed away from B and $(j_\alpha, j_\beta, 0)$ is the expression of j in the coordinate system (α, β, ν) as introduced with the expression (2.11). Substituting $\mathbf{v} = \nabla \times \mathbf{A}$ in the above relation, $\mathbf{J}$ is known to be given in terms of $\mathbf{v}$ as

$$\mathbf{J} = \nabla \times \mathbf{v} \quad \text{in B.}$$

In view of the property

$$\left(\nabla \times \int_S j \frac{1}{r} dS\right)_{\text{on S}}$$
$$= \pm 2\pi j \times \mathbf{n} + \oint_S \nabla\left(\frac{1}{r}\right) \times j dS \qquad (3.23)$$

which is essentially the same expression as (2.13), bringing the expression (3.22a) onto S yields the following condition to be satisfied by j:

$$j \times \mathbf{n} - \frac{1}{2\pi} \int_S \nabla\left(\frac{1}{r}\right) \times j dS$$
$$= \frac{1}{2\pi} \nabla \times \int_B (\nabla \times \mathbf{v})_Q \frac{1}{r} dV_Q - 2\mathbf{v}.$$

It is easy to see that the left-hand side of this equation coincides with that of our basic equation (3.19) if j is identified with the vortex vector λ defined by (2.12) and the equivalence (3.18) is noticed. Kress multiplied both sides of the above equation with n vectorially to obtain the integral equation for j as

$$j - \frac{1}{2\pi} \mathbf{n} \times \int_S \nabla\left(\frac{1}{r}\right) \times j dS$$
$$= \mathbf{n} \times \left[\frac{1}{2\pi} \nabla \times \int_B \frac{\nabla \times \mathbf{v}}{r} dV - 2\mathbf{v}\right]. \qquad (3.24)$$

Kress has shown that the vector j satisfying this equation together with $\mathbf{J} = \nabla \times \mathbf{v}$, solves the Prager problem for the given $\mathbf{v}$.

Now consider an irrotational flow of incompressible fluid past a body. Let B denote the region inside the body, and $\mathbf{V}_\infty$ be the velocity of the oncoming stream. Let, further, j be the solution of the Prager problem for the given vector field $-\mathbf{V}_\infty$ in B. Since $\mathbf{V}_\infty$ is irrotational, the equation for j reads

$$j - \frac{1}{2\pi} \mathbf{n} \times \oint_S \nabla\left(\frac{1}{r}\right) \times j dS = 2\mathbf{n} \times \mathbf{V}_\infty. \qquad (3.25)$$

This is equivalent to our equation (3.19) provided j satisfies the equation

$$\frac{1}{4\pi} \mathbf{n} \cdot \oint_S \nabla\left(\frac{1}{r}\right) \times j dS = \mathbf{n} \cdot \mathbf{V}_\infty \qquad (3.26)$$

which in turn is equivalent to the equation (3.15). Let, then, $\mathbf{v}_J$ be the flow induced by the surface vortex distribution j satisfying (3.25):

$$\mathbf{v}_J = \nabla \times \frac{1}{4\pi} \int_S j \frac{1}{r} dS$$

and construct the velocity field $\mathbf{V}$ by superposing $\mathbf{v}_J$ and $\mathbf{V}_\infty$:

$$\mathbf{V} = \mathbf{v}_J + \mathbf{V}_\infty$$

By construction, $\mathbf{V}$ vanishes identically in the region inside the body. Specifically, $\mathbf{V} \cdot \mathbf{n} = 0$ on the inner side of the body surface S. But $\mathbf{v}_J \cdot \mathbf{n}$, and hence $\mathbf{V} \cdot \mathbf{n}$, is continuous across S as is readily seen from the boundary property (3.23). Thus $\mathbf{V}$ represents the velocity field of potential flow around the body immersed in the oncoming stream of velocity $\mathbf{V}_\infty$.

There is little difference between Kress' formulation and ours in that both derive the basic equation from considerations on the condition to be satisfied for the region inside the body. Difference, in fact, resides in the process of reasoning which led to respective formulations. We strove to obtain a Fredholm integral equation of the second kind for the doublet strengths and attained this by replacing the original boundary condition $\partial\phi/\partial n = 0$ on S^+ with $\partial\phi/\partial t = 0$ on S^-, which was effected by taking advantage of the continuity of $\partial\phi/\partial n$ across the boundary S when the potential ϕ is given in terms of a doublet distribution on S. On the other hand, Kress' application of the solution method for the Prager problem may presuppose the knowledge that the vortex distribution over S which represents the flow around a body moving with a uniform speed of $-\mathbf{V}_\infty$ in a quiescent fluid induces a velocity field of $-\mathbf{V}_\infty$ within the body, which fact has been utilized by F. Vandrey[9] in formulating the potential flow around an axisymmetric body at zero incidence.

3.6 Some Properties of the Derived Integral Equation

Kress' another contribution is the analysis[19] of the properties of the integral operators appearing in (3.19) and (3.25). Let the operator in (3.25) be indicated by A:

$$Aj \equiv \frac{1}{2\pi}\, n \times \oint_S \nabla\left(\frac{1}{r}\right) \times j\, dS \qquad (3.27)$$

Also, let B be the integral operator defined by

$$\begin{aligned} Bi &\equiv \mathbf{n} \times A(\mathbf{n} \times i) \\ &= \frac{1}{2\pi}\oint_S \nabla\left(\frac{1}{r}\right) \times (i \times n)\, dS \\ &\quad - \frac{1}{2\pi}\left[\mathbf{n}\cdot\oint_S \nabla\left(\frac{1}{r}\right) \times (i \times \mathbf{n})\, dS\right]\mathbf{n} \qquad (3.28) \end{aligned}$$

It is noted that our basic equation (3.16) is written as

$$\mathbf{D}\mu + B(\mathbf{D}\mu) = \nabla\phi_\infty - (\mathbf{n}\cdot\nabla\phi_\infty)\mathbf{n}$$

of which both sides represent vectors confined within S. The operators A and B are adjoint to each other in the sense that the following equation holds for arbitrary vectors i and j lying on S:

$$\int_S (Aj)\cdot i\, dS = \int_S j\cdot(Bi)\, dS.$$

Now consider the integral equations for the surface vectors i and j given by

$$j - \lambda Aj = h \qquad (3.29)$$

and

$$i - \lambda Bi = k \qquad (3.30)$$

where λ is the parameter of integral equation. Kress has shown that $\lambda=+1$ and $\lambda=-1$ are $(q-1)$-fold eigenvalues of both operators A and B. That is, there are $q-1$ linearly independent solutions of either

$$j \pm Aj = 0 \qquad (3.31)$$

or

$$i \pm Bi = 0 \qquad (3.32)$$

where q is the number of multiplicity of the three-dimensional region bounded by S. For instance, when the flow past a system of bodies consisting of a number of disjoint solid bodies is considered, any one of the inside regions or the outside region bounded by respective body surfaces is simply connected, and hence both equations (3.29) and (3.30) possess unique solutions respectively, if they are solvable at all. On the other hand, when the flow inside a closed duct such as a wind tunnel circuit is considered, the equations (3.29) and (3.30) either possess no solution or solutions of the type $Cj_H + j_P$, say, where j_H is a solution for the homogeneous equation (3.31), j_P is a particular solution for (3.29), and C is an arbitrary constant. This is because the region of flow under consideration is doubly connected one.

Concerning the solvability of (3.29) and (3.30), the following theorem (Fredholm's Alternatives) is relevant:

either both (3.29) and (3.30) are uniquely solvable for arbitrarily given right-hand sides **h** and **k** respectively, or the homogeneous equations (3.31) and (3.32) possess the same number of linearly independent solutions j_ν and i_ν, $\nu = 1, 2, \ldots, q-1$.

In the latter case, the inhomogeneous equation (3.29) and (3.30) are solvable if and only if the conditions

$$\left.\begin{aligned} &\int_S i_\nu \cdot h\, dS = 0 \\ \text{and}\quad &\int_S j_\nu \cdot k\, dS = 0 \end{aligned}\right\}\ \nu = 1, 2, \ldots, q-1$$

are satisfied by **h** and **k** respectively.

This is an extension of the classical Fredholm's Alternatives, which is originally applicable to the integral equations with continuous kernels, to the equations (3.29) and (3.30), whose operators A and B possess singular kernels.

The results given so far are obtained by Kress[19] based on the assumptions, among others, that the curvature is continuous everywhere on S and that the vortex strength j is

Hölder-continuous on S. When the potential flow around a wing with sharp trailing edge is considered, obviously the former assumption is not met. There is neither assurance that the flow velocity remains Hölder-continuous in the trailing-edge proximity. Moreover, the existence of the trailing vortex sheet adds further complication to the problem. It seems that no definite statements have yet been given about the nature of the solution of either (3.19) or (3.25) when these equations are applied to the problem of the potential flow around a three-dimensional lifting wing. A local and preliminary attempt to study the behaviour of the integral equation at the trailing edge will be made in Section 6 in relation to a consideration on the implication of the Kutta condition.

4. OTHER RELEVANT FORMULATIONS OF POTENTIAL FLOW PROBLEMS

A general expression of the disturbance potential φ has been given by (3.1) as a combination of source- and doublet distributions over the body surface S. As was mentioned at the beginning of the preceding section, either the source strength σ or the doublet strength μ, or a combination thereof, can be specified arbitrarily and the rest is determined so that the potential satisfies the boundary condition on the body surface.

Our approach described in 3.3 is an embodiment of the intention to achieve a formulation based on an exclusive use of doublets. This exclusive use of doublets, however, does not necessarily leads to our formulation: other forms are also possible. Still other formulations can be established by assigning to the source strength a value which is different from zero. Evidently many possibilities exist within this class of formulations. A few of them are described in the present section.

Some of the formulations recently published (e.g. Refs. 22, 23 and 27) make a contrast to the older ones[1)–7)] in that they use the doublets as the primary variable whereas the sources features the scene in the older formulations with the doublets playing an auxiliary yet indispensable role of modelling the trailing vortex sheet. Brief notes of these recent doublet-based formulations will be given in the course of present discussion.

When the disturbance potential $\varphi=\phi-\phi_\infty$ is expressed as in (3.1), the source- and doublet strengths are related to the jumps in the boundary values of φ or ϕ and their normal derivatives as

$$\sigma = \left(\frac{\partial\varphi}{\partial n}\right)^{+} - \left(\frac{\partial\varphi}{\partial n}\right)^{-} = \left(\frac{\partial\phi}{\partial n}\right)^{+} - \left(\frac{\partial\phi}{\partial n}\right)^{-} \quad (4.1)$$

$$\mu = \varphi^{+} - \varphi^{-} = \phi^{+} - \phi^{-} \quad (4.2)$$

where the superscripts + and − indicate the values associated with the sides S^+ and S^- of the surface S respectively. The normal n to S along which the differentiation $\partial/\partial n$ is taken is directed from S^- to S^+.

Combining the expressions (2.6) and (2.13), it is seen that the gradient of φ given by (3.1) takes the following form on S:

$$\nabla\varphi = \pm\frac{1}{2}(D\mu + \sigma n) + v^{\mu} - v^{\sigma} \quad (4.3)$$

with

$$v^{\mu} = \frac{1}{4\pi}\oint_S (D\mu \times n) \times \nabla\left(\frac{1}{r}\right) dS$$

and

$$v^{\sigma} = \frac{1}{4\pi}\oint_S \nabla\left(\frac{1}{r}\right) dS.$$

That is, the disturbance velocity components on the body surface are given as

$$v_t^{+} = \frac{1}{2}D\mu + (v^{\mu} - v^{\sigma})_t\,, \quad (4.4)$$

$$v_t^{-} = -\frac{1}{2}D\mu + (v^{\mu} - v^{\sigma})_t\,, \quad (4.5)$$

$$v_n^{+} = \frac{1}{2}\sigma + (v^{\mu} - v^{\sigma})_n\,, \quad (4.6)$$

and

$$v_n^{-} = -\frac{1}{2}\sigma + (v^{\mu} - v^{\sigma})_n \quad (4.7)$$

where the subscripts t and n indicate the tangential- and normal components to S respectively. These are the formulae for the induced velocity components due to combined source- and doublet distributions. It is noted that the velocity components induced by a vortex distribution are also expressed by the above formulae once the equivalence rule (2.12) is observed.

Of the doublet-based formulations found in the open literature, two classes are discriminated according as how the source strength σ is specified to obtain an equation for the doublet strength μ or its gradient $D\mu$. In one class σ is put equal to zero everywhere on S, and in the other it is identified with the normal component of the disturbance velocity $(\partial\varphi/\partial n)^+$.

