UDC 533. 6. 69. 048

TECHNICAL REPORT OF NATIONAL AEROSPACE LABORATORY

TR-629T

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September 1980

NATIONAL AEROSPACE LABORATORY

CHŌFU, TOKYO, JAPAN

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ABSTRACT

An ideal lattice arrangement of the vortex-lattice method for rectangular wings is obtained by applying Stark's quadrature formula for a Cauchy integral to the integral equation of the lifting-surface theory. The relation between the circulation of a lattice vortex and the corresponding local lift is thereby clearly defined, and the spanwise distribution of the induced drag is shown to be calculated accurately.

概　要

揚力面理論の積分方程式に Cauchy 積分に対する Stark の数値積分公式を適用することによって，矩形翼に対する vortex lattice 法の理想的な格子配置を得た。これによって，格子渦の循環と対応する局所揚力との関係を明確に定義することができ，さらにこれからの当然の帰結として誘導抵抗の翼幅方向分布を精度よく計算できることを示した。

NOMENCLATURE

A	aspect ratio
c	chord length
C_{Di}	total induced drag coefficient
C_{Di}'	induced drag coefficient per unit span
C_L	total lift coefficient
C_L'	lift coefficient per unit span
D_i	total induced drag
D_i'	induced drag per unit span
$D_{\Delta p}$	drag component of normal force (per unit span)
K	$\pi A C_{Di}/C_L^2$ (vortex drag factor)
M	number of chordwise discretization
N	number of spanwise discretization
s	semispan
S	leading edge suction (per unit span)
U	main stream velocity
w	upwash on wing surface
X_{ac}	overall centre of pressure
X_{ac}'	section centre of pressure
α	angle of attack
γ	local circulation (difference between streamwise velocity components of upper and lower surfaces)
Γ	circulation of lattice vortex
ΔC_p	pressure difference coefficient
ρ	density of air

1. INTRODUCTION

The vortex-lattice method based on the so called 1/4–3/4 rule for calculating steady subsonic lift distributions of lifting surfaces[1] has not only been successfully extended to cases of oscillating surfaces[2,3] but also been applied to various complex configurations, which proves its great versatility[4]. Further studies, however, appear to be desirable with regard to the basis for the method. It is indeed remarkable that the simple 1/4–3/4 rule gives the correct lift and moment in the two-dimensional equally spaced panel arrangement[1,5]. This rule seems, however, to give no rigorous basis for the relation between the circulation of the lattice vortex and the local lift. The usually assumed relation results in errors in the local lift immediate to the leading and trailing edges no matter how fine the discretization may be[5,6]. This is in fact the reason why the leading-edge suction cannot be predicted by the equidistant vortex-lattice method[7–9].

A solution for this problem was given by Lan who proposed a non-equidistant lattice arrangement by the aid of a mechanical quadrature for a singular integral of Cauchy type[10]. However, his lattice arrangement, in which control points are located also at the trailing edge, has little resemblence to the conventional lattice of 1/4–3/4 type. An arrangement showing more resemblence to the conventional one had practically been given by Stark earlier[11], but unfortunately his contribution does not seem to have been noticed in connection with the vortex-lattice method, probably because this was shown only as an

* Received July 23, 1980.
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example of his general quadrature formula for Cauchy integrals.

The purpose of the present paper is to show an ideal lattice arrangement for rectangular wings on the basis of Stark's quadrature formula. This arrangement enables us to define a clear relation between the local lift and the circulation of the lattice vortex. It is also shown that the leading edge suction and therefore the spanwise distribution of the induced drag can be computed accurately by this arrangement.

2. TWO-DIMENSIONAL WING

In Ref. 11, a formula of the form

$$\int_{-1}^{1} \frac{W(\xi)f(\xi)}{\xi-x_i}d\xi=\sum_{j=1}^{I}\frac{a_j W(\xi_i)f(\xi_j)}{\xi_j-x_i} \tag{1}$$

has been proved to be exactly valid for an arbitrary weight function $W(x)$ when $f(x)$ is a polynomial of degree $\leq 2I$. Eq. (1) is used, in the following, with two weight functions $\sqrt{(1-x)/(1+x)}$ and $1/\sqrt{1-x^2}$. For $W(x)=\sqrt{(1-x)/(1+x)}$,

$$\begin{aligned} a_j&=\frac{2\pi\sqrt{1-\xi_j^2}}{2I+1}\\ \xi_j&=-\cos\frac{(2j-1)\pi}{2I+1}\\ x_i&=-\cos\frac{2i\pi}{2I+1} \qquad i=1(1)I \end{aligned} \tag{2}$$

and for $W(x)=1/\sqrt{1-x^2}$,

$$\begin{aligned} a_j&=\frac{\pi\sqrt{1-\xi_j^2}}{I}\\ \xi_j&=-\cos\frac{(2j-1)\pi}{2I}\\ x_i&=-\cos\frac{i\pi}{I} \qquad i=1(1)I-1 \end{aligned} \tag{3}$$

A great advantage of Eq. (1) is that the weights a_j and abscissas ξ_j are identical with those of the Gaussian quadrature formula for the same weight function

$$\int_{-1}^{1} W(x)f(x)dx=\sum_{j=1}^{I}a_j W(\xi_j)f(\xi_j) \tag{4}$$

The integral equation for two-dimensional thin wings is of the form

$$w(x)=-\frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma(\xi)}{x-\xi}d\xi \tag{5}$$

where w is the upwash on the wing surface and γ the local circulation. If Eq. (1) with the weight function $\sqrt{(1-x)/(1+x)}$ is applied to Eq. (5), we have

$$w(x_n)=-\frac{1}{\pi c}\sum_{\nu=1}^{N}\frac{\Gamma_\nu}{x_n-\xi_\nu} \qquad n=1(1)N \tag{6}$$

