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OPTIMAL LOW-THRUST INTERPLANETARY ORBIT TRANSFER INCLUDING EARTH ESCAPE SPIRAL TRAJECTORY

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ABSTRACT

A numerical analysis has been carried out on minimum-time low-thrust Earth-Mars transfer including Earth escape spiral trajectory. This is a three-point boundary-value problem with a constraint at the interior point $t=t_1$ when the hyperbolic velocity is attained in the geocentric force field, and the terminal constraints at the final time $t=t_f$ ($=t_1 + t_2$).

Minimal time t_1^* for the Earth escape problem is obtained here by the authors in a manner similar to that in Ref. 3, and t_2^* for the Earth-Mars heliocentric transfer problem is well-known (e.g., Ref. 2, 16).

A three-dimensional search procedure using Δt_1, Δt_f, and the control correction length α as three parameters is developed to solve the present complicated problem numerically.

The obtained total mission time t_f is slightly shorter than the sum of t_1^* and t_2^*. The control history in the escape portion is quite different from that in an optimal escape problem, but in the interplanetary portion it is similar to that in an optimal interplanetary transfer problem.

概　　要

微小推力を用いて，地球周回円軌道からスパイラル・レイジング（spiral raising）により脱出軌道速度を達成し，それ以後太陽の中心力場を飛行して目標惑星の公転軌道に到達する最適（最短時間）軌道遷移問題を設定した。新たに開発した3次元探索アルゴリズムを適用して，大型電子計算機により数値解を得た。

本問題は，初期時刻 $t_0 = 0$ において初期値（出発軌道の位置，速度），途中の時刻 t_1 において総エネルギー E，終端時刻 $t_f (= t_1 + t_2)$ において終端値（目標惑星公転軌道の位置，速度）の拘束条件を有しており，通常の2点境界値問題ではなく，3点境界値問題に帰着する。

質点としての宇宙船の運動は，同一平面内とし，地球引力圏脱出までは地球の中心力場の
み，それ以後は太陽の中心力場のみを考慮するそれぞれ二体問題として扱った。こうした単
純化にも拘らず，本問題は，運動方程式が時刻 t_1 で切換り，同時に座標系も切換えるので，
位置，速度などの状態変数に見かけ上不連続が生ずるなどの複雑さを含んでいる。

推力の大きさ，燃料消費率は一定と仮定しているので，本最適問題のパラメタは操舵角 u
の探索方向の長さ α 及び t_f, t_1 の修正分 Δt_f, Δt_1 の3つである。

この為，3つのパラメタを変動させて評価関数の極値を求めていく3次元探索アルゴリズ
ムを新たに開発して，本問題に適用した。

ところで，惑星間飛行の最短時間問題は数多くの研究者によって扱われており，その解
t_2^{*} はよく知られている。最短時間地球引力圏脱出問題の解 t_1^{*} は，今回，参考文献3と同様
の方法で得られた。

数値計算の結果，$t_1 > t_1^{*}$, $t_2 < t_2^{*}$ であるが，t_1^{*} と t_2^{*} の和よりも短い $t_f (=t_1+t_2)$
が得られた。操舵角の履歴は，引力圏脱出部分で最適脱出問題の解とは極めて異った，又惑
星間飛行部分では類似の特徴をそれぞれ示す。

1. INTRODUCTION

The low-thrust orbit transfers may be classified into the following three categories;

I) Transfers between geocentric (planet-centered) orbits—The total energy E of the spacecraft (kinetic energy plus potential energy in the central force field) is negative through the transfer. Both ascent and descent are included.

II) Interplanetary transfers in the heliocentric force field—This is the same type of I) in essence from a point of view of the transfers in one central force field, but is classified especially from a point of view of the Earth being our mother planet. The transfers to the outer-planets and the inner-planets are equivalent to the ascent and descent in category I) respectively.

III) Spiral escape (settling) trajectories from (into) the planet—centered orbit to (from) the hyperbolic volocity—This is a transient region which connects I) and II). E changes the sign from negative (posi-

tive) to positive (negative).

It is well-known that a lot of studies have been carried out on I) and II), and a few studies on III). Authors[1,2,3] have also been studying some transfer problems on I), II), and III).