4.1 Formulation where σ is specified as identically zero

From the induced-velocity formulae (4.6) and (4.7), the following relations are obtained when σ is identically zero:

$$v_n^+ = v_n^- \tag{4.8}$$

$$v_n^+ = (v^\mu)_n = v^\mu \cdot n. \tag{4.9}$$

The former also results from (4.1). The latter serves as a condition by which $D\mu$ is determined when v_n^+ is given. For instance, the impermeability condition $(\partial\phi/\partial n)^+ = 0$ on S leads to

$$(v^\mu + V_\infty)\cdot n = 0$$

which has already appeared as the equation (3.15), or equivalently as the equation (3.26) when written in terms of vortex distribution. This equation was adopted by Djojodihardjo & Widnall[20] and by Maskew[21], though in the former computation was carried out in terms of the doublet strength μ rather than of its gradient $D\mu$, and in the latter the vortex distribution is replaced by a system of vortex lattice, i.e. loops of line vortices. This usage of vortex lattices amounts to assuming constant μ over each panel into which the body surface is divided, cf. the expression (2.10).

Our formulation, on the other hand, is related to the other induced velocity formulae (4.4) and (4.5). As was said in the foregoing, the expressions (4.1) or (4.6) and (4.7) indicate the relation

$$\left(\frac{\partial\varphi}{\partial n}\right)^+ = \left(\frac{\partial\varphi}{\partial n}\right)^- \quad \text{or} \quad \left(\frac{\partial\phi}{\partial n}\right)^+ = \left(\frac{\partial\phi}{\partial n}\right)^- \quad \text{on S}$$

when σ is identically put equal to zero on S. As was shown in Section 3.3, the boundary condition $(\partial\phi/\partial n)^+=0$ then leads to $(\partial\phi/\partial n)^-=0$ which in turn results in ϕ=const. in the region inside the body. Specifically

$$\phi^- = \text{const.} \tag{4.10}$$

and hence

$$\mu = \phi^+ + \text{const.} \tag{4.11}$$

by virtue of the relation (4.2). The properties (4.10) and (4.11) respectively lead to

$$0 = v_t^- + (V_\infty)_t \tag{4.12}$$

and

$$D\mu = v_t^+ + (V_\infty)_t\,. \tag{4.13}$$

The conditions (4.10) through (4.13) are another expressions of the boundary condition $(\partial\phi/\partial n)^+=0$ on S. Now these results are combined with the induced velocity formulae (4.4) and (4.5): elimination of v_t^- from (4.5) and (4.12) results in our basic equation (3.16). Elimination of v_t^+ from (4.4) and (4.13) leads to the identical result.

Stopping short of elimination, on the other hand, an iteration procedure suggests itself when the expressions (4.4) and (4.13) are deemed as a system of simultaneous equations for $D\mu$ and v_t^+: given an approximation to v_t^+, the corresponding approximation to $D\mu$ is obtained from (4.13) and then a renewed approximation to v_t^+ is calculated by the induced velocity formula (4.4). Using this iteration procedure, Raj & Gray[22] computed the flow Field around the tip of nonlifting wing of symmetrical section. Needless to say, convergence of this procedure is problem-dependent. To the author's experience, the procedure easily fails to converge when applied to the problem of a lifting wing with sharp trailing edge. It is an open question whether a suitable choice of the relaxation factor redresses the malfunction of the iteration scheme.

4.2. Case where σ is identified with $(\partial\varphi/\partial n)^+$

As is readily seen from the process of reasoning, the derivation of our basic equation (3.16) hinges on the impermeability boundary condition $\partial\phi/\partial n=0$ on S. It may be advantageous if our equation is extended so that it is applicable to cases where the boundary condition of non-zero normal derivative $\partial\phi/\partial n$ is prescribed, e.g. the case where the boundary layer effects are taken into the inviscid flow model by expressing the viscous displacement effects in terms of an equivalent source distribution. This extension is in fact achieved by considering the equation for the disturbance velocity instead of the total velocity as will be shown below.

Consider the case where the source strength in (3.1) is specified to be equal to the normal component of the disturbance velocity $(\partial\varphi/\partial n)^+$ on the body surface. In this case, the induced velocity formulae (4.6) and (4.7) lead to

$$\sigma \equiv v_n^+ = 2(v^\mu - v^\sigma)_n \tag{4.14}$$

and

$$v_n^- = 0. \tag{4.15}$$

That is, putting σ equal to $(\partial\varphi/\partial n)^+$ implies vanishing normal derivative of φ along the inner side of the boundary surface S as is also evident from the relation (4.1) This condition immediately leads to the fact that φ=const. in the region inside the body, i.e.

$$\varphi^- = \text{const} \tag{4.16}$$

and hence from (4.2) to the fact that

$$\mu = \varphi^+ + \text{const.}, \tag{4.17}$$

which form a parallelism with the expressions (4.10) and (4.11). In terms of the total potential ϕ, the condition (4.16) means

$$\phi^- = \phi_\infty + \text{const.}$$

From (4.16) and (4.17) it follows that

$$v_t^- = 0 \tag{4.18}$$

and

$$v_t^+ = \mathbf{D}\mu. \tag{4.19}$$

Combined with (4.5) and (4.4) respectively, these relations yield the following integral equation for v_t^+ when v_n^+ is given on S:

$$v_t^+ - \frac{1}{2\pi}\oint_S (v_t^+ \times n) \times \nabla\left(\frac{1}{r}\right) dS\,\Big|_t = -\frac{1}{2\pi}\oint_S v_n^+ \nabla\left(\frac{1}{r}\right) dS\,\Big|_t \tag{4.20}$$

where the notation $|_t$ indicates taking the tangential components only.

It is noted that this equation, unlike the corresponding equation (3.16) or (3.19), does not postulate the impermeability boundary condition $(\partial\phi/\partial n)^+ = 0$, and is valid for arbitrarily given v_n^+. Based on this equation, then, extension of the equation (3.16) or (3.19) to cases with arbitrary Neumann boundary condition is carried out as follows: One observes that, in view of the relation (4.19), the conditions (4.4) and (4.14) are unified as

$$v = 2(v^\mu - v^\sigma) \tag{4.21}$$

where the superscript + on **v** has been dropped with the understanding that **v** represents the disturbance flow velocity along the body surface. Using the relations $\sigma = \mathbf{v}\cdot\mathbf{n}$ and $\mathbf{D}\mu = v_t$, the expression $\mathbf{v}^\mu - \mathbf{v}^\sigma$ on the right-hand side of (4.21) is written as

$$\mathbf{v}^\mu - \mathbf{v}^\sigma = \frac{1}{4\pi}\oint_S \left[(\mathbf{v}\times\mathbf{n})\times\nabla\left(\frac{1}{r}\right) - (\mathbf{v}\cdot\mathbf{n})\,\nabla\left(\frac{1}{r}\right)\right] dS. \tag{4.22a}$$

Substituting $\mathbf{V}-\mathbf{V}_\infty$ for **v** in the integrand of the above expression, one is led to

$$\mathbf{v}^\mu - \mathbf{v}^\sigma = \frac{1}{2}\mathbf{V}_\infty + \frac{1}{4\pi}\oint_S \left[(\mathbf{V}\times\mathbf{n})\times\nabla\left(\frac{1}{r}\right) - (\mathbf{V}\cdot\mathbf{n})\,\nabla\left(\frac{1}{r}\right)\right] dS \tag{4.22b}$$

where the following three identities have been utilized:

$$(\mathbf{V}_\infty \times \mathbf{n})\times\nabla\left(\frac{1}{r}\right) - (\mathbf{V}_\infty\cdot\mathbf{n})\,\nabla\left(\frac{1}{r}\right)$$

$$= \mathbf{V}_\infty \times [\, \mathbf{n} \times \nabla(\frac{1}{r})\,] - \mathbf{V}_\infty \mathbf{n} \cdot [\, \mathbf{n} \cdot \nabla(\frac{1}{r})\,],$$

$$\oint_S \mathbf{n}_Q \cdot \nabla_P \left(\frac{1}{r_{PQ}}\right) dS_Q = 2\pi \qquad (4.23)$$

for P lying on S,

and

$$\oint_S \mathbf{n}_Q \times \nabla_P \left(\frac{1}{r_{PQ}}\right) dS = 0 \qquad (4.24)$$

for arbitrarily located point P. Proofs of (4.23) and (4.24) are given in Appendix I. Now using (4.22b) to (4.21), one obtains the following equation for **V**:

$$\mathbf{V}(P) - \frac{1}{2\pi}\oint_S (\mathbf{V} \times \mathbf{n})_Q \times \nabla_P \left(\frac{1}{r_{PQ}}\right) dS_Q$$
$$= 2\mathbf{V}_\infty - \frac{1}{2\pi}\oint_S (\mathbf{V} \cdot \mathbf{n})_Q \nabla_P \left(\frac{1}{r_{PQ}}\right) dS_Q \qquad (4.25)$$

which equation has been referred to in Section 3.3 as the equation (3.19a). The corresponding equation written in terms of **v** is obtained by using (4.22a) to (4.21):

$$\mathbf{v}(P) - \frac{1}{2\pi}\oint_S (\mathbf{v} \times \mathbf{n})_Q \times \nabla_P \left(\frac{1}{r_{PQ}}\right) dS_Q$$

$$= -\frac{1}{2\pi}\oint_S (\mathbf{v} \cdot \mathbf{n})_Q \nabla_P \left(\frac{1}{r_{PQ}}\right) dS_Q. \qquad (4.26)$$

The tangential components of this equation have already been given as (4.20).

Apart from the source- and doublet distribution approaches, either Green's third identity or Green's boundary formula is also utilized to formulate the lifting flow problem around a wing. Applying Green's third identity in the form of (2.4) to the disturbance potential φ, one obtains the expression

$$\int_S [\varphi^+(Q) \frac{\partial}{\partial n_Q}\left(\frac{1}{r_{PQ}}\right) - \left(\frac{\partial \varphi}{\partial n}\right)_Q^+ \frac{1}{r_{PQ}}] dS_Q = 0 \qquad (4.27)$$

which is valid for P belonging to the region inside the body. This expression can be utilized to determine φ^+ when $(\partial\varphi/\partial n)^+$ is specified. Bristow & Grose[23] applied the panel procedure to solve this equation numerically by enforcing this identity at representative points inside the body which were selected to be very close to the control points of panels.

When the point P is brought onto the body surface from within the body, the equation (4.27) takes the form

$$-2\pi\varphi^+ + \oint_S [\varphi^+ \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \left(\frac{\partial\varphi}{\partial n}\right)^+ \frac{1}{r}] dS = 0 \qquad (4.28)$$

which is known as Green's boundary formula. This expression, too, can be regarded as an integral equation for φ^+ when $(\partial\varphi/\partial n)^+$ is known.[24), 25)] Morino[26)] and Kuo & Morino[27)] employed this formula to determine the value of φ on S for the known $(\partial\varphi/\partial n)^+ = -(\partial\phi_\infty/\partial n)$. The surface velocity components were then derived by numerical differentiation. It is readily seen that our equation (4.20) bears the relation to (4.28) as the differentiated form of the latter.

4.3 Formulations with Both Unknown Sources and Doublets

Computational performance of a numerical scheme for a lifting potential flow problem may be enhanced by designing proper combination of source- and doublet distributions. One can see this in the examples of successful panel-method procedures applied to the problem. For instance, bulk of the representative codes[1)–7)] of the first generation panel method employ a combination of sources and doublets, the former to express the wing displacement effects while the latter to model the downwash effects associated with the wake vortex layer. They inherit this technique from the classical thin-wing theory, and it is no wonder that, as long as the control points are chosen not too close to the trailing edge, they engender results which are not far from those predicted by the classical linearized theory, which in turn gives results that agree with experiments in a broad sense.

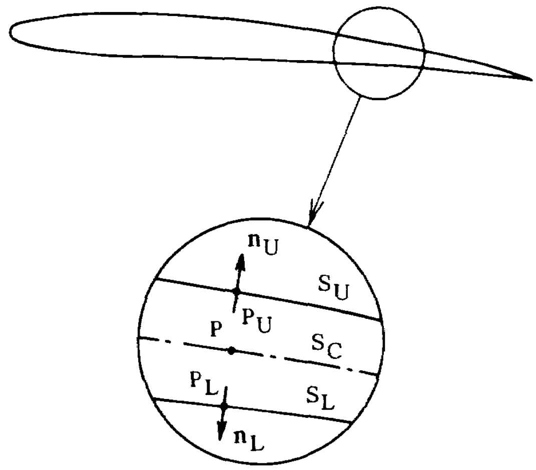
An inherent difficulty in numerical treatment of the potential flow around a lifting wing with sharp trailing edge is that the proximity of the upper- and lower surfaces at the trailing edge region brings about the situation where sources and/or doublets placed on one surface exert strong influence upon those on the other surface, and hence errors brought in during the discretization- and solution processes tend to be reflected as amplified deviation of the numerical solution from the true behaviour. Therefore a control may be required on the variation of strengths of sources and/or doublets in order to maintain the accuracy of numerical results. An example of such control measure is devised by Maskew & Woodward[28] for the two-dimensional problems.