where

$$\begin{aligned} x_n&=-\cos\frac{2n\pi}{2N+1}\\ \xi_\nu&=-\cos\frac{(2\nu-1)\pi}{2N+1} \end{aligned} \tag{7}$$

$$\Gamma_\nu=\frac{\pi c\sqrt{1-\xi_\nu^2}\gamma(\xi_\nu)}{2N+1} \tag{8}$$

and c is the chord length. If the following kth-order moment M_k is considered, and if Eq. (4) with the weight function $\sqrt{(1-x)/(1+x)}$ and further Eq. (8) are used, then we have

$$\begin{aligned} M_k&=\rho U\left(\frac{c}{2}\right)^{k+1}\int_{-1}^{1}\xi^k\gamma(\xi)d\xi\\ &=\rho U\left(\frac{c}{2}\right)^{k}\sum_{\nu=1}^{N}\xi_\nu^k\Gamma_\nu \end{aligned} \tag{9}$$

where ρ is the density of air and U the velocity of main stream. Eqs. (6) and (9) are nothing but the equations of vortex-lattice method, where Γ_ν represents the circulation of the lattice vortex. It should be noted that, for the simplest case of $N=1$, Eqs. (7) give $\xi_1=-1/2$ and $x_1=1/2$ which correspond to the 1/4 and 3/4 chord points respectively. One may therefore regard this vortex-lattice discretization rather than the ordinary equally spaced panel discretization as a genuine generalization of the 3/4 chordpoint formula of Pistolesi. The above was suggested by Stark rather modestly[11]. Eq. (8) defines the relation between the circulation of the lattice vortex and the local circulation, while there is no such clear relation in the equidistant discretization.

It will be instructive to consider here the equidistant discretization from the standpoint of the quadrature formula. We use for this purpose a transformation by which ξ_ν and x_n in Eqs. (7) are transformed into 1/4 and 3/4 chord points respectively in equally spaced panels. The leading and trailing edges must of course be unchanged by the transformation. If we denote this transformation by

$$x^*=f(x) \tag{10}$$

then

$$\begin{aligned} f(x_i)&=x_i^*\equiv-1+\frac{2i-1}{2N} \qquad i=1(1)2N\\ f(x_0)&=x_0^*\equiv-1\\ f(x_{2N+1})&=x_{2N+1}^*\equiv 1 \end{aligned} \tag{11}$$

where

$$x_i=-\cos\frac{i\pi}{2N+1} \qquad i=0(1)2N+1 \tag{12}$$

We can construct such a transformation for example using the Lagrangian interpolation as

$$f(x)=\sum_{i=0}^{2N+1}{}'' F_i(x)x_i^* \tag{13}$$

the dashes in $\sum''$ denoting the inclusion of a factor 1/2 for $i=0$ and $2N+1$. The interpolation coefficients $F_i(x)$ are given by

$$F_i(-\cos\theta)=\frac{-(-1)^i\sin\theta\sin(2N+1)\theta}{(2N+1)(\cos\theta-\cos\theta_i)}$$
$$\theta_i=\frac{i\pi}{2N+1} \tag{14}$$

Let Eq. (5) be rewritten as

$$w(x^*)=-\frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma(\xi^*)}{x^*-\xi^*}d\xi^* \tag{15}$$

and let Eq. (10) be applied, then

$$w(x^*)=-\frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma(\xi^*)}{x-\xi}\left\{\frac{x-\xi}{f(x)-f(\xi)}\right\}f'(\xi)d\xi \tag{16}$$

Since we may assume that $\gamma(\xi^*)=\sqrt{(1-\xi^*)/(1+\xi^*)}$ $\times$(a polynimial of ξ^*), and since $1/\sqrt{1+\xi^*}\approx 1/(\sqrt{f'(-1)}\sqrt{1+\xi})$ for $\xi\to-1$ and $\sqrt{1-\xi^*}\approx\sqrt{f'(1)}\sqrt{1-\xi}$ for $\xi\to 1$, it may be said that $\gamma(\xi^*)=\sqrt{(1-\xi)/(1+\xi)}\times$(a regular function of ξ) if the conditions $f'(-1)\neq 0$ and $f'(1)\neq 0$ are assumed to be satisfied. $(x-\xi)/\{f(x)-f(\xi)\}$ may have no poles and $f'(\xi)$ is a polynomial of ξ. We may therefore apply approximately Eq. (1) with the weight function $\sqrt{(1-x)/(1+x)}$ to Eq. (16). Then we have

$$w(x_n^*)=-\frac{1}{2N+1}\sum_{\nu=1}^{N}\frac{\gamma(\xi_\nu^*)\sqrt{1-\xi_\nu^2}f'(\xi_\nu)}{x_n^*-\xi_\nu^*} \tag{17}$$

where

$$x_n^*=f(x_n)=-1+\frac{4n-1}{2N}$$
$$\xi_\nu^*=x(\xi_\nu)=-1+\frac{4\nu-3}{2N} \tag{18}$$

Comparing Eq. (17) with the equation of the equidistant vortex-lattice method

$$w(x_n^*)=-\frac{1}{\pi c}\sum_{\nu=1}^{N}\frac{\Gamma_\nu^*}{x_n^*-\xi_\nu^*} \tag{19}$$

we obtain the relation between the circulation of the lattice vortex Γ_ν^* and the local circulation

$$\Gamma_\nu^*=\frac{\pi c\sqrt{1-\xi_\nu^2}\,f'(\xi_\nu)\gamma(\xi_\nu^*)}{2N+1} \tag{20}$$

The kth-order moment is obtained by using Eqs. (4) and (20) as

$$M_k=\rho U\left(\frac{c}{2}\right)^{k+1}\int_{-1}^{1}(\xi^*)^k\gamma(\xi^*)f'(\xi)d\xi$$
$$=\rho U\left(\frac{c}{2}\right)^{k}\sum_{\nu=1}^{N}(\xi_\nu^*)^k\Gamma_\nu^* \tag{21}$$

which has the same form as Eq (9). Eq. (21), however, holds only approximately, while Eq. (9) is a rigorous expression as long as $w(x)$ is a polynomial and N is sufficiently large.