On the contrary, few studies have been carried out on the orbit transfer problems covering more than two categories among the above-mentioned three.

Moeckel[4,5,6] studied the Earth-Mars one-way trip and round-trip. The parameters, e.g., E, radial distance, velocity, steering angle, and angular distance, are calculated of the spiral escape (settling) trajectories with a constant tangential thrust of wide range of thrust-mass ratio of $10 \sim 10^{-4}$, and shown in many charts. It is simulated an eight-man Earth-Mars round-trip expedition with initial acceleration of 1.6 mm·s^{-2} from the geocentric circular orbit to the Mars-centered circular orbit including the exploration of Mars surface by the chemical-rocket vehicle. In the computation, the 1068-day long expe-

dition is divided into seven phases as follows:

1) Acceleration from the geocentric orbit to the hyperbolic velocity for minimum-energy interplanetary transfer.

2) Coasting along the minimum-energy path to the Mars orbit.

3) Settling into the orbit round Mars from the hyperbolic approaching velocity.

4) Waiting period and exploration at Mars surface.

5) Acceleration from the orbit round Mars to the hyperbolic velocity for minimum-energy interplaneary transfer.

6) Coasting along the minimum-energy ellipse to Earth's orbit.

7) Settling into the geocentric orbit from the hyperbolic approaching velocity.

It is assumed that the spacecraft is accelerated (decelerated) by a tangential thrust throughout the powered flight. Moeckel[5,6] studied also the round-trip including the powered interplanetary flight and reduced the mission time with indirect trajectory.*

Fox[7] also studied the one-way trip and round-trip with constant tangential acceleration between the Earth and Venus, Mars, and Jupiter respectively. The coasting is included, and the proper moments for thrust shutdown and startup are determined using the concept of the osculating orbit.

Recently, some missions[8~14] using the solar electric propulsion systems have been studied of flyby and rendezvous to some celestial bodies such as the comets

Encke, Halley, the asteroids, and the planets. However, in all missions, the chemical rocket launching vehicles such as Titan/Centaur are to be used until the eascape velocity is attained.

As mentioned previously, the studies are few in number on the interplanetary transfer starting from the geocentric orbit with the low thrust. Even so, in almost all of them, it is assumed that the thrust is acted in the tangential direction throughout the powered flight. Therefore, they are near-optimum problems, but not optimal ones. It is quite clear that such an optimal problem has one or more interior-point constraints, e.g., total energy, radial distance, and velocity, in addition to the constraints at the initial time t_0 and the final time t_f. This is consequently reduced to a complicated multi-point boundary-value (MPBV) problem[15], but not to a conventional two-point boundary-value (TPBV) problem. Moreover, the motion of the spacecraft should be considered in more than two different central force fields one after another. Even if the motion is approximated as a two-body problem throughout the flight, the equations of motion should be switched at the moment when the spacecraft goes out of one central force field and enters into another. At the same time, the state variables such as the distance and the velocity jump because of the different coordinate systems, e.g., the planet-centered coordinate system and the heliocentric one. The apparent discontinuities are caused also by the different normalization units as shown later.

Even if solving the conventional TPBV problem numerically, we often encounter

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*transfer to the high (low) altitude orbit descending (ascending) in the beginning of the flight.

the difficulties such as a bad convergence or even a divergence. The above mentioned complexities and the difficulties in obtaining a numerical solution would explain why such an MPBV problem has not been studied yet numerically.

Authors have defined and tried to solve a very simple optimal problem among the MPBV problems, i.e., an optimal orbit transfer from the geocentric orbit to the heliocentric planetary orbit including Earth escape spiral trajectory. The space-craft attains to the hyperbolic velocity starting from the geocentric circular orbit through spiral raising, and then continues the powered flight to the target-planet orbit in the heliocentric force field. This problem has an interior-point constraint on the total energy E at $t=t_1$ besides the initial conditions at $t=t_0$ and the terminal constraints at $t=t_f$ ($=t_1+t_2$). The state variables have an apparent discontinuity at $t=t_1$, and the equations of motion are switched also at the same time.

The defined optimal problem is a minimum-time low-constant-thrust coplanar orbit transfer with constant mass flow (fuel consumption) rate. The performance index of the present problem is $J=t_f$, and a function of three parameters, i,e., Δt_f, Δt_1, and α, the corrections of t_f and t_1, and the control (steering angle) correction length along the search direction respectively.