In the Maskew-Woodward procedure, both sources and vortices are placed along the aerofoil contour with equal strengths at corresponding points on the upper- and lower surfaces. Near the trailing edge, then, the magnitudes of source strengths are kept small, as is required for good numerical accuracy, by this constraint because a sum of the upper- and lower surface source strengths is independently controlled to be small by the imposition of the boundary condition of vanishing normal velocity components. Extension of this procedure to the three-dimensional case has been made by Petrie.[29]

A measure like this to counter undue large variations in singularity strengths may be indispensable to keep numerical errors within certain bounds. Our formulation in terms of the surface flow velocity components as the variables may facilitate control of numerical errors because, at least qualitatively, their modes of variation near the trailing edge can be judged on the physical ground. How to realize such control will be the crux of the numerical implementation of our formulation.

5. LIMITING CASE OF ZERO-WING THICKNESS

It is anticipated that, by nature of construction, our basic equation (3.16) would become



Sketch 2.
Notations about the Limiting Process of Reducing Wing Thickness Indefinitely

increasingly more ill-conditioned as the wing thickness gets smaller and smaller. To study this aspect, the limiting form of our equation as the wing thickness goes to zero is examined.

Consider a very thin wing and denote its upper- and lower surfaces by S_U and S_L respectively. Let P_U and P_L be points on S_U and S_L respectively. In the process of reducing wing thickness indefinitely, let them tend to a common point P on the camber surface S_C to which both S_U and S_L degenerate in the limit, cf. Sketch 2.

Suppose that the perturbation flow field is expressed in terms of a potential of doublet distribution over the wing surface $S_U + S_L$ and the trailing vortex sheet S_W:

$$\varphi(P) = \frac{1}{4\pi}\int_{S_U+S_L+S_W} \mu(Q)\frac{\partial}{\partial n_Q}\left(\frac{1}{r_{PQ}}\right)dS_Q$$

The gradient of φ at P_U and P_L respectively are then given by

$$\nabla\varphi^{\pm}(P_U) = \pm\frac{1}{2}D\mu(P_U) + \frac{1}{4\pi}[U_U + U_L + W(P_U)]$$

$$\nabla\varphi^{\pm}(P_L) = \pm\frac{1}{2}D\mu(P_L) + \frac{1}{4\pi}[L_U + L_L + W(P_L)]$$

where

$$U_U(P_U) = \oint_{S_U} (D\mu \times n)_U \times \nabla(\frac{1}{r}) dS,$$

$$U_L(P_U) = \int_{S_L} (D\mu \times n)_L \times \nabla(\frac{1}{r}) dS,$$

$$L_U(P_L) = \int_{S_U} (D\mu \times n)_U \times \nabla(\frac{1}{r}) dS,$$

$$L_L(P_L) = \oint_{S_L} (D\mu \times n)_L \times \nabla(\frac{1}{r}) dS,$$

and

$$W(P) = \int_{S_W} (D\mu \times n)_Q \times \nabla_P(\frac{1}{r_{PQ}}) dS_Q,$$

where the subscripts U and L refer to the integration domains of S_U and S_L respectively. Let $D\mu_U$ and $D\mu_L$ denote the limiting values of $D\mu(P_U)$ and $D\mu(P_L)$ respectively as P_U and P_L approach the Point P on S_C. To make the matter simple, let us assume that firstly S_U is brought to coincide with S_C and then S_L is done so. In the latter process, the points at which L_U and U_L are to be evaluated, viz. P_L for L_U and P_U for U_L, are brought onto the surface over which the integration is carried out. Noting that P_L approaches P from the side of S_C that is opposite to the one into which the normal n_U is directed (the normal n is directed into the flow field, cf. Sketch 2), one sees that

$$\lim_{S_L \to S_C} L_U = -2\pi D\mu_U + \oint_{S_C} (D\mu \times n)_U \times \nabla(\frac{1}{r}) dS.$$

Similarly

$$\lim_{S_L \to S_C} U_L = -2\pi D\mu_L + \oint_{S_C} (D\mu \times n)_L \times \nabla(\frac{1}{r}) dS.$$

Hence, in the limit of S_U and S_L approaching S_C, one obtains the following expressions for $\nabla\varphi(P_U)$ and $\nabla\varphi(P_L)$:

$$\nabla\varphi_U^{\pm} \equiv \lim \nabla\varphi(P_U) = \pm\frac{1}{2} D\mu_U - \frac{1}{2} D\mu_L + B$$

$$\nabla\varphi_L^{\pm} \equiv \lim \nabla\varphi(P_L) = \pm\frac{1}{2} D\mu_L - \frac{1}{2} D\mu_U + B$$

where

$$B = \frac{1}{4\pi}\oint_{S_C} [(D\mu_U - D\mu_L) \times n_U] \times \nabla(\frac{1}{r}) dS + \frac{1}{4\pi}\int_{S_W} (D\mu \times n) \times \nabla(\frac{1}{r}) dS.$$

Introducing $D\mu_S$ and $D\mu_A$ defined by

$$D\mu_S = \frac{1}{2}(D\mu_U + D\mu_L) \tag{5.1}$$

and

$$D\mu_A = \frac{1}{2}(D\mu_U - D\mu_L), \tag{5.2}$$

$\nabla\varphi_U$ and $\nabla\varphi_L$ are written as

$$\nabla\varphi_U^{+} = D\mu_A + B, \tag{5.3a}$$

$$\nabla\varphi_U^{-} = -D\mu_S + B, \tag{5.3b}$$

$$\nabla\varphi_L^{+} = -D\mu_A + B, \tag{5.3c}$$

and

$$\nabla\varphi_L^{-} = -D\mu_S + B \tag{5.3d}$$

where

$$B = \frac{1}{2\pi}\oint_{S_C} (D\mu_A \times n_U) \times \nabla(\frac{1}{r}) dS + \frac{1}{4\pi}\int_{S_W} (D\mu \times n) \times \nabla(\frac{1}{r}) dS. \tag{5.4}$$

Our basic equation (3.16) has been derived from the condition

$$\frac{\partial\phi^-}{\partial t} \equiv (\nabla\varphi^- + \nabla\phi_\infty) \cdot t = 0.$$

In the present case this condition reduces to

$$(-D\mu_S + B + \nabla\phi_\infty) \cdot t = 0 \text{ on } S_C \tag{5.5}$$

whereas the boundary condition of vanishing normal velocity component reads

$$(B + \nabla\phi_\infty) \cdot n = 0 \text{ on } S_C \tag{5.6}$$

because $D\mu$ lies on the tangent plane to S_C. The equation (5.5) contains two unknowns $D\mu_S$ and $D\mu_A$. This reflects the fact the originally distinct surfaces S_U and S_L merge to the single surface S_C while the unknowns $D\mu_U$ on S_U and $D\mu_L$ on S_L remain distinct with each other after merging. Evidently one is unable to determine both $D\mu_S$ and $D\mu_A$ from the equation (5.5) only. On the other hand, the equation (5.6) contains $D\mu_A$ only, and hence can be used to determine the latter. In fact this equation reduces to the basic equation of the lifting surface theory

$$\lim_{z\to 0} \frac{\partial}{\partial z} \int_{S_{C_0}} \frac{\partial \mu_A}{\partial \xi} \frac{z}{(y-\eta)^2+z^2} \left[1+\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\right] d\xi d\eta = -2\pi \left(\frac{\partial \phi_\infty}{\partial z}\right)_{z=0}$$

if the approximation inherent in the theory are introduced to transform (5.6), where S_{Co} indicates the projection of S_C onto the plane z=0, and where $\partial\mu_A/\partial\xi$ is half of the difference between the upper- and lower surface chordwise velocity components (cf. relations (5.2) and (3.18)) that is related to the pressure difference ΔCp across the wing surface via

$$\Delta Cp = C_P(\text{lower surface}) - C_P(\text{upper surface}) = 4\frac{\partial \mu_A}{\partial \xi}$$

Once $D\mu_A$ is found, then $D\mu_S$ is obtained from the condition (5.5). But there is no practical meaning in obtaining $D\mu_S$ in its own since it is irrelevant to aerodynamic force calculation.

Thus, it is seen that our basic equation (3.16) degenerates into the equation (5.5) when the body thickness is reduced to zero, which fails to produce a solution by itself. One must be aware of this feature when one deals with the flow around a wing with thin trailing-edge region by using the basic equation (3.16).

6. BEHAVIOUR OF THE BASIC EQUATION AT THE TRAILING EDGE OF A WING–IMPLICATION OF THE KUTTA CONDITION

Our basic equation (3.16) is based on the formula (2.13) for the derivatives of the potential due to a doublet distribution over a surface. This formula is derived on the assumption that the doublet gradient $D\mu$ is Hölder-continuous and the surface S over which the doublet is distributed possesses continuous curvature (e.g. cf. Ref. 14, p.164). When a wing with sharp trailing edge is considered, it is an open question whether the equation (3.16) is valid even at the trailing edge. In this context it may be worthwhile to see how the equation (3.16) behaves when the trailing edge is approached.

A study in this direction leads to a condition to be satisfied at the trailing edge by the flow velocities on the wing upper- and lower surfaces and the doublet gradient on the trailing vortex sheet. It turns out that this condition, combined with the force-free condition for the trailing vortex sheet, ensures the continuity of pressure at the trailing edge. In other words, the Kutta condition emerges as a consequence of the condition that our basic equation is consistent at the trailing edge.

6.1 Basic Equation for the Case of a Lifting Wing

Since the trailing vortex sheet is an essential part for the representation of a potential flow field around a lifting wing, the expression for the potential must be given in a form including the contribution from this sheet. Thus φ_D to be used in the present section is assumed to be given

$$\varphi_D = \frac{1}{4\pi}\int_{S_B} \mu \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \frac{1}{4\pi}\int_{S_W} \mu \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS \qquad (6.1)$$

where S_B and S_W denote the wing surface and the trailing vortex sheet respectively. Repeating the argument made in Section 3.3 to obtain the equation (3.19) in this time using φ_D given by (6.1), the basic equation for the lifting-wing case is found to be

$$V(P) - \frac{1}{2\pi}\oint_{S_B} (V \times n)_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q = 2V_\infty + \frac{1}{2\pi}\int_{S_W} (D\mu \times n)_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q \qquad (6.2)$$

for a point P lying on the wing surface S_B. In fact, line integrals associated with the trailing edge must be included in the process of deriving this expression. The discussion concerning with these line integrals will be given in the next section, wherein it is shown that the condition for a finite velocity at the trailing edge render these line integrals vanish identically.

Now, in addition to the surface velocity **V** along S_B, the shape of the trailing vortex sheet S_W and the doublet gradient $D\mu$ on S_W are also unknowns of the problem. These latter two are essentially determined from the following two conditions:

(1) the trailing vortex sheet is parallel to the flow velocities on its both sides, and
(2) the fluid pressure is continuous across the sheet.

To obtain mathematical expressions of these conditions, the flow velocity at a point on the trailing vortex sheet is sought. Applying (2.13) to (6.1), it is given as

$$q^\pm \equiv \nabla(\varphi_D + \phi_\infty)^\pm = V_\infty \pm \frac{1}{2} D\mu + \frac{1}{4\pi}\int_{S_B} (V \times n) \times \nabla\left(\frac{1}{r}\right) dS + \frac{1}{4\pi}\oint_{S_W} (D\mu \times n) \times \nabla\left(\frac{1}{r}\right) dS \qquad (6.3)$$

where $D\mu$ on S_B has been replaced by the flow velocity **V**. Introducing q_m defined by

$$q_m = V_\infty + \frac{1}{4\pi}\int_{S_B} (V \times n) \times \nabla\left(\frac{1}{r}\right) dS + \frac{1}{4\pi}\oint_{S_W} (D\mu \times n) \times \nabla\left(\frac{1}{r}\right) dS \qquad (6.4)$$

the expression (6.3) is restated as

$$q^\pm = q_m \pm \frac{1}{2} D\mu \qquad (6.5)$$

where q^+ is the flow velocity along the positive side of the sheet (the normal **n** to S_W is directed from the negative- to positive sides) while q^- is that along the other side. This expression for $q^\pm$ shows that the flow velocity suffers a jump of $D\mu$ when the trailing vortex sheet is crossed in the direction of the normal **n** to the sheet. It follows that any streamline cannot cross the sheet because the velocity must be continuous along a streamline. This leads to the condition (1) stated above, which is now expressed as

$$n \cdot q^\pm = n \cdot q_m = 0 \qquad (6.6)$$

because the doublet gradient $D\mu$ is tangent to S_W, cf. (2.11). The second condition, on the other hand, is expressed by virtue of the Bernoulli's equation as

$$D\mu \cdot q_m = 0 \qquad (6.7)$$

which follows immediately from the condition $q^+ \cdot q^+ = q^- \cdot q^-$.