The relation between the local circulation and the circulation of the lattice vortex is usually assumed as

$$\gamma_e(\xi_\nu^*)=(N/c)\Gamma_\nu^* \tag{22}$$

The ratio of $\gamma(\xi_\nu^*)$ obtained from Eq. (20) to $\gamma_e(\xi_\nu^*)$ is

$$\frac{\gamma(\xi_\nu^*)}{\gamma_e(\xi_\nu^*)}=\frac{2N+1}{N\pi\sqrt{1-\xi_\nu^2}f'(\xi_\nu)} \tag{23}$$

Eq. (23) gives an approximate correction factor for the local circulation (or the local lift) obtained by the equidistant vortex-lattice method, and notably it does not depend on $w(x)$. Values of the correction factor for the case of $N=10$ are shown in Table 1 in which they are compared with DeYoung's result[6] obtained by solving Eq. (19) for $w(x)=1$.

3. RECTANGULAR WING

The integral equation for a rectangular wing is written as

$$w(x,y)=-\frac{1}{4\pi A}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\gamma(\xi,\eta)/\partial\eta}{y-\eta}\times\left\{1+\frac{\sqrt{(x-\xi)^2+A^2(y-\eta)^2}}{x-\xi}\right\}d\xi d\eta \tag{24}$$

where A is the aspect ratio. If Eq. (1) with the weight functions $\sqrt{(1-x)/(1+x)}$ and $1/\sqrt{1-x^2}$ is applied to the integrations with respect to ξ and η, respectively, in Eq. (24), we have

$$w(x_n,y_m)=-\frac{\pi}{2(M+1)(2N+1)A}\sum_{\mu=1}^{M+1}\frac{\sqrt{1-\eta_\mu^2}}{y_m-y_\mu}\times\sum_{\nu=1}^{N}\gamma_\eta(\xi_\nu,\eta_\mu)\sqrt{1-\xi_\nu^2}\times\left\{1+\frac{\sqrt{(x_n-\xi_\nu)^2+A^2(y_m-\eta_\mu)^2}}{x_n-\xi_\nu}\right\}$$
$$n=1(1)N,\quad m=1(1)M \tag{25}$$

where

$$x_n=-\cos\frac{2n\pi}{2N+1}$$
$$\xi_\nu=-\cos\frac{(2\nu-1)\pi}{2N+1}$$
$$y_m=-\cos\frac{m\pi}{M+1}$$
$$\eta_\mu=-\cos\frac{(2\mu-1)\pi}{2(M+1)} \tag{26}$$

and $\gamma_\eta(\xi_\nu,\eta_\mu)$ means $\partial\gamma/\partial\eta|_{\xi=\xi_\nu,\,\eta=\eta_\mu}$.

We can construct the corresponding vortex-lattice model by locating each discretized bound-vortex on a segment connecting adjacent two integration points in the spanwise direction such as points (ξ_ν, η_μ) and $(\xi_\nu, \eta_{\mu+1})$ as shown in Fig. 1, and by putting control points at (x_n, y_m). The equation of this model is

$$w(x_n, y_m)=\frac{1}{4\pi s}\sum_{\mu=1}^{M}\sum_{\nu=1}^{N}\Gamma_{\nu\mu}$$
$$\times\left[\frac{1}{y_m-\eta_{\mu+1}}\left\{1+\frac{\sqrt{(x_n-\xi_\nu)^2+A^2(y_m-\eta_{\mu+1})^2}}{x_n-\xi_\nu}\right\}\right.$$
$$\left.-\frac{1}{y_m-\eta_\mu}\left\{1+\frac{\sqrt{(x_n-\xi_\nu)^2+A^2(y_m-\eta_\mu)^2}}{x_n-\xi_\nu}\right\}\right] \tag{27}$$

where s is the semispan and $\Gamma_{\nu\mu}$ is the circulation of a horseshoe vortex whose bound vortex is located between points (ξ_ν, η_μ) and $(\xi_\nu, \eta_{\mu+1})$. By introducing $\Gamma_{\nu0}=\Gamma_{\nu, M+1}=0$, Eq. (27) can be rewritten in the form

$$w(x_n, y_m)=-\frac{1}{4\pi s}\sum_{\mu=1}^{M+1}\frac{1}{y_m-\eta_\mu}\sum_{\nu=1}^{N}(\Gamma_{\nu\mu}-\Gamma_{\nu,\mu-1})$$
$$\times\left\{1+\frac{\sqrt{(x_n-\xi_\nu)^2+A^2(y_m-\eta_\mu)^2}}{x_n-\xi_\nu}\right\} \tag{28}$$

Comparing Eq. (28) with Eq. (25), we have

$$\Gamma_{\nu\mu}-\Gamma_{\nu,\mu-1}$$
$$=\frac{\pi^2 c}{(M+1)(2N+1)}\sqrt{1-\xi_\nu^2}\sqrt{1-\eta_\mu^2}\,\gamma_\eta(\xi_\nu, \eta_\mu) \tag{29}$$

or

$$\Gamma_{\nu\mu}=\frac{\pi^2 c}{(M+1)(2N+1)}\sqrt{1-\xi_\nu^2}$$
$$\times\sum_{\lambda=1}^{\mu}\sqrt{1-\eta_\lambda^2}\,\gamma_\eta(\xi_\nu, \eta_\lambda) \tag{30}$$