In Ref. 16, Powers and Shieh developed the two-dimensional search procedure (2-DSP) to approximate the minimum of a function of two parameters $J(\Delta t_f, \alpha)$. The procedure with the conjugate gradient method employing the penalty functions was applied to a minimum-time inter-

planetary orbit transfer problem with terminal equality constraints while improving considerably the convergence rate.

Authors have developed a three-dimensional search procedure (3-DSP) with Δt_f, Δt_1, and α as three parameters. The 3-DSP with the gradient method employing the penalty functions is applied to the present problem.

Since the solution t_2^* for a minimum-time Earth-Mars heliocentric orbit transfer is well-known by numerous investigations (e.g., Refs. 2, 16), Mars is chosen as the target planet from a point of view of comparison. The solution t_1^* for a minimum-time Earth escape problem from the geosynchronous orbit has been obtained here using 2-DSP similarly in Ref. 3.

A number of simulations have shown that it is difficult in many cases to get a good convergence using only 3-DSP. Taking account of the fact that the magnitude of α is extremely small compared with Δt_f and Δt_1, a conventional one-dimensional search procedure (1-DSP) is introduced with $\Delta t_f=0$ and $\Delta t_1=0$. 3-DSP and 1-DSP are used in series. A number of combinations have been tested of search procedures and the penalty functions.

The total mission time $t_f(=t_1+t_2)$ is obtained which is slightly shorter than the sum of t_1^* and t_2^*. t_1 is longer than t_1^*, but t_2 is shorter than t_2^*. In some case, the resultant t_f is as short as 4.2 percents. It is very interesting that the minimal time for the total flight is shorter than the sum of the minimal times for the partitioned flights, i.e., the escape portion and the interplanetary portion respectively. The control history in the escape portion is

quite different from that in an optimal escape problem, but in the interplanetary portion it is similar to that in an optimal interplanetary transfer problem. The trajectories are rather similar in both portions.

NOMENCLATURE

A : ratio of t_e to t_s, $t_e/t_s =$ $\sqrt{r_0^3 / \mu_e} / \sqrt{R_{AU}^3 / \mu_s}$

B : 4×4 square matrix, $dX(t_1^+)$ $=B \cdot dx(t_1^-)$

$B_{11} \sim B_{44}$: elements of B

$C_1 \sim C_{10}$: coefficients of a quadratic function $\bar{J}(\alpha, \Delta t_1, \Delta t_f)$

E : total energy in the central force field

E_e : specified total energy

F, f : equations of motion in vectoral form for $t \in [t_1^+, t_f]$ and for $t \in [t_0, t_1^-]$

G, g : co-state equations in vectoral form for $t \in [t_1^+, t_f]$ and for $t \in [t_0, t_1^-]$

$H_e[t], H_e[s]$: Hamiltonian for $t \in [t_0, t_1^-]$ and for $s \in [0, 1^-]$, $H_e[s] = t_1 \cdot H_e[t]$

$H_s[t], H_s[s]$: Hamiltonian for $t \in [t_1^+, t_f]$ and for $s \in [1^+, 2]$, $H_s[s] = (t_f - t_1) \cdot H_s[t]$

J : Jacobian matrix

J : performance index

$\bar{J}$: augmented performance index

$\bar{J}_{t_1}, \bar{J}_{t_f}, \bar{J}_\alpha$: partial derivatives of $\bar{J}$ with respect to $\Delta t_1, \Delta t_f$, and α respectively

m_0 : initial mass

m_c : mass flow (fuel consumption) rate

$\hat{m}_{ce}, \hat{m}_{cs}$: normalized mass flow rates in the geocentric and in the heliocentric force field, $m_c/(m_0\sqrt{\mu_e/r_0^3})$, $m_c/(m_0\sqrt{\mu_s/R_{AU}^3})$

$P_1 \sim P_4$: penalty functions

$p(s)$: control search direction (gradient direction)