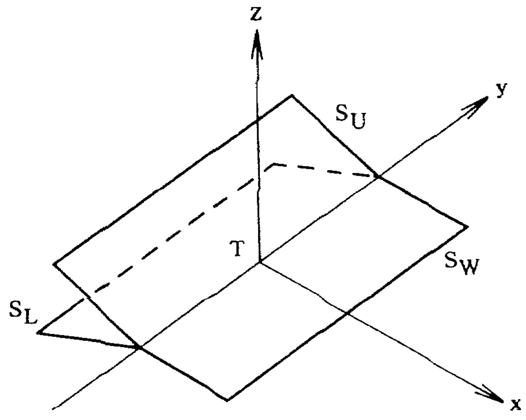
The conditions (6.2), (6.6) and (6.7) constitute three equations for the three unknowns **V** on S_B, $D\mu$ on S_W, and S_W itself. Specifically, the conditions (6.6) and (6.7) dictate that S_W is ruled out by the curves which emanate from the trailing edge and are aligned with the mean velocity q_m on the trailing vortex sheet, and that the doublet strength μ is constant along each of these curves. The characteristics of the trailing vortex sheet are determined once the wing surface velocity distribution is known because the doublet strength at the trailing edge is fixed when the velocity distribution over the wing is given (cf. the expressions (6.16a) and (6.16b)), and because the mean trailing vortex sheet velocity q_m is given at the trailing edge in terms of the flow velocities on the wing surface (cf. the relations (6.30a) and (6.30b)). The classical Prandtl's model for a lifting surface eliminates S_W from the list of the unknowns by use of the approximation that the local mean velocity q_m on S_W be replaced by the free-stream velocity V_∞. According to the condition (6.6), then, the trailing vortex sheet is formed by the straight lines which start at the wing trailing edge and extend downstream in the direction of V_∞. However, the shape of S_W immediately downstream of the trailing edge bears essential relation, at least conceptually, to the fulfilment of the Kutta condition. This aspect will be expounded in Section 6.3.

6.2 Regularity Conditions for the Derivatives of φ_D at the Trailing Edge

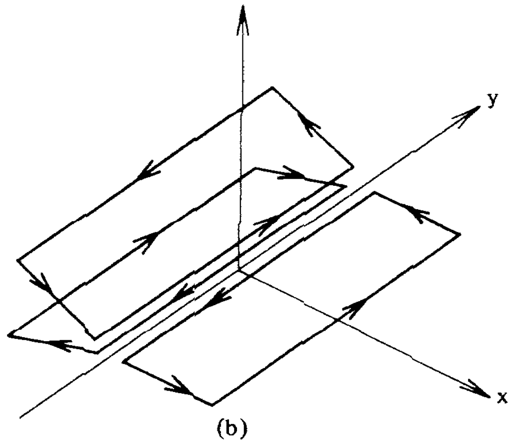
Consider the potential φ_D given by (6.1). Its derivatives exhibit certain singularities as the wing trailing edge is approached. This singular behaviour has already been examined in Ref. 16, the part whereof relevant to the present discussion is reiterated here.

The problem to be dealt with is to clarify how $\nabla\varphi_D(P)$ behaves as the point P tends to a point T lying on the trailing edge. Three distinct surfaces, viz. the wing upper surface, the wing lower surface and the trailing vortex sheet, meet together at the trailing edge making, in general, finite angles with each other. Restricting the consideration to a small region around the point T, and assuming that the trailing edge is straight in this region, each part of the three surfaces contained in the region can be approximated by a rectangular plane with the trailing edge as the common part of the peripheries, see Sketch 3, (a). Since the doublet strength may not be continuous across the trailing edge, a cut is brought in along the trailing edge so that each of the three surfaces is disjoint from others and is accordingly treated independently, cf. Sketch 3, (b).

Now, combining (2.8) and (2.10), $\nabla\varphi_D$ is given as



(a)



(b)

Sketch 3. Simplification of Surfaces in the Trailing-edge Region

$$\nabla\varphi_D(P) = \frac{1}{4\pi}\Sigma_{S_i}\nabla \times \left(\int_{S_i}\frac{n\times D\mu}{r}\,dS - \int_{\partial S_i}\frac{\mu}{r}\,t\,ds\right) \qquad (6.8)$$

where S_i ranges over the wing upper surface S_U, the lower surface S_L, and the trailing vortex sheet S_W. The symbol Σ_{S_i} indicates taking summation over S_i. In examining the behaviour of $\nabla\varphi_D(P)$ as P approaches T, consideration is divided into that of the line-integral part and that of the surface-integral part.

(1) line-integral part

Obviously, the singular terms arising from the line integrals on the right-hand side of (6.8) as P → T are associated with the integration path along the trailing edge. Introducing a coordinate system such that the origin is taken at T and that y-axis is aligned with the trailing edge directed outboard, the line-integral part along the trailing edge takes the form

$$\frac{1}{4\pi}\nabla\times\{[\int_{-b}^{b}(\frac{\mu}{r})_U d\eta - \int_{-b}^{b}(\frac{\mu}{r})_L d\eta - \int_{-b}^{b}(\frac{\mu}{r})_W d\eta]\, j\}$$

where **j** is the unit vector in the y-direction and the subscripts U, L and W respectively indicate the quantities associated with the wing upper-, lower surfaces and the trailing vortex sheet.

Let us assume that the doublet gradient $D\mu$ is Hölder-continuous at T. Then, each of the doublet strengths μ on S_U, S_L and S_W can be expressed along the trailing edge as

$$\mu(\eta) = \mu_0 + (\frac{d\mu}{d\eta})_0 \eta + O(\eta^{1+\nu}),\ \nu > 0$$

where the subscript o denotes the values at T. Consider the following two integrals:

$$L_1 = \int_{-b}^{b} \frac{\mu}{r}\, d\eta$$

and

$$L_2 = \int_{-b}^{b} \frac{\mu_0 + (d\mu/d\eta)_0\eta}{r}\, d\eta.$$

Since the following estimations hold:

$$|L_1 - L_2| < C\int_{-b}^{b}\frac{|\eta^{1+\nu}|}{r}\, d\eta \text{ and}$$

$$|\nabla(\int_{-b}^{b}\frac{|\eta^{1+\nu}|}{r}\, d\eta)| < C'\int_{-b}^{b}|\eta^{\nu-1}|\, d\eta$$

the singular behaviour in L_1 is the same as that in L_2. The latter can be readily determined by performing the integration actually. The result is

$$S\{\nabla\times(\Sigma_{S_i}\int_{\partial S_i}\frac{\mu}{r}\, t\, ds)\} = 2(\mu_U - \mu_L - \mu_W)\, j\times a\frac{1}{\rho} \qquad (6.9)$$

where the symbol S{ } indicates the singular part of the quantity inside the bracket, ρ the distance of P to T, **a** the unit vector in the direction from T to P, and μ_U, μ_L and μ_W are respectively the values at T of the doublet strengths μ associated with S_U, S_L and S_W.

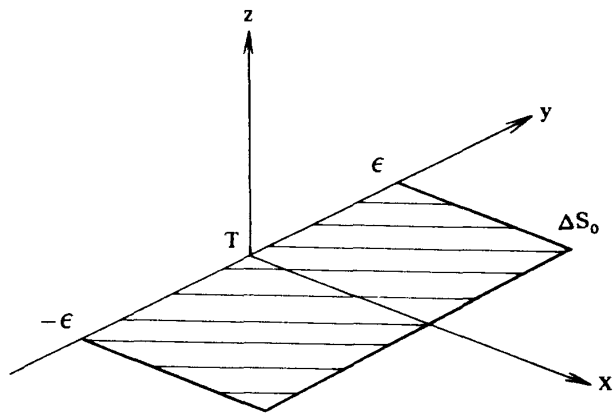
(2) surface-integral part

As is shown in Appendix I, the singular behaviour of a surface integral

$$I(P;S) = \int_S \alpha(Q)\nabla_P(\frac{1}{r_{PQ}})dS_Q$$

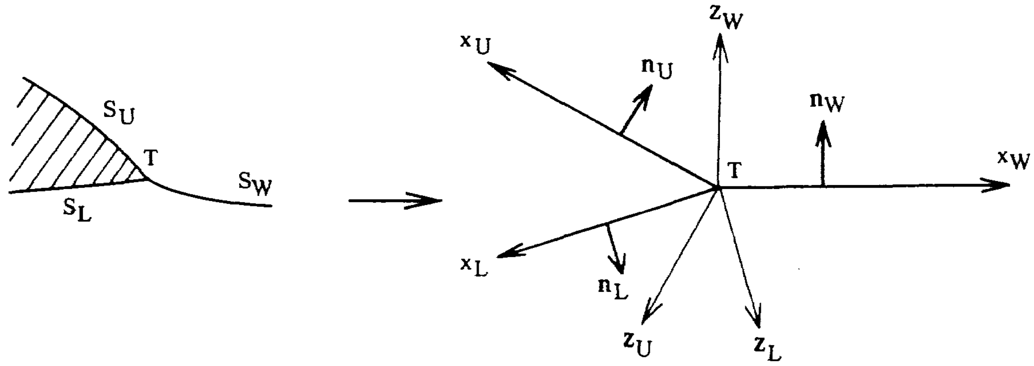
as P tends to T is given by

$$S\{I(P;S)\} = \alpha(T)[-\log\rho^2 i + 2(\varphi-\pi)k] \qquad (6.10)$$



Sketch 4. Integration Domain for I (P ; S)

provided $\alpha(Q)$ is Hölder-continuous at T. The expression on the right-hand side has been referred to the coordinate system such that the origin is taken at T, the y-axis is taken along the trailing edge, and the x-axis lies on the tangent plane to S at T so that the essential part of S can be approximated by the plane region $0 < x < \epsilon$, $-\epsilon < y < \epsilon$ and z=0, cf. Sketch 4. In the expression (6.10), i and **k** are the unit vectors along the x- and z-axes respectively. In deriving (6.10), the point P has been assumed to approach T along the straight line $x\sin\varphi - z\cos\varphi = 0$ and y=0, $0 < \varphi < 2\pi$, and ρ is the distance of P to T: $\rho = \sqrt{x^2+z^2}$. The expression (6.10) thus indicates that the surface integral I(P;S) exhibits a logarithmical singularity in the x-component and an approaching path-dependent behaviour in the z-component



Sketch 5. Base Vectors in Respective Coordinate Systems

while none singular behaviour in the y-component.

Now, since

$$\nabla_P \times [\frac{\lambda(Q)}{r_{PQ}}] = \nabla_P(\frac{1}{r_{PQ}}) \times \lambda(Q)$$

it follows from (6.10) that

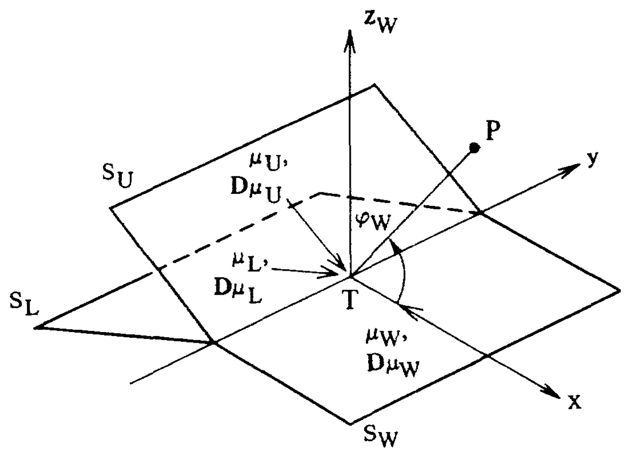
$$\underset{P\to T}{S}\{\nabla \times \int \frac{\lambda}{r} dS\}$$
$$= [-\log\rho^2 i + 2(\varphi - \pi)k] \times \lambda(T).$$

Hence

$$\underset{P\to T}{S}\{\nabla \times \int \frac{n \times D\mu}{r}\} = (D\mu \times n)_T$$
$$\times [-\log\rho^2 i + 2(\varphi - \pi)k]$$
$$= (n \times i) \times D\mu_T \log\rho^2 - 2(\varphi - \pi)(k \cdot n) D\mu_T \qquad (6.11)$$

because $D\mu \cdot n = 0$, where the subscript T denotes the values at T.

In order to apply the expression (6.11) to obtain respective contributions to $\nabla\varphi_D$ from the surfaces S_U, S_L and S_W, the coordinate systems associated with each of these surfaces are introduced as depicted in Sketch 5. Let (i_U, j, k_U), (i_L, j, k_L) and (i_W, j, k_W) be the sets of the base vectors of S_U-, S_L- and S_W-systems respectively with j being common to all systems. Then, noting that $n=k_L$ and $n=k_W$ for S_L and S_W respectively but $n=-k_U$ for S_U, application of (6.11) to the surface-integral part of (6.8) leads to



Sketch 6.
Quantities Associated with the Limiting Process of P Approaching T

$$\underset{P\to T}{S}\{(\Sigma_{S_i}\nabla \times \int_{S_i} \frac{n \times D\mu}{r} dS)\}$$
$$= (D\mu_U - D\mu_L - D\mu_W) \times j \log\rho^2$$
$$+ 2(\varphi_U - \pi)D\mu_U - 2(\varphi_L - \pi)D\mu_L$$
$$- 2(\varphi_W - \pi)D\mu_W \qquad (6.12)$$

where $D\mu_U$ etc are the limits of $D\mu$ on S_U etc as P tends to T, and φ_U etc are the angles which the approaching path of P towards T makes with the x_U-axis etc, cf. Sketch 6. It is noted here that these angles φ_U, φ_L and φ_W should take the values lying between 0 and 2π because the expression (6.10) has been obtained on the postulation that the angle φ lies in this interval.