In order to see the relation between $\Gamma_{\nu\mu}$ and the local circulation $\gamma(\xi, \eta)$ more clearly, we may assume that the latter is of the form

$$\gamma(\xi, \eta)=a_r(\xi)\sin r\phi \qquad (r=1, 2, \cdots) \tag{31}$$

where $\eta=-\cos\phi$. Then the sum in Eq. (30) becomes

$$\sum_{\lambda=1}^{\mu}\sqrt{1-\eta_\lambda^2}\,\gamma_\eta(\xi_\nu, \eta_\lambda)$$
$$=ra_r(\xi_\nu)\sum_{\lambda=1}^{\mu}\cos\frac{r(2\lambda-1)\pi}{2(M+1)}$$
$$=\frac{1}{2}ra_r(\xi_\nu)\left(\sin\frac{r\mu\pi}{M+1}\right)\Big/\sin\frac{r\pi}{2(M+1)}$$
$$\approx\frac{M+1}{\pi}a_r(\xi_\nu)\sin\frac{r\mu\pi}{M+1}$$
$$=\frac{M+1}{\pi}\gamma\left(\xi_\nu, -\cos\frac{\mu\pi}{M+1}\right) \tag{32}$$

where we have used the approximation $\sin\{r\pi/[2(M+1)]\}\approx r\pi/[2(M+1)]$ for lor large M. We then have

$$\Gamma_{\nu\mu}\approx\frac{\pi c\sqrt{1-\xi_\nu^2}\,\gamma(\xi_\nu, y_\mu)}{2N+1} \tag{33}$$

where $y_\mu=-\cos[\mu\pi/(M+1)]$. It is seen from Eqs. (26) that $\eta_\mu<y_\mu<\eta_{\mu+1}$. Eq. (33) means that the relation between the circulation of discretized vortex and the true local circulation for a rectangular wing is quite similar to that for a two-dimensional wing, Eq. (8), provided that $\Gamma_{\nu\mu}$ is the converged solution of Eq. (27).

Convergence of the total lift and moments will be better than that of the local circulation. Consider a general form of moment

$$\overline{M}_{pq}=\frac{M_{pq}}{\rho U(c/2)^{p+1}s^{q+1}}$$
$$=\int_{-1}^{1}\int_{-1}^{1}\xi^p\eta^q\gamma(\xi, \eta)d\xi d\eta$$
$$=-\frac{1}{q+1}\int_{-1}^{1}\xi^p d\xi\int_{-1}^{1}\eta^{q+1}\frac{\partial\gamma}{\partial\eta} \tag{34}$$

By applying Eq. (4) with the weight functions $\sqrt{(1-x)/(1+x)}$ and $1/\sqrt{1-x^2}$ to the integrations with respect to ξ and η, respectively, in Eq. (34), and by using Eq. (29), we have

$$\overline{M}_{pq}=-\frac{2}{(q+1)c}\sum_{\nu=1}^{N}\sum_{\mu+1}^{M+1}\xi_\nu{}^p\eta_\mu{}^{q+1}(\Gamma_{\nu\mu}-\Gamma_{\nu,\mu-1})$$
$$=\frac{2}{(q+1)c}\sum_{\nu=1}^{N}\sum_{\mu=1}^{M}\xi_\nu{}^p(\eta_{\mu+1}{}^{q+1}-\eta_\mu{}^{q+1})\Gamma_{\nu\mu} \tag{35}$$

or

$$M_{pq}=\rho U\left(\frac{c}{2}\right)^p s^{q+1}\sum_{\nu=1}^{N}\sum_{\mu=1}^{M}\xi_\nu{}^p(\eta_{\mu,\mu+1})^q(\eta_{\mu+1}-\eta_\mu)\Gamma_{\nu\mu} \tag{36}$$

where $\eta_{\mu,\mu+1}$ lies between η_μ and $\eta_{\mu+1}$, and is defined by

$$(\eta_{\mu,\mu+1})^q=\frac{\eta_{\mu+1}{}^{q+1}-\eta_\mu{}^{q+1}}{(q+1)(\eta_{\mu+1}-\eta_\mu)}$$
$$=\frac{1}{q+1c}(\eta_\mu{}^q+\eta_\mu{}^{q-1}\eta_{\mu+1}+\cdots$$
$$+\eta_\mu\eta_{\mu+1}{}^{q-1}+\eta_{\mu+1}{}^q) \tag{37}$$

Eq. (36) states that the total moment of any order is given by simply applying the law of Kutta-Joukowski to the discretized bound vortices. This is a generalization of Eq. (9) for the two-dimensional case. It is to be noted that Eq. (29) instead of Eq. (33) has been used in deriving Eq. (36). This is the reason why the convergence of general moments is thought to be better than that of the local circulation.

It will be of some interest to consider the lift of a rectangular flat-plate wing predicted by the simplest vortex lattice, that is, a single horseshoe vortex. With $w(x_1, y_1)=-U\alpha$, α being the angle of attack, Eq. (27) with $N=M=1$ gives Γ_{11}. The lift-curve slope is then obtained from Eq. (36) as

$$C_{L\alpha}=\frac{\pi A}{1+\sqrt{1-A^2/2}} \tag{38}$$

Eq. (38) is erroneous when the aspect ratio A is large, but it gives the correct value $(\pi/2)A$ when A approaches zero.

The control points (x_n, y_m) and the integration points (ξ_ν, η_μ) given by Eqs. (26) are not new; in fact, (x_n, y_m) are the same as the so-called optimum control points which have been widely used since Multhopp[12,13], while (ξ_ν, η_μ) are the same as those employed by P. T. Hsu[14], although the previous derivations were based on different standpoints.