R : ratio of r_0 to R_{AU}, r_0/R_{AU}

R_{AU} : astronomical unit

r : radial distance

r_0 : radius of the initial geocentric circular orbit

s : new independent variable introduced by Long's transformation

T : transformations of $x(t_1^-)$ into $X(t_1^+)$

T : thrust magnitude

$\hat{T}_e, \hat{T}_s$: normalized thrust magnitudes in the geocentric and in the heliocentric force field

t_e : time unit in the geocentric force field, $r_0/v_0 = \sqrt{r_0^3/\mu_e}$

$\hat{t}_e$: normalized time by t_e, $t/\sqrt{r_0^3/\mu_e}$

t_f : final time, total mission time $(=t_1 + t_2)$

t_s : time unit in the heliocentric force field, $R_{AU}/v_e = \sqrt{R_{AU}^3/\mu_s}$

$\hat{t}_s$: normalized time by t_s, $t/\sqrt{R_{AU}^3/\mu_s}$

t_0 : initial time

t_1 : time for Earth escape

t_1^- : $t=t_1$ when the escape condition is attained

t_1^+ : $t=t_1$ when the interplanetary transfer is started

t_2 : time for interplanetary

		transfer
Δt_f	:	correction of t_f
Δt_1	:	correction of t_1
u	:	control (steering angle)
V	:	ratio of v_0 to v_e, v_0/v_e
v_e	:	Earth's orbital velocity round the sun, $\sqrt{\mu_s/R_{AU}}$
v_r	:	radial velocity
v_θ	:	circumferential velocity
v_0	:	orbital velocity of the initial geocentric circular orbit, $\sqrt{\mu_e/r_0}$
X	:	column vector of the normalized state variables for $t \in [t_1^+, t_f]$, $X^T = (X_1, X_2, X_3, X_4)$
X_{1f}, X_{3f}, X_{4f}	:	specified state variables at $t = t_f$
x	:	column vector of the normalized state variables for $t \in [t_0, t_1^-]$, $x^T = (x_1, x_2, x_3, x_4)$

(Greek letters)

α	:	control correction length along the search direction
β	:	angle between velocity vector and thrust vector
γ	:	angle between local horizontal direction and velocity vector
ϵ	:	convergence criterion
ζ	:	angle between Earth's orbital velocity v_e and $v(t_1^-)$
θ	:	transfer angle
θ_e	:	transfer angle of the Earth in the heliocentric coordinate system
θ_Q	:	angle between the Earth and the sun seen from the spacecraft
θ_s	:	angle between the Earth

		and the spacecraft seen from the sun
λ	:	column vector of Lagrange multipliers, $\lambda^T = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$
μ	:	mass of a central body × universal gravitational constant
μ_e	:	mass of the Earth × universal gravitational constant
μ_s	:	mass of the sun × universal gravitational constant
$\psi_1 \sim \psi_4$	:	boundary constraints

(Symbols)

$d(\ )$	:	differential
$\delta(\ )$	:	variation
$(\dot{\ })$	:	derivative with respect to s

2. EQUATIONS OF MOTION

Before deriving the equations of motion of the spacecraft as a point mass, authors have adopted the following assumptions:

1) Coplanar motion throughout the spiral escape and the interplanetary transfer.

The spiral escape problem is usually considered in the equatorial plane, and the interplanetary transfer between the circular planetary orbits in the ecliptic plane. Although the inclination angle between both planes is about 23°27′ (obliquity of ecliptic)[17], a coplanar motion throughout the flight is assumed like in Refs. 4, 5, 6, and 7.

2) Two-body problems throughout the flight, i.e., in the geocentric force field until the hyperbolic velocity is attained, and in the heliocentric force field thereafter.

It is not a simple matter to estimate where the planet-centered force field

should be switched to the heliocentric one. Moeckel[4] calculated and discussed about the "sphere of influence" for several planets under the condition of the effective acceleration due to the heliocentric force field being equal to that due to the planet-centered force field, but adopted finally the above assumption for simplicity. Fox[7] studied the moon's effect on the motion through some computer runs, and concluded that it was small if the "time phasing" was properly carried out.

The state variables are defined in Figure 1. The angles u, γ, and β are a steering angle measured from local horizontal direction, an angle between local horizontal direction and velocity vector, and an angle between velocity vector and thrust vector respectively.