So far, the singular terms in the right-hand side of (6.8) are extracted by dividing the consideration into the line-integral- and the surface-integral parts. Now assembling these contributions given by (6.9) and (6.12) respectively, the singular parts in $\nabla\varphi_D(P)$ as P approaches T are expressed as

$$\underset{P\to T}{S}\{\nabla\varphi_D(P)\} = \frac{1}{2\pi}\{[(\mu_U - \mu_L - \mu_W) \times (k_W \cos\varphi_W - i_w \sin\varphi_w)]\frac{1}{\rho} + [(D\mu_U - D\mu_L - D\mu_W) \times j] \log\rho + [\varphi_U D\mu_U - \varphi_L D\mu_L - \varphi_w D\mu_w - \pi(D\mu_U - D\mu_L - D\mu_w)]\} . \quad (6.13)$$

The conditions for $\nabla\varphi_D$ to remain finite as P → T ($\rho \to 0$) are then

$$\mu_U - \mu_L - \mu_w = 0$$

and

$$(D\mu_U - D\mu_L - D\mu_w) \times j = 0.$$

Since T is an arbitrary point on the trailing edge, these two conditions must hold along the whole length of the trailing edge. Differentiating the first condition along the trailing edge, one obtains

$$(D\mu_U - D\mu_L - D\mu_w) \cdot j = 0$$

which combines with the second condition in the above to yield

$$D\mu_U - D\mu_L - D\mu_w = 0 \quad (6.14)$$

along the whole trailing edge.

It has been shown that $D\mu$ on the wing surface is identical with the flow velocity **V** along the surface, cf. (3.18). In particular, one possesses the relations

$$D\mu_U = V_U \quad \text{and} \quad D\mu_L = V_L$$

where V_U and V_L are the limits of the wing upper- and lower surface velocities as the trailing edge is approached. Using this fact, the non-divergence conditions for $\nabla\varphi_D$ are expressed as

$$\mu_U - \mu_L - \mu_w = 0 \quad (6.15a)$$

and

$$V_U - V_L - D\mu_w = 0 \quad (6.15b)$$

along the trailing edge.

The condition (6.15a) indicates that the line integrals in (6.8) arising from the trailing edge boundaries of S_U, S_L and S_W completely cancel as a whole. Another implication of (6.15a) is as follows: the circulation around a wing section C(T) which passes through the point T on the trailing edge is given by

$$\Gamma(T) = \oint_{C(T)} V \cdot ds = \oint_{C(T)} D\mu \cdot ds = \mu_U - \mu_L \quad (6.16a)$$

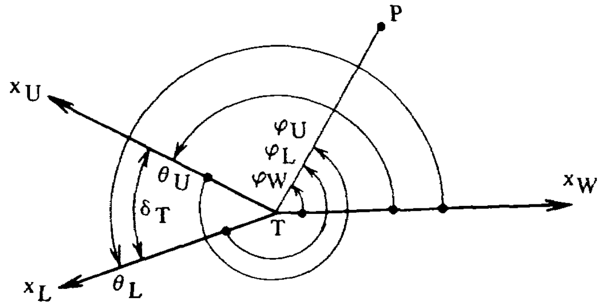
Using this Γ, the condition (6.15a) is written as

$$\Gamma = \mu_w, \quad (6.16b)$$

that is, the doublet strength of the trailing vortex sheet at the trailing edge is equal to the circulation around the wing, which is a well-known statement. Now Γ vanishes at the wing tips. So does μ_W there. It follows that the doublet strength along the side edges of the trailing vortex sheet vanishes identically since the side edges should coincide with the mean streamlines emanating from the wing tips, and since the doublet strengths are constant along these mean streamlines, see the discussion at the end of the preceding section.

Thus it is established that no line integrals need to be involved in applying the relation (2.8) and (2.10) to the doublet potential (6.1) because, although the wing surface must be treated as an open surface by applying a cut along the trailing edge where a discontinuity in the doublet strengths is expected, the line integrals along the trailing edge arising from this cut are exactly cancelled by that arising from the trailing vortex sheet counterpart by virtue of the condition (6.15a), and because the line integrals associated with other edges of the trailing vortex sheet vanish identically since the doublet strengths are nil along these boundaries.

Returning to the expression (6.13), one observes that $\nabla\varphi_D$ contains a term that is dependent on the path along which P tends to T. Assuming that the conditions (6.15a) and (6.15b) are satisfied, the expression (6.13) becomes



Sketch 7. Definition of Relevant Angles

$$\underset{P\to T}{S}\{\nabla\varphi_D(P)\} = \frac{1}{2\pi}(\varphi_U V_U - \varphi_L V_L - \varphi_w D\mu_w) \tag{6.17}$$

Let θ_U and θ_L be the angles which the x_U- and x_L-axes respectively make against the x_W-axis. If it is assumed that the trailing vortex sheet emanates from the trailing edge in a direction such that its tangent plane, when extended upstream, enters the inside of the wing, then θ_U and θ_L fall in the ranges.

$$\pi - \delta_T \leq \theta_U \leq \pi \tag{6.18a}$$

and

$$\pi \leq \theta_L \leq \pi + \delta_T \tag{6.18b}$$

respectively, where δ_T designates the trailing edge angle, cf. Sketch 7. Now, in studying the condition (6.17), let us first consider the case in which P approaches T from 'above', i.e. along a path such that the path angle φ_W measured from the x_W-axis is in the range $0 < \varphi_W < \theta_U$. In this case, φ_U and φ_L are given as (cf. sketch 7)

$$\varphi_U = 2\pi - \theta_U + \varphi_w$$

and

$$\varphi_L = 2\pi - \theta_L + \varphi_w.$$

Then (6.17) reduces to

$$\underset{P\to T}{S}\{\nabla\varphi_D(P)\} = V_U - V_L + \frac{1}{2\pi}(\theta_L V_L - \theta_U V_U). \tag{6.19}$$

Next, consider the case where P approaches T from 'below', i.e. along a path such that φ_W is in the range $\theta_L < \varphi_W < 2\pi$. Then φ_U and φ_L are given by

$$\varphi_U = \varphi_W - \theta_U$$

and

$$\varphi_L = \varphi_W - \theta_L,$$

and (6.17) become

$$S\{\nabla\varphi_D(P)\} = \frac{1}{2\pi}(\theta_L V_L - \theta_V V_U) \tag{6.20}$$

The expressions (6.19) and (6.20) indicate that $\nabla\varphi_D(P)$ is in fact independent of the approaching path but exhibits a difference according as P approaches the trailing edge either from 'above' or from 'below'. This difference is given by

$$\nabla\varphi_D(P)\,|_{P\downarrow T} - \nabla\varphi_D(P)\,|_{P\uparrow T} = V_U - V_L$$

and is equal to $D\mu_w$ in view of the condition (6.15b).

Thus, one obtains the relation

$$q_U - q_L = V_U - V_L = D\mu_w \tag{6.21}$$

where q_U and q_L are the limits of the flow velocity at the trailing edge when the edge is approached from 'above' and 'below' respectively. In the preceding section, it has been shown that the flow velocity suffers a jump of $D\mu$ across the trailing vortex sheet, see the expression (6.5). The condition (6.15b) ensures that this jump relations holds continuously up to the trailing edge.

Though the discussion has been rather lengthy, the essential result in the present section is solely the conditions (6.15a) and (6.15b). These conditions ensure that the flow velocity expressed in terms of the doublet distribution potential (6.1) remains finite as the wing trailing edge is approached provided the doublet gradient $D\mu$ is Hölder-continuous along the trailing edge.

6.3 Behaviour of the Integral Equation at the Trailing Edge

Reiterating our integral equation (6.2) for the surface velocity component **V** on a lifting wing, it is rewritten as

$$\mathbf{V}(P) = 2\mathbf{V}_\infty + \frac{1}{2\pi}\oint_{S_B} (\mathbf{V} \times \mathbf{n})_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q + \frac{1}{2\pi}\int'_{S_W} (\mathbf{D}\mu \times \mathbf{n})_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q. \quad (6.22)$$

Also, the surface velocity $\mathbf{q}^+$ or $\mathbf{q}^-$ at a point on the trailing vortex sheet has been given in (6.3) and is reproduced here as

$$\mathbf{q}^{\pm}(P) = \mathbf{V}_\infty \pm \frac{1}{2}\mathbf{D}\mu + \frac{1}{4\pi}\int_{S_B} (\mathbf{V} \times \mathbf{n})_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q + \frac{1}{4\pi}\oint'_{S_W} (\mathbf{D}\mu \times \mathbf{n})_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q \quad (6.23)$$

It is noted here that the condition (6.15a) has been assumed in deriving these equations starting from the doublet potential (6.1) so that the line-integral terms in $\nabla\varphi_D$ as appeared in (6.8) can be eliminated from the outset.

Now the behaviour of the right-hand sides of (6.22) and (6.23) is examined as the evaluation point P approaches a point T on the tailing edge. In order to do this a formula corresponding to (6.10) and valid for the case of a Cauchy integral is needed. This is derived, as is shown in Appendix **I**, as

$$\underset{P\to T}{S}\left\{\oint_S \alpha(Q)\nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q\right\} = -\alpha(T)\log\rho^2 \mathbf{i} \quad (6.24)$$

for P lying on S provided $\alpha(Q)$ is Hölder-continuous at the point T. The notation in (6.24) is the same as used with the formula (6.10). Correspondingly, the expression (6.11) for the case of a Cauchy integral becomes

$$\underset{P\to T}{S}\left\{\oint_S (\mathbf{D}\mu \times \mathbf{n})_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right) dS_Q\right\} = (\mathbf{n} \times \mathbf{i}) \times \mathbf{D}\mu_T \log\rho^2. \quad (6.25)$$

Let P_U, P_L and P_W be points lying on the wing upper surface S_U, the lower surface S_L and the trailing vortex sheet S_W respectively. Consider, for instance, the limit of the right-hand side of (6.22) as $P{=}P_U$ approaches the trailing edge. Let us denote by $S\{\mathbf{V}(T_U)\}$ the non-regular part of the right-hand side of (6.22) arising in the limiting process of $P_U \to T$. Utilization of (6.25) as well as (6.11) leads to

$$S\{\mathbf{V}(T_U)\} = \frac{1}{\pi}[(\mathbf{V}_U - \mathbf{V}_L - \mathbf{D}\mu_W) \times \mathbf{j}\log\rho - (\pi + \theta_U - \theta_L)\mathbf{V}_L - (\theta_U - \pi)\mathbf{D}\mu_W] \quad (6.26a)$$

because φ_L and φ_W in this case are given as

$$\varphi_L = 2\pi - \theta_L + \theta_U$$

and

$$\varphi_W = \theta_U,$$

see Sketch 8, (a). Similarly, one obtains the following results for $P_L \to T$ and $P_W \to T$:

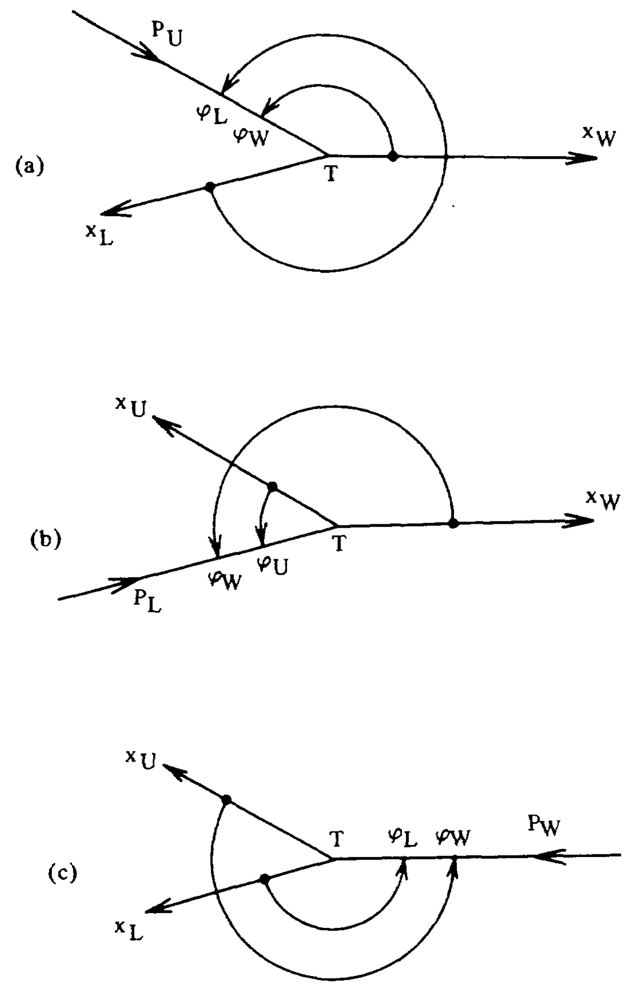
$$S\{\mathbf{V}(T_L)\} = \frac{1}{\pi}[(\mathbf{V}_U - \mathbf{V}_L - \mathbf{D}\mu_W) \times \mathbf{j}\log\rho + (\theta_L - \theta_U - \pi)\mathbf{V}_U - (\theta_L - \pi)\mathbf{D}\mu_W] \quad (6.26b)$$

because $\varphi_U = \theta_L - \theta_U$ and $\varphi_W = \theta_L$ for $P = P_L$ (cf. Sketch 8, (b)), and

$$S\{\mathbf{q}^{\pm}(T)\} = \frac{1}{2\pi}[(\mathbf{V}_U - \mathbf{V}_L - \mathbf{D}\mu_W) \times \mathbf{j}\log\rho + (\pi - \theta_U)\mathbf{V}_U - (\pi - \theta_L)\mathbf{V}_L] \quad (6.26c)$$

because $\varphi_U = 2\pi - \theta_U$ and $\varphi_L = 2\pi - \theta_L$ for $P{=}P_W$ (Sketch 8, (c)).