In the two-dimensional case, Eq. (8) holds exactly as long as $w(x)$ is a polynomial whose degree does not exceed $2N$; that is, solving Eq. (6) with respect to Γ_ν, we can find exact values of $\gamma(\xi_\nu)$ from Eq. (8). In the case of rectangular wing, on the other hand, Eq. (33) is by no means exact due not only to the approximation introduced to the sine term in Eq. (32) but also to the fact that there is a square-root term in Eq. (24) to which Eq. (1) cannot be applied exactly. Convergence of the solution will still be guaranteed in Eq. (33) because the square root can be approximated by a polynomial to any accuracy; that is, if Eq. (27) is solved with respect to $\Gamma_{\nu\mu}$ and then values of $\gamma(\xi_\nu, y_\mu)$ are found from Eq. (33), they will unlimitedly approach to the exact values as N and M become large. In this sense, the above vortex-lattice model may be said to be an ideal one. We can see, however, that the ideal arrangement is not defined uniquely. For example, the chordwise arrangement of the present model may be replaced by Lan's one[10] without losing accuracy. Further, we may use a spanwise arrangement by Borja and Brakhage[15] who applied their quadrature formula to Eq. (24). Although they did not give any consideration about the vortex-lattice method, it is clear from the above discussion that this can be constructed from their method. According to their quadrature formula, two kinds of spanwise arrangements are possible, one of which is the same as that of the present model where the lattice does not extend to the wing tip. The other is such that the lattice does extend to the tip. In this arrangement, y_m and η_μ in Eq. (26) should be replaced by

$$y_m = -\cos\frac{(2m-1)\pi}{2M}, \qquad \eta_\mu = -\cos\frac{(\mu-1)\pi}{M} \tag{39}$$

and in Eq. (25), a factor 1/2 should be included when $\mu=1$ and $\mu=M+1$ in the summation with respect to μ. By similar considerations to the above, we have the same relation as Eq. (33) but, in this case, y_μ should be replaced by $y_\mu = -\cos[(2\mu-1)\pi/(2M)]$.

4. INDUCED DRAG

The spanwise distribution of the induced drag is expressed as the difference between the drag component of the mormal force and the leading-edge suction[12]. The drag component of the normal force per unit span at $y_\mu = -\cos[\mu\pi/(M+1)]$ is

$$D_{\Delta p}(y_\mu) = -\frac{\rho c}{2}\int_{-1}^{1} \gamma(\xi, y_\mu) w(\xi, y_\mu) d\xi \tag{40}$$

Applying the quadrature formula Eq. (4) for the weight function $W(x)=\sqrt{(1-x)/(1+x)}$ to Eq. (40), and using Eq. (33) we obtain

$$D_{\Delta p}(y_\mu) = -\rho\sum_{\nu=1}^{N} \Gamma_{\nu\mu} w(\xi_\nu, y_\mu) \tag{41}$$

It will be more convenient to express $D_{\Delta p}(y_\mu)$ in terms of upwash values at control points $w(x_n, y_\mu)$ instead of the values at points on the bound vortices $w(\xi_\nu, y_\mu)$. This is accomplished by an interpolation. The appropriate Lagrangian interpolation formula is

$$w(\xi, y) = \sum_{n=1}^{N} w(x_n, y) g_n(\xi) \tag{42}$$

where

$$g_n(-\cos\theta') = \frac{-(-1)^n \sin(\theta_n/2)\sin\theta_n \sin(N+1/2)\theta'}{(N+1/2)(\cos\theta' - \cos\theta_n)\sin(\theta'/2)}, \qquad \theta_n = \frac{2n\pi}{2N+1}, \quad x_n = -\cos\theta_n \tag{43}$$

Substitution of Eq. (42) into Eq. (41) gives

$$D_{\Delta p}(y_\mu) = -\rho\sum_{\nu=1}^{N}\Gamma_{\nu\mu}\sum_{n=1}^{N} w(x_n, y_\mu) g_n(\xi_\nu) \tag{44}$$

where

$$g_n(\xi_\nu) = \frac{(-1)^{n+\nu}\sin(\theta_n/2)\sin\theta_n}{(N+1/2)(\cos\theta_\nu' - \cos\theta_n)\sin(\theta_\nu'/2)}, \qquad \theta_\nu' = \frac{(2\nu-1)\pi}{2N+1}, \quad \xi_\nu = -\cos\theta_\nu' \tag{45}$$

The leading-edge suction per unit span at y_μ is given by[16]

$$S(y_\mu) = \lim_{\xi\to -1}\frac{\pi\rho c}{8}[\gamma(\xi, y_\mu)]^2(1+\xi) \tag{46}$$

To express $\gamma(\xi, y_\mu)$ in terms of $\Gamma_{\nu\mu}$, an interpolation is again essential. The appropriate Lagrangian interpolation formula in this case is

$$\sqrt{\frac{1+\xi}{1-\xi}}\gamma(\xi, y_\mu) = \sum_{\nu=1}^{N}\sqrt{\frac{1+\xi_\nu}{1+\xi_\nu}}\gamma(\xi_\nu, y_\mu) f_\nu(\xi) \tag{47}$$

or

$$\gamma(\xi, y_\nu)=\sum_{\nu=1}^{N}\gamma(\xi_\nu, y_\mu)f_\nu(\xi) \tag{48}$$

where

$$\begin{aligned}&f_\nu(-\cos\theta')\\&=\frac{\cot(\theta'/2)}{\cot(\theta_\nu'/2)}\bar{f}_\nu(-\cos\theta')\\&=\frac{-(-1)^\nu(1-\cos\theta_\nu')\cos(\theta_\nu'/2)\cos(N+1/2)\theta'}{(N+1/2)(\cos\theta'-\cos\theta_\nu')\sin(\theta'/2)}\end{aligned} \tag{49}$$

Substituting Eq. (48) into Eq. (46) and using Eq. (33), we have

$$\begin{aligned}S(y_\mu)&=\lim_{\theta'\to 0}\frac{\pi\rho c}{4}\left\{\sum_{\nu=1}^{N}\gamma(\xi_\nu, y_\mu)f_\nu(-\cos\theta')\sin\frac{\theta'}{2}\right\}^2\\&=\frac{\rho}{4\pi c}\left\{\sum_{\nu=1}^{N}(-1)^\nu\Gamma_{\nu\mu}\operatorname{cosec}\frac{\theta_\nu'}{2}\right\}^2\end{aligned} \tag{50}$$