The equations of coplanar motion of a point mass in the polar coordinate system with an origin at the mass center of a central body are well-known (e.g., Ref. 18):

$$\frac{dr}{dt} = v_r \tag{1}$$

$$\frac{d\theta}{dt} = v_\theta / r \tag{2}$$
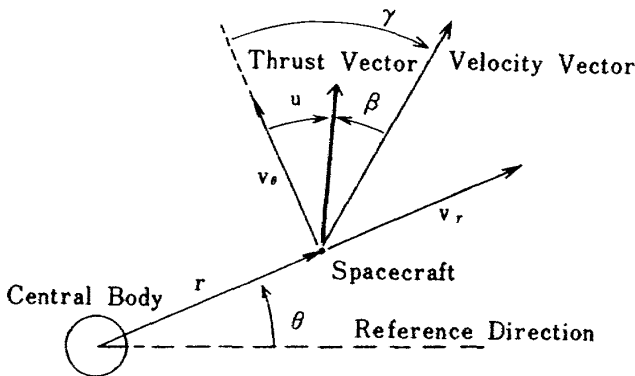
Figure 1. Polar Coordinate System for the Coplanar Motion

$$\frac{dv_r}{dt} = v_\theta^2 / r - \mu / r^2 + T \cdot \sin u / (m_0 - m_c \cdot t) \tag{3}$$

$$\frac{dv_\theta}{dt} = - v_r v_\theta / r + T \cdot \cos u / (m_0 - m_c \cdot t) \tag{4}$$

The above equations are normalized using $r_u, v_u (= \sqrt{\mu / r_u}), t_u, m_u$, the units of radial distance, velocity, time, and mass respectively.

In geocentric force field

The following normalized equations are obtained using the radius r_0 and the velocity $v_0 (= \sqrt{\mu_e / r_0})$ of the initial circular orbit, $t_e (= r_0 / v_0)$, and m_0:

$$\frac{dx_1}{d\hat{t}_e} = x_3 \tag{5}$$

$$\frac{dx_2}{d\hat{t}_e} = x_4 / x_1 \tag{6}$$

$$\frac{dx_3}{d\hat{t}_e} = x_4^2 / x_1 - 1 / x_1^2 + \hat{T}_e \cdot \sin u / (1 - \hat{m}_{ce} \cdot \hat{t}_e) \tag{7}$$

$$\frac{dx_4}{d\hat{t}_e} = - x_3 x_4 / x_1 + \hat{T}_e \cdot \cos u / (1 - \hat{m}_{ce} \cdot \hat{t}_e) \tag{8}$$

In heliocentric force field

The normalized equations are

$$\frac{dX_1}{d\hat{t}_s} = X_3 \tag{9}$$

$$\frac{dX_2}{d\hat{t}_s} = X_4 / X_1 \tag{10}$$

$$\frac{dX_3}{d\hat{t}_s} = X_4^2 / X_1 - 1 / X_1^2 + \hat{T}_s \cdot \sin u / (1 - \hat{m}_{cs} \cdot \hat{t}_s) \tag{11}$$

$$\frac{dX_4}{d\hat{t}_s} = -X_3 X_4 / X_1$$
$$+ \hat{T}_s \cdot \cos u/(1 - \hat{m}_{cs} \cdot \hat{t}_s) \qquad (12)$$

The units are an astronomical unit R_{AU}, Earth's orbital velocity v_e $(=\sqrt{\mu_s/R_{AU}})$ round the sun, $t_s(=R_{AU}/v_e)$, and m_0 respectively.

Since the motion of the spacecraft is continuous in essence through the transition from spiral escape to interplanetary flight, the state variables are, too. However, in the derived equations, the state variables have an apparent discontinuity at $t=t_1$, because of the different coordinate systems and the different normalization units. To more complicated, an independent variable t has, too. As for the state variables, the transformations of $x(t_1^-)$ into $X(t_1^+)$ can be derived as follows:

$$X_1 = \sqrt{1 + R^2 x_1^2 + 2Rx_1 \cos(\theta_e - x_2)} \qquad (13)$$

$$X_2 = \theta_e - \tan^{-1}\left\{ \frac{2Rx_1 \sin(\theta_e - x_2)}{1 + X_1^2 - R^2 x_1^2} \right\} \qquad (14)$$