Let **R** denote the non-singular portion on the right-hand side of (6.22) as **P** approaches the trailing edge. Obviously, then, the non-



Sketch 8.
Identification of Approach Path Angles

singular portion of (6.23) apart from the term $\pm \frac{1}{2}\mathbf{D}\mu$ is given by $(1/2)\mathbf{R}$. Now, assuming that the condition (6.15b) is satisfied, the limiting forms of (6.22) and (6.23) as the trailing edge is approached become

$$\mathbf{V}_U = \mathbf{R} - \frac{1}{\pi}[(\theta_U - \pi)\mathbf{V}_U + (2\pi - \theta_L)\mathbf{V}_L], \quad (6.27a)$$

$$\mathbf{V}_L = \mathbf{R} - \frac{1}{\pi}[\theta_U \mathbf{V}_U + (\pi - \theta_L)\mathbf{V}_L], \quad (6.27b)$$

$$\mathbf{q}_U = \frac{1}{2}\mathbf{R} + \frac{1}{2}\mathbf{D}\mu_W - \frac{1}{2\pi}[(\theta_U - \pi)\mathbf{V}_U + (\pi - \theta_L)\mathbf{V}_L], \quad (6.28a)$$

and

$$\mathbf{q}_L = \frac{1}{2}\mathbf{R} - \frac{1}{2}\mathbf{D}\mu_W - \frac{1}{2\pi}[(\theta_U - \pi)\mathbf{V}_U + (\pi - \theta_L)\mathbf{V}_L]. \quad (6.28b)$$

Note that the Hölder-continuity in $\mathbf{V}$ and $\mathbf{D}\mu$ has been underlying assumption to derive these results.

The first two equations in the above reduce to the single equation

$$\mathbf{R} = \frac{1}{\pi}[\theta_U \mathbf{V}_U + (2\pi - \theta_L)\mathbf{V}_L] \quad (6.29)$$

while the latter two equations reduce to, respectively,

$$\mathbf{q}_U = \mathbf{V}_U \quad (6.30a)$$

and

$$\mathbf{q}_L = \mathbf{V}_L. \quad (6.30b)$$

That is, the condition (6.15b) assures that at the trailing edge, the wing upper- and lower surface velocities $\mathbf{V}_U$ and $\mathbf{V}_L$ enjoy continuous transition to the trailing-vortex-sheet upper- and lower velocities $\mathbf{q}_U$ and $\mathbf{q}_L$ respectively.

The expressions (6.27a) and (6.27b) indicate that the integral equation (6.22) written for $\mathbf{V}_U$ and $\mathbf{V}_L$ degenerates into the single equation (6.29) at the trailing edge. Therefore one more condition is needed, in principle, so that the two quantities $\mathbf{V}_U$ and $\mathbf{V}_L$ are determined. This freedom may be exploited to incorporate the Kutta condition into the formulation. How this is done will be discussed in the next subsection.

6.4 Implication of the Kutta Condition

The Kutta condition at the trailing edge is expressed, for the steady-flow case, as

$$(\mathbf{V}_U + \mathbf{V}_L) \cdot (\mathbf{V}_U - \mathbf{V}_L) = 0. \quad (6.31)$$

Now consider the following iteration process:
(1) assume $\mathbf{D}\mu_W$, i.e. the value of $\mathbf{D}\mu$ at the trailing edge,
(2) fix a tentative trailing vortex sheet such

that the doublet strength distribution upon it conforms to the assumed $\mathbf{D}\mu_W$,

(3) solve the integral equation (6.22) to obtain the wing surface velocity **V** at points off the trailing edge,

(4) determine $\mathbf{V}_U$ and $\mathbf{V}_L$ i.e. the wing upper- and lower surface velocities at the trailing edge, by use of (6.15b) and (6.29),

(5) from **V** thus obtained, calculate the circulation Γ around the wing at each point T on the trailing edge to obtain $\mu_W(T)$, i.e. the doublet strength at T of the trailing vortex sheet,

(6) determine the distribution of the doublet strength μ just downstream of the trailing edge by assigning the value of $\mu_W(T)$ to each curve which emanates from each point on the trailing edge in the direction tangent to the mean velocity at the trailing edge $\mathbf{V}_m = (\mathbf{V}_U + \mathbf{V}_L)/2$,

(7) determine $\mathbf{D}\mu_W$ from this μ-distribution,

(8) compare $\mathbf{D}\mu_W$ thus obtained with the one assumed at the outset. If they disagree, determine a corrected value of $\mathbf{D}\mu_W$ in an appropriate way and return to the Step 2.

A detailed discussion about the convergence of this iteration process cannot be offered at this stage. At least, however, one can assert that the existence of such $\mathbf{D}\mu_W$ is conceptually not inconsistent.

This $\mathbf{D}\mu_W$ by nature satisfies the following two conditions:

$$\mathbf{D}\mu_W = \mathbf{V}_U - \mathbf{V}_L \tag{6.32a}$$

and

$$\mathbf{D}\mu_W \cdot \mathbf{V}_m = 0. \tag{6.32b}$$

Then, since $\mathbf{V}_m = (\mathbf{V}_U + \mathbf{V}_L)/2$, the Kutta condition (6.31) immediately follows. That is, the Kutta condition is a natural consequence of the requirements that the non-singularity condition (6.15b) is observed, and that the trailing vortex sheet evolves from the trailing edge in the manner that the direction of the sheet is tangent to the mean velocity and the doublet strength upon it is so distributed that the condition (6.7) is met.

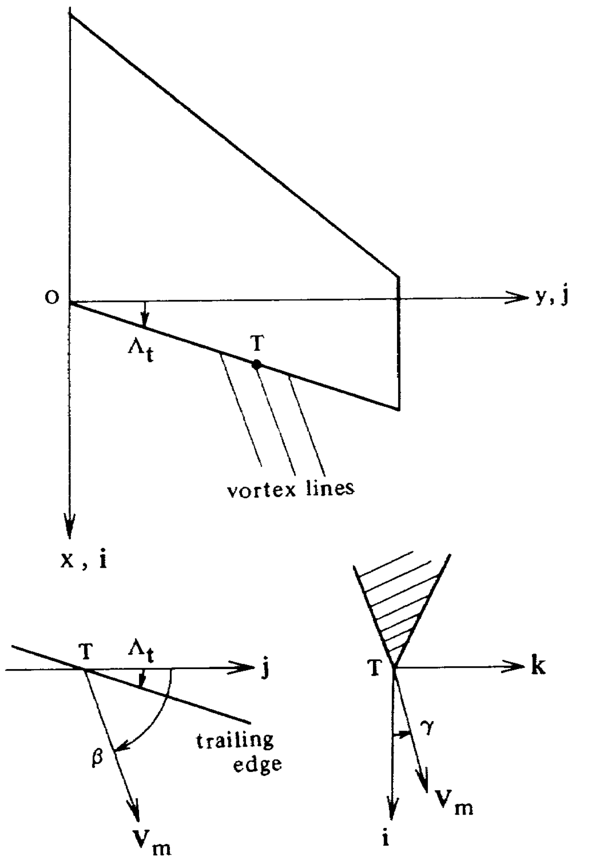
Thus, in our formulation, the fulfilment of the Kutta condition is tightly coupled with the trailing vortex sheet characteristics at the trailing edge. The situation in the majority of the existing formulations for the lifting wing potential flow field makes a contrast to ours in this respect since the starting direction of the trailing vortex sheet at the trailing edge can be arbitrarily specified in them. An advantage in those formulations is that this freedom can be exploited to match the computed results with the observations in the real flow field. In fact, with those formulations, the geometry of the trailing vortex sheet is fixed in accordance with the Prandtl's model and the doublet strengths are constant along the straight lines which are aligned with the free-stream direction. Matching with the real flow is made through other agencies such as the location of the Kutta-condition control points.

If one sticks to a rigorous representation, the conditions (6.32a) and (6.32b) must be observed by any formulation as long as the Kutta condition is to be satisfied because, the condition (6.32a) is necessary for the flow velocity to remain continuous at the trailing edge while the condition (6.32b) is the Kutta condition written by use of this velocity-continuity condition.

Calculation by a simple model may illustrate the effects of accounting for the condition (6.32b) for the starting geometry of the trailing vortex sheet. Consider a wing with trailing-edge sweep-back angle of Λ_t. Let us introduce the coordinate system (x, y, z) such that the origin is taken at the root trailing edge, the x-axis is aligned with the free-stream direction pointing downstream, and the y-axis lying on the wing plane and directed starboard, cf. Sketch 9. Let, further, t be the distance along the trailing edge measured from the origin. Then a point $T(x_t, y_t, z_t)$ lying on the trailing edge is expressed as

$$(x_t, y_t, z_t) = (t \sin \Lambda_t, t \cos \Lambda_t, 0).$$

Next, let the direction of the mean velocity $\mathbf{V}_m$ at the trailing edge be specified by the two



Sketch 9.
Definition of Trailing Vortex Sheet Geometry

angles β and γ, which the projections of $\mathbf{V}_m$ upon the x-y- and x-z planes make with the y- and x-axes respectively, cf. Sketch 9. As was already stated (see, e.g. the discussion at the end of Section 6.1), the doublet strength μ is constant along each of the lines starting from the trailing edge downstream in the direction of $\mathbf{V}_m$. These lines (henceforth called the vortex lines because the vortex vector defined by (2.12) is always tangent to these lines) are expressed as

$$x = x(s, t) = x_t + s \sin\beta \cos\gamma$$
$$y = y(s, t) = y_t + s \cos\beta$$
$$z = z(s, t) = z_t + s \sin\beta \sin\gamma$$

where s denotes the distance along each of the lines measured from the trailing edge. These expressions as a set can be deemed a two-parameter representation of the trailing vortex sheet.

Let **t** and **s** be the vectors defined by

$$\mathbf{t} = (\partial x/\partial t, \partial y/\partial t, \partial z/\partial t) = (\sin \Lambda_t, \cos \Lambda_t, 0)$$

$$\mathbf{s} = (\partial x/\partial s, \partial y/\partial s, \partial z/\partial s) = (\sin\beta \cos\gamma, \cos\beta, \sin\beta \sin\gamma)$$

Then, since $\mathbf{D}\mu$ lies on the trailing vortex sheet and is normal to the vortex lines, it satisfies the following two conditions:

$$\mathbf{D}\mu \cdot (\mathbf{t} \times \mathbf{s}) = 0$$

and

$$\mathbf{D}\mu \cdot \mathbf{s} = 0.$$

Now suppose that the doublet strength along the trailing edge is given as $\mu = \mu_W(t)$. Then one sees that

$$(\mathbf{D}\mu \cdot \mathbf{t})_{s=0} = \frac{d\mu_W}{dt}$$

From these three conditions, $\mathbf{D}\mu_W$ is determined as

$$\mathbf{D}\mu_W = \frac{d\mu_W}{dt} \frac{1}{\sin^2\Omega}(\mathbf{t} - \mathbf{s} \cos\Omega)$$

where

$$\cos\Omega = \mathbf{t} \cdot \mathbf{s} = \sin\Lambda_t \sin\beta \cos\gamma + \cos\Lambda_t \cos\beta.$$

Since the angle γ is at most of the order of the trailing-edge angle, one may put approximately $\cos\gamma=1$ and $\sin\gamma=0$. Observing that

$$\frac{d\mu_W}{dt} = \cos\Lambda_t \frac{d\mu_W}{dy} = \cos\Lambda_t \frac{d\Gamma}{dy}$$

one obtains the expression of $\mathbf{D}\mu_W$ as

$$\mathbf{D}\mu_W = \frac{d\Gamma}{dy} \cdot \frac{\cos\Lambda_t}{\sin(\beta - \Lambda_t)}(-\cos\beta, \sin\beta, 0). \tag{6.33}$$

Obviously, this $\mathbf{D}\mu_W$ agrees with that given by the Prandtl's model when β is equal to $\pi/2$. Compared to the latter model, the vortex strength in the present example is greater when $\mathbf{V}_m$ is directed outboard (i.e. when $\beta<\pi/2$). As $\mathbf{V}_m$ rotates from the free-stream direction towards the trailing edge direction ($\beta \rightarrow \Lambda_t$) the strength increases indefinitely.