Now that $D_{\Delta p}(y_\mu)$ and $S(y_\mu)$ have been known, the induced drag per unit span at y_μ is given by

$$D_i'(y_\mu)=D_{\Delta p}(y_\mu)-S(y_\mu) \tag{51}$$

Eq. (50) can be put into another form if we realize the fact that in two-dimensional wings the leading-edge suction just offsets the drag component of the normal force. Referring to Eq. (6), we have "the two-dimensional upwash due to $\Gamma_{\nu\mu}$"

$$\bar{w}(x_n, y_\mu)=-\frac{1}{\pi c}\sum_{\nu=1}^{N}\frac{\Gamma_{\nu\mu}}{x_n-\xi_\nu} \tag{52}$$

Replacing $w(x_n, y_\mu)$ in Eq. (44) by $\bar{w}(x_n, y_\mu)$, we have an alternative form of $S(y_\mu)$

$$S(y_\mu)=\frac{\rho}{\pi c}\sum_{\nu=1}^{N}\sum_{\lambda=1}^{N}\Gamma_{\nu\mu}\Gamma_{\lambda\mu}\sum_{n=1}^{N}\frac{g_n(\xi_\nu)}{x_n-\xi_\lambda} \tag{53}$$

When the form of Eq. (53) for $S(y_\mu)$ is used, Eq. (51) becomes

$$\begin{aligned}D_i'(y_\mu)=&-\rho\sum_{\nu=1}^{N}\Gamma_{\nu\mu}\sum_{n=1}^{N}g_n(\xi_\nu)\\&\times\left\{w(x_n, y_\mu)+\frac{1}{\pi c}\sum_{\lambda=1}^{N}\frac{\Gamma_{\lambda\mu}}{x_n-\xi_\lambda}\right\}\end{aligned} \tag{54}$$

We can show that the two expressions for $S(y_\mu)$ are actually identical. Consider the term $\sum_{n=1}^{N}g_n(\xi_\nu)/(x_n-\xi_\lambda)$ in Eq. (53). When $\xi\neq\xi_\nu$, it can be shown that

$$\begin{aligned}\frac{g_n(\xi_\nu)}{x_n-\xi}=&\frac{1}{\xi_\nu-\xi}\\&\times\left\{g_n(\xi_\nu)+\frac{(-1)^\nu\sin(\theta'/2)}{\sin(\theta_\nu'/2)\sin(N+1/2)\theta'}g_n(\xi)\right\}\end{aligned} \tag{55}$$

where $\xi=-\cos\theta'$. Using the obvious relation $\sum_{n=1}^{N}g_n(\xi)=1$, we obtain

$$\begin{aligned}&\sum_{n=1}^{N}\frac{g_n(\xi_\nu)}{x_n-\xi_\lambda}\\&=\begin{cases}\dfrac{1}{\xi_\nu-\xi_\lambda}\left\{1-(-1)^{\nu-\lambda}\dfrac{\sin(\theta_\lambda'/2)}{\sin(\theta_\nu'/2)}\right\} & (\lambda\neq\nu)\\[2ex]\dfrac{1}{2(1-\cos\theta_\nu')}\end{cases}\end{aligned} \tag{56}$$

where the limiting value has been taken when $\lambda=\nu$. Eq. (56) can further be converted into a single equation:

$$\begin{aligned}&\sum_{n=1}^{N}\frac{g_n(\xi_\nu)}{x_n-\xi_\lambda}\\&=[2\{(-1)^\nu\sin(\theta_\nu'/2)+(-1)^\lambda\sin(\theta_\lambda'/2)\}\\&\quad\times(-1)^\nu\sin(\theta_\nu'/2)]^{-1}\end{aligned} \tag{57}$$

Since ν and λ are interchangeable in Eq. (53), $S(y_\mu)$ is expressed also by the following added mean

$$\begin{aligned}S(y_\mu)=&\frac{\rho}{2\pi c}\sum_{\nu=1}^{N}\sum_{\lambda=1}^{N}\Gamma_{\nu\mu}\Gamma_{\lambda\mu}\\&\times\left\{\sum_{n=1}^{N}\frac{g_n(\xi_\nu)}{x_n-\xi_\lambda}+\sum_{n=1}^{N}\frac{g_n(\xi_\lambda)}{x_n-\xi_\nu}\right\}\end{aligned} \tag{58}$$

If Eq. (57) is substituted into Eq. (58), we find that the resulting equation is exactly the same as Eq. (50).

5. NUMERICAL EXAMPLE

In order to see accuracy of the lift and induced drag distribution to be obtained by the present lattice arrangement, computations were carried out on a rectangular flatplate wing of aspect ratio 2. The results are compared with those of a lifting-surface theory[17] in Tables 2 to 6. In Table 2 are shown chordwise distributions of the local lift coefficient for unit angle of attack $\Delta C_p/\alpha$ at various spanwise locations. Since values of the local lift are computed only at points $\xi=\xi_\nu$, an interpolation of the form of Eq. (48) was used to obtain values at points in the table. Tables 3 to 5 show, respectively, spanwise variations of the lift coefficient per unit span referred to the total lift coefficient C_L'/C_L, of the section centre of pressure X_{ac}', and of the induced drag coefficient per unit span referred to the square of the total lift coefficient C_{Di}'/C_L^2. Finally in Table 6 are shown the total lift coefficient C_L, the total moment coefficient C_m, the overall centre of pressure X_{ac}, and the vortex drag factor $K=\pi AC_{Di}/C_L^2$, C_{Di} being the total induced drag coefficient. The total induced drag was obtained by the equation

$$D_i=s\sum_{\mu=1}^{M}(\eta_{\mu+1}-\eta_\mu)D_i'(y_\mu) \tag{59}$$

which is a simple sum and is not based on the

Gaussian quadrature formula.