$$X_3 = -\frac{Rx_1 \sin(\theta_e - x_2)}{X_1}$$
$$+ V\left\{ \frac{x_3(X_1^2 + R^2 x_1^2 - 1)}{2Rx_1 X_1} \right.$$
$$\left. + \frac{x_4 \sin(\theta_e - x_2)}{X_1} \right\} \qquad (15)$$

$$X_4 = \frac{X_1^2 - R^2 x_1^2 + 1}{2X_1}$$
$$+ V\left\{ -\frac{x_3 \cdot \sin(\theta_e - x_2)}{X_1} \right.$$
$$\left. + \frac{x_4(X_1^2 + R^2 x_1^2 - 1)}{2Rx_1 X_1} \right\} \qquad (16)$$

where $R=r_0/R_{AU}$ and $V=v_0/v_e$. For the details, see Appendix A. The same procedure cannot be applied to as for time t. Because an independent variable should be continuous throughout the equations of motion. Therefore, time unit should be t_e also at this time, but not t_s.

Now, the normalized equations of motion are

$$\frac{dX_1}{d\hat{t}_e} = A \cdot \left\{ X_3 \right\} \qquad (17)$$

$$\frac{dX_2}{d\hat{t}_e} = A \cdot \left\{ X_4 / X_1 \right\} \qquad (18)$$

$$\frac{dX_3}{d\hat{t}_e} = A \cdot \left\{ X_4^2 / X_1 - 1/X_1^2 \right.$$
$$\left. + \hat{T}_s \cdot \sin u/(1 - \hat{m}_{ce} \cdot \hat{t}_e) \right\} \qquad (19)$$

$$\frac{dX_4}{d\hat{t}_e} = A \cdot \left\{ -X_3 X_4 / X_1 \right.$$
$$\left. + \hat{T}_s \cdot \cos u/(1 - \hat{m}_{ce} \cdot \hat{t}_e) \right\} \qquad (20)$$

where $A = \sqrt{r_0^3/\mu_e} \sqrt{R_{AU}^3/\mu_s}$. Hereafter, the normalized time $\hat{t}_e$ will be expressed simply by t without confusion. The equations of motion and the transformations are written compactly in vectoral form using the state variables vectors.

$$x^T(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \qquad (21)$$

$$X^T(t) = (X_1(t), X_2(t), X_3(t), X_4(t)) \qquad (22)$$

$$\frac{dx(t)}{dt} = f(x,u,t), \quad t \in [t_0, t_1^-] \qquad (23)$$

$$\frac{dX(t)}{dt} = F(X,u,t), \quad t \in [t_1^+, t_f] \qquad (24)$$

$$X(t_1^+) = T(X_1(t_1^+), x(t_1^-)) \qquad (25)$$

where f, F, and T are the column vectors whose elements are the right-hand sides of

Eqs. (5)~(8), Eqs. (17)~(20), and Eqs. (13)~(16) respectively. t_1^- means $t=t_1$ when the escape condition is attained, and t_1^+ when the interplanetary flight is started.

3. PROBLEM DIFINITIONS AND THREE-POINT BOUNDARY-VALUE PROBLEM

3.1. PROBLEM DEFINITIONS

The present problem is a minimum-time orbit transfer with a constant low thrust and mass flow rate without coasting from the geocentric circular orbit to the heliocentric planetary orbit. The steering angle is a control which minimizes the total mission time t_f. It is assumed that there is no constraint on the steering angle.

The performance index of the present problem is defined by

$$J = t_f \qquad (26)$$

The constraints are given at the initial time t_0, the final time t_f, and the interior time t_1.

Initial conditions at $t=t_0$

For the initial geocentric circular orbit,

$x_1(0) = 1.0$,
$x_2(0)$: arbitrary (normally $= 0$),
$x_3(0) = 0.0$,
$x_4(0) = 1.0$ $\qquad (27)$

Terminal conditions at $t=t_f$

Since the final target is a heliocentric planetary orbit, the conditions are

$$\psi_1 = X_1(t_f) - X_{1f} = 0 \qquad (28)$$

$$\psi_2 = X_3(t_f) - X_{3f} = 0 \qquad (29)$$

$$\psi_3 = X_4(t_f) - X_{4f} = 0 \qquad (30)$$

where X_{1f}, X_{3f}, and X_{4f} are a radius, radial and circumferential velocity of the target orbit respectively. X_{3f} is zero for any target planet since the circular orbit is assumed.