The angle β is among the unknowns of the problem and is determined as a part of the solution. As for the angle γ, though it is also a variable of the problem, its latitude of variation can be fairly narrowed down when reference is made to the Mangler & Smith's analysis.[30] Their analysis is recapitulated in the following.

The wing surface velocities $\mathbf{V}_U$ and $\mathbf{V}_L$ at the trailing edge and the doublet gradient $\mathbf{D}\mu_W$ are the vectors confined within the respective surface S_U, S_L and S_W. Suppose that they are resolved as, for example,

$$\mathbf{V}_U = V_U^f \, f_U + V_U^t \, \mathbf{t}$$

where the unit vector $\mathbf{t}$ is along the trailing edge while f_U is the unit vector lying on S_U and normal to t. Likewise $\mathbf{V}_L$ and $\mathbf{D}\mu_W$ are expressed as

$$\mathbf{V}_L = V_L^f f_L + V_L^t \, \mathbf{t}$$

and

$$\mathbf{D}\mu_W = D\mu_W^f f_W + D\mu_W^t \, \mathbf{t}.$$

Using these expressions, the Kutta condition (6.31) is written as

$$(V_U^f)^2 + (V_U^t)^2 = (V_L^t)^2 + (V_L^t)^2$$

while the trailing edge-wise component of the condition (6.15b) reads

$$V_U^t - V_L^t = D\mu_w^t = \frac{d\Gamma}{dt}$$

where t is a distance measured along the trailing edge. Defining V_m^t by

$$V_m^t = (V_U^t + V_L^t)/2 = \mathbf{V}_m \cdot \mathbf{t},$$

the Kutta condition is rewritten as

$$2V_m^t \frac{d\Gamma}{dt} = (V_L^f)^2 - (V_U^f)^2. \qquad (6.34)$$

Now, on the assumption that the flow in the plane normal to the trailing edge is featured by a two-dimensional flow past corners, Mangler & Smith argue that the three surfaces S_U, S_L and S_W cannot form a convex corner at the trailing edge because if so the flow velocity goes to infinity at such a corner. Then, only three alternatives exist for the direction of the trailing vortex sheet at the trailing edge, viz.

Case A: the sheet emerges between the two wing surfaces, i.e.,

$$\pi - \delta_T < \theta_U < \pi \text{ and } \pi < \theta_L < \pi + \delta_T ;$$

where θ_U is the angle between the wing upper surface and the trailing vortex sheet, and θ_L is the one between the wing lower surface and the trailing vortex sheet, see Sketch 7 in page 27.

Case B: the sheet emerges in the tangential direction to the upper surface, i.e.,

$$\theta_U = \pi \text{ and } \theta_L = \pi + \delta_T ;$$

Case C: the sheet emerges in the tangential direction to the lower surface, i.e.,

$$\theta_U = \pi - \delta_T \text{ and } \theta_L = \pi.$$

In Case A, V_U^f and V_L^f must vanish because the three surfaces S_U, S_L and S_W form two concave corners at the trailing edge, and hence $V_m^t \, d\Gamma/dt = 0$ from (6.34). Similarly, V_L^f vanishes in Case B whence $V_m^t \, d\Gamma/dt < 0$ whereas in Case C, V_U^f must be zero leading to $V_m^t \, d\Gamma/dt > 0$. Thus, the sign of the product $V_m^t \, d\Gamma/dt$ determines which of the three alternative cases actually takes place at each point on the trailing edge. Mangler & Smith maintain that the Case A is exceptional since it corresponds to the condition either of no shed vorticity ($d\Gamma/dt = 0$) or of no mean velocity ($V_U^f = V_L^f = V_m^t = 0$) at the trailing edge. In other words, what usually occurs is the Case B or C, and hence γ is given by a combination of the angle β, the trailing edge sweep angle Λ_t, the incidence angle α and the wing geometry at the trailing edge.

This Mangler & Smith's result is inescapable as long as the assumption is accepted that the flow in the plane normal to the trailing edge can be identified with the two-dimensional flow past corners. This assumption looks plausible: one can hardly choose otherwise if one adheres to a flow which is physically realizable.

In spite of the fact that the potential flow formulation inevitably relies on the trailing-edge condition in order that its solution as a whole is determined so as to represent the real flow with reasonable accuracy, its solution is in general of little use in describing the actual flow situation in the proximity of the trailing edge. As a consequence, an accurate representation of the trailing vortex sheet may be unnecessary in practical calculations unless the global solution is seriously affected by approximate treatment of the sheet.

To what degree of accuracy the sheet must be represented will depend on the problem in hand and the method of formulation to be used. In our preliminary study of the effects of accounting for the variable β and γ in formulating the trailing vortex sheet, it is observed that, as long as the trailing-edge sweep angle remains moderate, taking account of variable β brings a change in circulations, and hence in lift, of the order of a few percents relative to the Prandtl-model formulation while the effects of γ are smaller than those of β by more than one order of magnitudes. Naturally, the importance of accurate representation of the trailing vortex sheet will grow with increase in the trailing-edge sweep. It is again emphasized that the trailing vortex sheet representation depends on the particular formulation of the problem under consideration.

7. CONCLUDING REMARKS

A formulation of the incompressible potential flow around a body by exclusive use of doublet distribution over the body surface, if the body-boundary condition is applied in the direct manner, results in a Fredholm integral equation of the first kind to determine the unknown doublet strengths. The motive of the present research is to cast the doublet-based formulation into an equation of the second kind since the latter type of equation is expected to be more tractable both mathematically and computationally than the former. An analysis spurred by this motive has led to the following findings:

(1) construction of a Fredholm integral equation of the second kind is realized by extending the Prager-Vandrey-Martensen procedure to the general three-dimensional case, the result being the equation (3.16);

(2) it is found that the gradient of the doublet strength is identical with the flow velocity along the body surface, which fact is used to rewrite our basic equation in the form of (3.19) as a unifcation of our equation (3.16) and that which results from a direct application of the Neumann-type boundary condition;

(3) the basic equation (3.19) is generalized as (4.25) or equivalently (4.26) to cases where the prescribed normal velocity components along the body surface is not necessarily vanishing;

(4) since it is expected that singularities show up in the present formulation along sharp trailing edge of a wing, the behaviour of the doublet potential is examined as the trailing edge is approached from within the flow field, and the expressions (6.15a) and (6.15b) are derived as the conditions with which the assumption of finite velocity at the trailing edge remains consistent;

(5) because of the way of its derivation, our basic equation (3.19) lacks validity at the sharp trailing edge of a wing, and is to be replaced there by the equation (6.29); and

(6) the implication of the Kutta condition at the trailing edge of a lifting wing is studied in the context of the present formulation, and it is shown that the condition can be satisfied if the trailing vortex sheet characteristics are so determined that the relations (6.32a) and (6.32b) hold along the trailing edge.

An advantage of our formulation is in the feature that the unknown of the problem is the velocity components of flow along the body surface, which are the most pertinent quantities both in representing the nature of the flow in hand and in calculating the aerodynamic forces acting on the body under consideration. This feature also makes it feasible to get an insight, based on a knowledge of the physics of flow, as to the performance of the mathematical model employed to describe the flow in the crucial regions such as the trailing-edge proximity.

The disadvantage, on the other hand, is that the variable is a vector defined on a surface. That is, two unknown scalar quantities must be dealt with at each point on the surface instead of one scalar unknown which is the case with the formulation based on a source distribution. When the panel method is applied in order to obtain a numerical solution, the resulting system of simultaneous algebraic equations possesses a size approximately double of that of one-scalar formulations if the number of the panels is the same in both cases. It is noted here that the storage requirement for the necessary matrix coefficients, however, stands to a ratio of 4 to 3 because in the one-scalar formulations two extra matrices need to be stored in order to generate the two tangential velocity components along the body surface after the solution for the primary variable is found.

An application-oriented theoretical frame is of little use unless it is verified by the practical performance. Preliminary attempts of applying our scheme to the problem of lifting-wing potential flow field have revealed that physically plausible solutions are hard to obtain by a straightforward application of the discretizing procedure typical in the first-generation panel method. Specifically, the approximation of the surface flow velocity by constant-strength distributions over each panel does not seem to work very well. Tests with nonlifting symmetrical bodies exemplified by ellipsoids show that, when the constant-strength approximation over each panel is employed, our basic equation generates results slightly inferior to those due to the source-distribution approach in the region where the velocity gradient is very large. This feature may bear a link to the characteristics of the two-dimensional vortex distribution which produces accurate solutions when discretized as a system either of concentrated vortices or of distributed vortices of linearly varying strengths but which exhibits surprisingly poor performance when approximated by a system of piecewise constant strength vortices.[31)]

It is also pointed out that a proper implementation of the trailing edge conditions established by the present work may require more sophisticated than the constant-strength representation of singularity strengths over each panel. A work along this line is now underway.

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Appendix I. A Proof of Properties (4.23) and (4.24)

Let Σ be the part of a closed surface S which is contained within the sphere (henthforth denoted by L) of radius ρ centred at a point P on S. Then the Cauchy integral on S can be defined as

$$\oint_S f(P,Q)dS_Q \equiv \lim_{\rho\to 0}\int_{S-\Sigma} f(P,Q)dS_Q.$$

First, consider the integral

$$I = \oint_S \mathbf{n}_Q \cdot \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q$$

where $\mathbf{n}$ denotes the unit normal to S directed outward. Let Ω be the part of the surface of the sphere L which is contained within the region enclosed by S. Then one may write

$$I = \lim_{\rho\to 0}\left[\left(\int_{S-\Sigma+\Omega} - \int_\Omega\right)\mathbf{n}_Q \cdot \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q\right].$$

Since $S-\Sigma+\Omega$ is a closed surface and P lies outside the region V enclosed by this surface, application of Gauss' theorem leads to

$$\int_{S-\Sigma+\Omega} \mathbf{n}_Q \cdot \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q = \int_{S-\Sigma+\Omega} \mathbf{n}_Q \cdot \nabla_Q\left(\frac{1}{r_{PQ}}\right)dS_Q = -\int_V \nabla_Q^2\left(\frac{1}{r_{PQ}}\right)dV_Q = 0.$$

On the other hand,

$$\int_\Omega \mathbf{n}_Q \cdot \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q = -\int_\Omega \mathbf{n}_Q \cdot \nabla_Q\left(\frac{1}{r_{PQ}}\right)dS_Q = -\int_\Omega -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right)dS = -\int_\Omega \sin\theta\, d\theta\, d\varphi$$

where (θ, φ) is a polar coordinates on the surface of L. Provided S is sufficiently smooth in the neighbourhood of P, the last integral approaches -2π as the radius ρ tends to zero. Thus the identity (4.23) has been established.

Next, consider the integral

$$J = \oint_S \mathbf{n}_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q \equiv \lim_{\rho\to 0}\int_{S-\Omega} \mathbf{n}_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q.$$

From the expressions (2.9) and (2.10) in the main body of the present report, one has the identity

$$\int_{S-\Sigma} \mu\mathbf{n}_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q = \int_{S-\Sigma}\frac{\lambda(Q)}{r_{PQ}}dS_Q - \int_{\partial(S-\Sigma)}\frac{\mu(Q)}{r_{QP}}t_Q ds_Q$$

where $\lambda = \mathbf{n} \times D\mu$.

Now let us put $\mu \equiv 1$ in the above identity. Since the boundary $\partial(S-\Sigma)$ of the surface $S-\Sigma$ consists of the boundary of Σ, it follows that

$$\int_{S-\Sigma} \mathbf{n}_Q \times \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q = \int_{\partial\Sigma}\frac{1}{r_{PQ}}t_Q ds_Q$$

But under a proper condition for S, one possesses the estimation

$$\int_{\partial\Sigma}\frac{1}{r_{PQ}}t_Q ds_Q = \int_0^{2\pi}\left[\frac{1}{\rho}(\cos\varphi, \sin\varphi, 0) + 0(\rho^0)\right]\rho\, d\varphi$$

and hence

$$\lim_{\rho\to 0}\int_{\partial\Sigma}\frac{1}{r_{PQ}}t_Q ds_Q = 0$$

which proves the identity (4.24).