Throughout Tables 2 to 6, the results of the present vortex-lattice method correspond very well with those of the BAC lifting-surface theory. In Table 2, the effect of increasing N from 4 to 6 while maintaining $M=15$ is somewhat observed near the wing tip, whereas that of increasing M from 15 to 31 while maintaining $N=4$ is small. The same is true in Tables 4 and 5, but not in Table 3. Further, as for the overall values in Table 6, the former effect is seen to be reflected only in K. In C_L, C_m, and X_{ac} and in Table 3, this effect may have been averaged out by integration.

6. CONCLUDING REMARKS

It has been shown that the ideal lattice arrangements of the vortex-lattice method for rectangular wings are obtained by applying the quadrature formulas for the Cauchy integral to the integral equation of the lifting-surface theory, and that the relation between the circulation of the lattice vortex and the local lift is thereby clearly defined. As a natural consequence of this, the spanwise distribution of the induced drag has been computed accurately.

In the ordinary vortex-lattice method, the basis is found fundamentally in the equally spaced panel arrangement, although fairly flexible panel arrangements seem to be used in actual applications. According to the discussions in this paper, the lattice arrangement of the equally spaced panel can be though to be a deformation of the ideal arrangement. Although the former still have a small part of the advantages of the latter, most of the reasons of the inaccuracy in the latter are attributed to this deformation.

It may be difficult to give bases for more general planforms as well as for the rectangular wing. Favourable results may, however, be obtained in practical applications by using lattice arrangements as similar as possible to the ideal arrangement for the rectangular wing.

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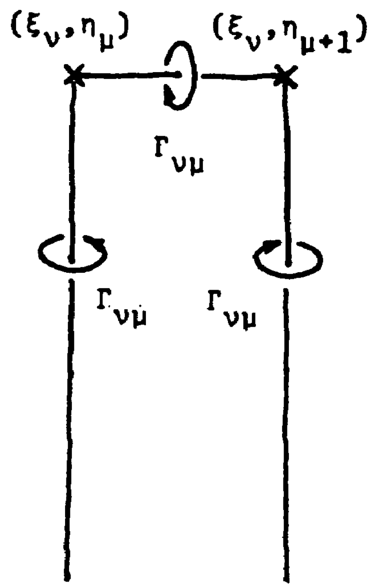

Fig. 1 A Lattice Horseshoe Vortex

Table 1. $\gamma(\xi_\nu^*)/\gamma_e(\xi_\nu^*)$, $N=10$

ν	Ref. 6	Eq. (23)
1	1.128	1.1283
2	1.009	1.0078
3	1.003	1.0018
4	1.001	1.0006
5	1.000	1.0002
6	1.000	0.9999
7	0.999	0.9996
8	0.998	0.9990
9	0.995	0.9965
10	0.977	0.9779

Table 2. Chordwise Distributions of Local Lift Coefficient for Unit Angle of Attack $\Delta C_p/\alpha$

(a) $\eta=0$

% chord	Present Method $M=15$ $N=4$	Present Method $M=15$ $N=6$	Present Method $M=31$ $N=4$	BAC Method[17] $m=13$ $N=4$	BAC Method[17] $m=13$ $N=6$
0.5	31.5492	31.5360	31.5311	31.5251	31.5171
1.25	19.7945	19.7891	19.7826	19.7792	19.7759
2.5	13.8105	13.8097	13.8016	13.7998	13.7992
5	9.5049	9.5076	9.4980	9.4973	9.4988
10	6.3603	6.3647	6.3547	6.3550	6.3573
15	4.9073	4.9112	4.9023	4.9030	4.9047
20	4.0096	4.0124	4.0050	4.0058	4.0067
30	2.8984	2.8992	2.8945	2.8953	2.8948
40	2.2031	2.2030	2.1998	2.2004	2.1994
50	1.7096	1.7096	1.7069	1.7070	1.7064
60	1.3312	1.3316	1.3291	1.3288	1.3289
70	1.0226	1.0232	1.0209	1.0202	1.0208
80	0.7525	0.7527	0.7513	0.7504	0.7509
90	0.4867	0.4860	0.4860	0.4850	0.4849
95	0.3310	0.3299	0.3305	0.3296	0.3292

Table 2. (continued)

(b) $\eta=0.3827$

% chord	Present Method $M=15$ $N=4$	Present Method $M=15$ $N=6$	Present Method $M=31$ $N=4$	BAC Method[17] $m=13$ $N=4$	BAC Method[17] $m=13$ $N=6$
0.5	30.1434	30.1161	30.1232	30.1132	30.0956
1.25	18.8898	18.8783	18.8765	18.8712	18.8639
2.5	13.1532	13.1510	13.1433	13.1408	13.1394
5	9.0175	9.0222	9.0097	9.0095	9.0124
10	5.9896	5.9974	5.9833	5.9847	5.9891
15	4.5892	4.5959	4.5837	4.5856	4.5889
20	3.7254	3.7300	3.7205	3.7225	3.7240
30	2.6623	2.6632	2.6583	2.6598	2.6587
40	2.0048	2.0041	2.0016	2.0022	2.0005
50	1.5445	1.5439	1.5420	1.5417	1.5408
60	1.1966	1.1966	1.1947	1.1937	1.1939
70	0.9163	0.9166	0.9149	0.9136	0.9144
80	0.6734	0.6733	0.6725	0.6710	0.6716
90	0.4356	0.4346	0.4350	0.4339	0.4336
95	0.2962	0.2952	0·2958	0.2951	0.2945

Table 2. (continued)