Interior condition at $t=t_1$

The total energy (kinetic energy plus potential energy) of the spacecraft at $t=t \le t_1$ in the geocentric force field is given by

$$E(t) = x_3^2(t) + x_4^2(t) - 2/x_1(t) \qquad (31)$$

In the escape problem, $E(t)=0$, i.e., condition of parabolic velocity, is usually adopted as the terminal condition. Since the trajectory starts from the geocentric orbit where E is negative, E remains negative near to zero even with good convergence in numerical solution. The solution might be sufficient for the escape problem. However, it is not for the present problem, where the interplanetary flight is to continue without interruption.

Therefore, hyperbolic velocity must be attained, even $E(t_1^-)$ is just slightly larger than zero. The condition is

$$\psi_4 = x_3^2(t_1^-) + x_4^2(t_1^-) - 2/x_1(t_1^-) - E_e$$
$$= 0 \qquad (32)$$

where E_e is a specified total energy (slightly larger than zero).

The present optimal problem is defined:

Minimize $\quad J=t_f$

Subject to Equations of motion (23) and (24) with Transformations (25)

Boundary conditions (27)~(30), and (32)

Well-known penalty function approach

is employed to deal with the boundary equality constraints.

The augmented performance index is defined by

$$\bar{J} \equiv J + \sum_{i=1}^{4} P_i \psi_i^2 \tag{33}$$

$$= t_f + P_1 \left\{ X_1(t_f) - X_{1f} \right\}^2$$

$$+ P_2 \left\{ X_3(t_f) - X_{3f} \right\}^2$$

$$+ P_3 \left\{ X_4(t_f) - X_{4f} \right\}^2$$

$$+ P_4 \left\{ x_3^2(t_1) + x_4^2(t_1) \right.$$

$$\left. -2/x_1(t_1) - E_e \right\}^2 \tag{34}$$

where $P_1 \sim P_4$ are the penalty functions. The problem previously defined is converted into the following one:

Minimize $\bar{J} = t_f + \sum_{i=1}^{4} P_i \psi_i^2$

Subject to

Equations of motion (23) and (24) with Transformations (25)

Without boundary conditions

3.2. THREE-POINT BOUNDARY-VALUE PROBLEM

The performance index J and the augmented pferformance index $\bar{J}$ are written in more general form in order to define the three-point boundary-value problem more generally.

$$J = \Phi[X(t_f), t_f] + \int_{t_0}^{t_1^-} L_e[x(t), u(t), t] dt$$

$$+ \int_{t_1^+}^{t_f} L_s[X(t), u(t), t] dt \tag{35}$$

$$\bar{J} = \Phi[X(t_f), t_f] + \sum_{i=1}^{3} P_i \psi_i^2 [X(t_f), t_f]$$

$$+ P_4 \psi_4^2 [x(t_1), t_1]$$

$$+ \int_{t_0}^{t_1^+} L_e[x(t), u(t), t] dt$$

$$+ \int_{t_1^+}^{t_f} L_s[X(t), u(t), t] dt \tag{36}$$

For the present problem,

$$L_e[x(t), u(t), t] = 0 \ , \ t \in [t_0, t_1^-] \tag{37}$$

$$L_s[X(t), u(t), t] = 0 \ , \ t \in [t_1^+, t_f] \tag{38}$$

$$\Phi[X(t_f), t_f] = t_f \tag{39}$$

Hamiltonian H_e, H_s are introduced by

$$H_e[x(t), u(t), t] = L_e[x(t), u(t), t]$$

$$+ \lambda^T(t) \cdot f[x(t), u(t), t] \tag{40}$$

$$H_s[X(t), u(t), t] = L_s[X(t), u(t), t]$$

$$+ \lambda^T(t) \cdot F[(X(t), u(t), t] \tag{41}$$

where λ is a column vector whose elements are Lagrange multipliers $\lambda_i(t)$ $i = 1 \sim 4$.