Appendix II. A Proof of Expressions (6.10) and (6.24)

The problem to be dealt with here is to examine the behaviour of the integrals

$$I(P;S) = \int_S \alpha(Q)\,\nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q$$

for P locating outside of S, and

$$J(P;S) = \oint_S \alpha(Q)\,\nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q$$

for P lying on S, as P approaches a point T on the trailing edge, where the integration domain S represents either the wing upper surface or the lower surface, or the trailing vortex sheet. Since only part of S which is in an immediate neighbourhood of T is relevant to the singular behaviour under consideration, one may devides S into $S-\Delta S$ and ΔS and leaves the former out of the consideration, where ΔS is a small portion of S containing a part of the trailing edge of which the middle point is T.

It has been shown in Appendix II of Ref. 16 that a point $Q(\xi, \eta, \zeta)$ on S in a neighbourhood of the trailing edge can be expressed as

$$\begin{aligned} \xi &= \xi_T + u + O(\rho^2) \\ \eta &= \eta_T + v + O(\rho^2) \\ \zeta &= \zeta_T \quad + O(\rho^2) \end{aligned}$$

provided S possesses continuous curvature in the region under consideration, where (ξ_T, η_T, ζ_T) denotes the point T, u is a distance measured along the intersection of S with the plane normal to the trailing edge and passing through the point Q while v is a distance measured parallel to the trailing edge, and $\rho^2 = u^2 + v^2$. Let us then assume that ΔS is the region defined by

$$\Delta S : 0 < u < \epsilon \text{ and } -\epsilon < v < \epsilon.$$

where ϵ is a small positive number.

Let ΔS_0 be a plane region which approximates ΔS in the manner that a point $Q_0(\xi_0, \eta_0, \zeta_0)$ upon it is expressed by

$$\begin{aligned} \xi_0 &= \xi_T + u \\ \eta_0 &= \eta_T + v \\ \zeta_0 &= \zeta_T \end{aligned}$$

with the ranges of variation of u and v being identical with those for ΔS.

Consider then the following two integrals:

$$I(P;\Delta S) = \int_{\Delta S} \alpha(Q)\,\nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q$$

and

$$I_0(P;\Delta S_0) = \alpha(T)\int_{\Delta S_0} \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q,$$

the difference of which is written as

$$\begin{aligned} I - I_0 = &\int_{\Delta S} \{\alpha(Q) - \alpha(T)\}\,\nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q \\ &+ \alpha(T)\left\{\int_{\Delta S} \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q - \int_{\Delta S_0} \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q\right\} \end{aligned}$$

It has also been shown in Appendix II of Ref.16 that both the first integral and the expression within the curly bracket on the right-hand side of the above equation exhibit no singularity as P approaches T provided that P is outside of the integration domains ΔS and ΔS_0, and that $\alpha(Q)$ satisfied the condition of Hölder-continuity. Hence $I(P;\ \Delta S)$ can be replaced by $I_0(P;\ \Delta S_0)$ when studying the singular behaviour of the former in the process of P approaching T.

As for the integral J corresponding to the case of P lying on S, one may proceed parallel to the preceding, viz. by considering the two integrals

$$J(P;\Delta S) = \oint_{\Delta S} \alpha(Q)\nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q$$

and

$$J_0(P;\Delta S_1) = \alpha(P)\oint_{\Delta S_1} \nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q$$

where ΔS_1 is the projection of ΔS onto the tangent plane to S at P, one is to show that the balance

$$J - J_0 = J_1 + \alpha(P)J_2,$$

with

$$J_1 = \oint_{\Delta S} [\alpha(Q) - \alpha(P)]\,\nabla_P\left(\frac{1}{r_{PQ}}\right)dS_Q$$

$$J_2 = \oint_{\Delta S} \nabla_P(\frac{1}{r_{PQ}})dS_Q - \oint_{\Delta S_1} \nabla_P(\frac{1}{r_{PQ}})dS_Q$$

is to be shown non-singular in the process of P tending to T. The non-singularity may be asserted if it is verified that both J_1 and J_2 are bounded. This can be shown in the following way: let us assume that P is sufficiently close to T such that both ΔS and ΔS_1 are contained within the sphere of radius 2ϵ whose centre is P. Since

$$|\alpha(Q) - \alpha(P)| < C r_{PQ}^{\nu}$$

and

$$|\nabla_P(\frac{1}{r_{PQ}})| = \frac{1}{r_{PQ}^2}$$

provided $\alpha(Q)$ is Hölder-continuous, one obtains an estimation

$$|J_1| < C\int_{\Delta S} \frac{1}{r_{PQ}^{2-\nu}} dS < C\int_0^{2\epsilon} \frac{1}{\rho^{2-\nu}} 2\pi\rho d\rho$$
$$= \frac{2\pi C}{\nu}(2\epsilon)^{\nu}$$

Hence J_1 exhibits no singular behaviour as P approaches T.

In order to deal with J_2, let us introduce a polar coordinate system (ρ, θ) on ΔS_1 and an orthogonal surface coordinate system (u, v) on ΔS such that the two are related by

$$(u, v) = \rho(\cos\theta, \sin\theta).$$

Let a point $Q(\xi, \eta, \zeta)$ on ΔS be expressed as

$$\xi = \xi_P + u + O(\rho^2)$$
$$\eta = \eta_P + v + O(\rho^2)$$
$$\zeta = \zeta_P + O(\rho^2)$$

where (ξ_P, η_P, ζ_P) indicates the point P. It follows then that

$$dS_Q = K\,dudv,\quad K = 1 + O(\rho),$$

and

$$J_2 = \oint_{\Delta S_0} [K\nabla_P(\frac{1}{r_{PQ}}) - \nabla_P(\frac{1}{\rho})]\rho d\rho d\theta.$$

Since

$$r_{PQ}^2 = (\xi - \xi_P)^2 + (\eta - \eta_P)^2 + (\zeta - \zeta_P)^2$$
$$= \rho^2[1 + O(\rho)],$$

$$\nabla_P(\frac{1}{r_{PQ}}) = \frac{[u + O(\rho^2)]\,i + [v + O(\rho^2)]\,j + O(\rho^2)\,k}{r_{PQ}^3}$$

and

$$\nabla_P(\frac{1}{\rho}) = \frac{ui + vj}{\rho^3},$$

the integrand of J_2 is estimated as

$$|K\nabla_P(\frac{1}{r_{PQ}}) - \nabla_P(\frac{1}{\rho})|$$
$$\leqq |(K\frac{\rho^3}{r_{PQ}^3} - 1)$$
$$\times |\frac{[u + O(\rho^2)]\,i + [v + O(\rho^2)]\,j + O(\rho^2)k}{\rho^3}|$$
$$+ \frac{O(\rho^2)}{\rho^3} < C\frac{1}{\rho}$$

for sufficiently large positive constant C.
Hence

$$|J_2| < C\int_0^{2\epsilon} \frac{2\pi\rho d\rho}{\rho} = 4\pi C\epsilon$$

which indicates that J_2 nor exhibits singularity as P approaches T. It also can be shown that the difference $J_0(P; \Delta S_1) - J_0(P; \Delta S_0)$ gives rise to no singularity in the process of P approaching T and hence $J_0(P; \Delta S_1)$ can be replaced by $J_0(P; \Delta S_0)$ in examining the singular behaviour of J(P; S).

Now that the integrals I(P ; S) and J(P ; S) have been replaced by $I_0(P ; \Delta S_0)$ and $J_0(P ; \Delta S_0)$ respectively, the singular behaviour of the latter two is examined by actually carrying out the integration. To do this, let us introduce a coordinate system such that the origin is taken at T, the y-axis is aligned with the tangent to the trailing edge at T and the x-axis lies on ΔS_0, see Sketch 4 in page 24. Then a point $Q(\xi, \eta, \zeta)$ on ΔS_0 is in the range such that $0 < \xi < \epsilon$, $-\epsilon < \eta < \epsilon$ and $\zeta = 0$. Since

$$\nabla_P(\frac{1}{r_{PQ}}) = \frac{(\xi - x)i + (\eta - y)j - zk}{r_{PQ}^3}$$

where (x, y, z) indicates the point P, and (**i**, **j**, **k**) is the set of the base vectors in respective coodinate directions, carrying out the integration yields

$$I_0(P;\Delta S_0) = \alpha(T)$$

$$\Big\{ \mathbf{i} \log \sqrt{\frac{(r_{11}-\epsilon+y)(r_{10}-\epsilon-y)(r_{01}+\epsilon-y)(r_{00}+\epsilon+y)}{(r_{11}+\epsilon-y)(r_{10}+\epsilon+y)(r_{01}-\epsilon+y)(r_{00}-\epsilon-y)}}$$

$$+ \mathbf{j} \log \Big[\frac{(r_{11}+x-\epsilon)(r_{00}+x)}{(r_{10}+x-\epsilon)(r_{01}+x)}\Big]$$

$$+ \mathbf{k} \Big[\mathrm{Tan}^{-1}\frac{(\epsilon-x)(y-\epsilon)}{zr_{11}} - \mathrm{Tan}^{-1}\frac{(\epsilon-x)(y+\epsilon)}{zr_{10}}$$

$$- \mathrm{Tan}^{-1}\frac{x(y-\epsilon)}{zr_{01}} + \mathrm{Tan}^{-1}\frac{x(y+\epsilon)}{zr_{00}}\Big]\Big\}$$

and

$$J_0(P;\Delta S_0) = \alpha(P) \times$$

$$\Big\{ \mathbf{i} \log \sqrt{\frac{(r_{11}-\epsilon+y)(r_{10}-\epsilon-y)(r_{01}+\epsilon-y)(r_{00}+\epsilon+y)}{(r_{11}+\epsilon-y)(r_{10}+\epsilon+y)(r_{01}-\epsilon+y)(r_{00}-\epsilon-y)}}$$

$$+ \mathbf{j} \log \Big[\frac{(r_{11}+x-\epsilon)(r_{00}+x)}{(r_{10}+x-\epsilon)(r_{01}+x)}\Big]\Big\}$$

where

$$r_{11} = \sqrt{(x-\epsilon)^2 + (y-\epsilon)^2 + z^2},$$

$$r_{10} = \sqrt{(x-\epsilon)^2 + (y+\epsilon)^2 + z^2},$$

$$r_{01} = \sqrt{x^2 + (y-\epsilon)^2 + z^2},$$

and

$$r_{00} = \sqrt{x^2 + (y+\epsilon)^2 + z^2}.$$

Hereafter y is put equal to zero without loss of generality. Using a polar coordinates (ρ, φ) in lieu of (x, z) defined by $\rho\,(\cos\varphi, \sin\varphi) = (x, y)$, one is led to the estimations

$$I_0(P;\Delta S_0)$$

$$= \mathbf{i}\Big[2\log\Big(\frac{2}{\sqrt{2}+1}\Big) - \log\kappa^2 + O(\kappa)\Big]$$

$$+ \mathbf{k}\,[2(\varphi - \pi)],$$

and

$$J_0(P;\Delta S_0)$$

$$= \mathbf{i}\Big[2\log\Big(\frac{2}{\sqrt{2}+1}\Big) - \log\kappa^2 + O(\kappa)\Big]$$

where $k = \rho/\epsilon$ and φ is taken to be in the interval $0 < \varphi < 2\pi$.

Let S(I) and **R**(I) denote the singular- and non-singular parts respectively in the limit of P approaching T. Then one obtains

$$S\{I(P;S)\} = \alpha(T)\,[-(\log\rho^2)\mathbf{i} + 2(\varphi-\pi)\mathbf{k}]$$

$$S\{J(P;S)\} = \alpha(T)\,[-(\log\rho^2)\mathbf{i}]$$

$$\mathbf{R}\{I(P;S)\}$$

$$= \lim_{P\to T}\Big\{ I(P;S-\Delta S)$$

$$+\int_{\Delta S}[\alpha(Q)-\alpha(T)]\,\nabla_P\Big(\frac{1}{r_{PQ}}\Big)dS_Q$$

$$+\alpha(T)\Big[\int_{\Delta S}\nabla_P\Big(\frac{1}{r_{PQ}}\Big)dS_Q$$

$$-\int_{\Delta S_0}\nabla_P\Big(\frac{1}{r_{PQ}}\Big)dS_Q$$

$$+2\log\Big(\frac{2}{\sqrt{2}+1}\Big)\mathbf{i}\Big]\Big\}$$

and

$$\mathbf{R}\{J(P;S)\}$$

$$= \lim_{P\to T}\Big\{ J(P;S-\Delta S)$$

$$+\oint_{\Delta S}[\alpha(Q)-\alpha(T)]\,\nabla_P\Big(\frac{1}{r_{PQ}}\Big)dS_Q$$

$$+\alpha(P)\oint_{\Delta S}\nabla_P\Big(\frac{1}{r_{PQ}}\Big)dS_Q$$

$$-\alpha(T)\oint_{\Delta S_0}\nabla_P\Big(\frac{1}{r_{PQ}}\Big)dS_Q$$

$$+\alpha(T)\Big[2\log\Big(\frac{2}{\sqrt{2}+1}\Big)\Big]\mathbf{i}\Big\}$$

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