(c) $\eta=0.7071$

% chord	Present Method $M=15$ $N=4$	Present Method $M=15$ $N=6$	Present Method $M=31$ $N=4$	BAC Method[17] $m=13$ $N=4$	BAC Method[17] $m=13$ $N=6$
0.5	25.6965	25.6989	25.6625	25.6585	25.6668
1.25	16.0201	16.0221	15.9986	15.9976	16.0010
2.5	11.0607	11.0625	11.0454	11.0465	11.0469
5	7.4583	7.4600	7.4475	7.4503	7.4482
10	4.8015	4.8032	4.7940	4.7980	4.7944
15	3.5755	3.5775	3.5697	3.5737	3.5706
20	2.8295	2.8322	2.8249	2.8284	2.8266
30	1.9413	1.9450	1.9382	1.9400	1.9413
40	1.4234	1.4270	1.4215	1.4214	1.4246
50	1.0823	1.0840	1.0812	1.0796	1.0823
60	0.8365	0.8352	0.8360	0.8335	0.8338
70	0.6432	0.6391	0.6429	0.6402	0.6379
80	0.4748	0.4705	0.4746	0.4723	0.4694
90	0.3064	0.3057	0.3062	0.3049	0.3049
95	0.2072	0.2089	0.2070	0.2063	0.2084

Table 2. (concluded)

(d) $\eta=0.9239$

% chord	Present Method $M=15$ $N=4$	Present Method $M=15$ $N=6$	Present Method $M=31$ $N=4$	BAC Method[17] $m=13$ $N=4$	BAC Method[17] $m=13$ $N=6$
0.5	16.4681	17.3203	16.3999	16.5395	17.2606
1.25	10.1594	10.5463	10.1192	10.1966	10.5097
2.5	6.8926	7.0114	6.8674	6.9101	6.9872
5	4.4888	4.4110	4.4752	4.4904	4.3966
10	2.7002	2.5408	2.6954	2.6904	2.5349
15	1.8870	1.7589	1.8861	1.8739	1.7574
20	1.4113	1.3396	1.4122	1.3983	1.3404
30	0.8924	0.9108	0.8945	0.8835	0.9128
40	0.6358	0.6799	0.6377	0.6323	0.6708
50	0.4940	0.5180	0.4951	0.4945	0.5177
60	0.4018	0.3950	0.4022	0.4041	0.3942
70	0.3254	0.3034	0.3253	0.3267	0.3028
80	0.2461	0.2325	0.2459	0.2444	0.2320
90	0.1537	0.1593	0.1537	0.1487	0.1580
95	0.0988	0.1093	0.0989	0.0933	0.1074

Table 3. Lift Coefficient per Unit Span Referred to Total Lift Coefficient C_L'/C_L

η	Present Method $M=15$ $N=4$	Present Method $M=15$ $N=6$	Present Method $M=31$ $N=4$	BAC Method[17] $m=13$ $N=4$	BAC Method[17] $m=13$ $N=6$
0	1.2561	1.2560	1.2548	1.2543	1.2543
0.1951	1.2349	1.2348	1.2336	1.2331	1.2330
0.3827	1.1710	1.1710	1.1697	1.1692	1.1692
0.5556	1.0643	1.0643	1.0629	1.0625	1.0625
0.7071	0.9153	0.9154	0.9141	0.9137	0.9137
0.8315	0.7271	0.7272	0.7261	0.7257	0.7257
0.9239	0.5054	0.5055	0.5046	0.5044	0.5044
0.9808	0.2592	0.2593	0.2588	0.2586	0.2587

Table 4. Section Centre of Pressure X_{ac}'

η	Present Method $M=15$ $N=4$	Present Method $M=15$ $N=6$	Present Method $M=31$ $N=4$	BAC Method[17] $m=13$ $N=4$	BAC Method[17] $m=13$ $N=6$
0	0.2201	0.2200	0.2200	0.2199	0.2199
0.1951	0.2188	0.2188	0.2187	0.2187	0.2187
0.3827	0.2151	0.2150	0.2150	0.2149	0.2149
0.5556	0.2087	0.2086	0.2087	0.2085	0.2085
0.7071	0.1997	0.1996	0.1998	0.1996	0.1996
0.8315	0.1889	0.1885	0.1891	0.1886	0.1886
0.9239	0.1781	0.1769	0.1784	0.1773	0.1770
0.9808	0.1701	0.1674	0.1706	0.1685	0.1674

Table 5 Induced Drag Coefficient per Unit Span Referred to Square of Total Lift Coefficient C_{Di}'/C_L^2

η	Present Method $M=15$ $N=4$	Present Method $M=15$ $N=6$	Present Method $M=31$ $N=4$	BAC Method[17] $m=13$ $N=4$	BAC Method[17] $m=13$ $N=6$
0	0.1850	0.1853	0.1848	0.1848	0.1850
0.1951	0.1834	0.1838	0.1832	0.1832	0.1835
0.3827	0.1782	0.1789	0.1781	0.1781	0.1786
0.5556	0.1686	0.1695	0.1685	0.1686	0.1693
0.7071	0.1540	0.1540	0.1541	0.1541	0.1539
0.8315	0.1358	0.1312	0.1361	0.1353	0.1315
0.9239	0.1144	0.1030	0.1148	0.1131	0.1033
0.9808	0.0781	0.0705	0.0782	0.0770	0.0701

Table 6. Overall Values

	Present Method $M=15$ $N=4$	Present Method $M=15$ $N=6$	Present Method $M=31$ $N=4$	BAC Method[17] $m=13$ $N=4$	BAC Method[17] $m=13$ $N=6$
C_L	2.4735	2.4741	2.4736	2.4744	2.4744
$-C_m$	0.5185	0.5182	0.5185	0.5182	0.5181
X_{ac}	0.2096	0.2095	0.2096	0.2094	0.2094
K	1.0107	1.0028	1.0119	1.0108	1.0033

TECHNICAL REPORT OF NATIONAL
AEROSPACE LABORATORY
TR-629T

航空宇宙技術研究所報告629T号（欧文）

昭和55年9月 発行

発行所 航空宇宙技術研究所
東京都調布市深大寺町1880
電話武蔵野三鷹(0422)47-5911(大代表)〒182
印刷所 株式会社 東京プレス
東京都板橋区桜川2－27－12

Published by
NATIONAL AEROSPACE LABORATORY
1,880 Jindaiji, Chōfu, Tokyo
JAPAN