$$\lambda^T(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)) \tag{42}$$

The augmented performance index Eq. (36) is rewritten as

$$\bar{J} = \Phi[X(t_f), t_f] + \sum_{i=1}^{3} P_i \psi_i^2 [X(t_f), t_f]$$

$$+ P_4 \psi_4^2 [x(t_1), t_1]$$

$$+ \int_{t_0}^{t_1^-} \left\{ H_e[x(t), u(t), t] - \lambda^T(t) \right.$$

$$\left. \cdot f[x(t), u(t), t] \right\} dt$$

$$+ \int_{t_1^+}^{t_f} \left\{ H_s[X(t), u(t), t] - \lambda^T(t) \right.$$

$$\left. \cdot F[X(t), u(t), t] \right\} dt \tag{43}$$

Integrating by parts the fourth and fifth terms and taking the variation, the first variation of $\overline{J}$ is derived.

$$
\begin{aligned}
\delta \overline{J} = &\frac{\partial \Phi}{\partial X(t_f)} \, \delta X(t_f) + \frac{\partial \Phi}{\partial t_f} dt_f \\
&+ 2 \sum_{i=1}^{3} P_i \psi_i \left(\frac{\partial \psi_i}{\partial X(t_f)} \delta X(t_f) + \frac{\partial \psi_i}{\partial t_f} dt_f \right) \\
&+ 2 P_4 \psi_4 \left(\frac{\partial \psi_4}{\partial x(t_1)} \delta x(t_1) + \frac{\partial \psi_4}{\partial t_1} dt_1 \right) \\
&- [\lambda^T \cdot \delta x]_{t_0}^{t_1^-} - [\lambda^T \cdot \delta X]_{t_1^+}^{t_f} \\
&+ [(H_e - \lambda^T \cdot \frac{dx}{dt}) dt]_{t_0}^{t_1^-} \\
&+ [(H_s - \lambda^T \frac{dX}{dt}) dt]_{t_1^+}^{t_f} \\
&+ \int_{t_0}^{t_1^-} \left\{ (\frac{\partial H_e}{\partial x} + \frac{d\lambda^T}{dt}) \delta x \right. \\
&\left. + \frac{\partial H_e}{\partial u} \delta u \right\} dt + \int_{t_1^+}^{t_f} \left\{ (\frac{\partial H_s}{\partial X} + \frac{d\lambda^T}{dt}) \delta X \right. \\
&\left. + \frac{\partial H_s}{\partial u} \delta u \right\} dt
\end{aligned} \tag{44}
$$

The differentials $dx(t_1^-)$ and $dX(t_1^+)$ and the variations $\delta x(t_1^-)$ and $\delta X(t_1^+)$ are related to each other as follows:[15]

$$
dx(t_1^-) = \delta x(t_1^-) + \frac{dx(t_1^-)}{dt} dt \tag{45}
$$

$$
dX(t_1^+) = \delta X(t_1^+) + \frac{dX(t_1^+)}{dt} dt \tag{46}
$$

It is assumed without loss of generality that the initial time t_0 and the initial value of the state variables are given. Therefore,

$$
dt_0 = 0, \quad dx(t_0) = 0, \quad \delta x(t_0) = 0 \tag{47}
$$

Substituting Eqs. (45)~(47) into Eq. (44) and collecting terms, Eq. (44) yields

$$
\begin{aligned}
\delta \overline{J} = &[\frac{\partial \Phi}{\partial X(t_f)} + 2 \sum_{i=1}^{3} P_i \psi_i \frac{\partial \psi_i}{\partial X(t_f)} - \lambda^T(t_f)] \\
&\delta X(t_f) + [\frac{\partial \Phi}{\partial t_f} + 2 \sum_{i=1}^{3} P_i \psi_i \frac{\partial \psi_i}{\partial t_f} \\
&+ H_s(t_f) - \lambda^T(t_f) \cdot \frac{dX(t_f)}{dt}] \, dt_f \\
&+ [2P_4 \psi_4 \frac{\partial \psi_4}{\partial x(t_1^-)} - \lambda^T(t_1^-)] \, dx(t_1^-) \\
&+ \lambda^T(t_1^+) dX(t_1^+) \\
&+ [2P_4 \psi_4 (\frac{\partial \psi_4}{\partial t_1} - \frac{\partial \psi_4}{\partial x(t_1^-)} \cdot \frac{dx(t_1^-)}{dt}) \\
&+ H_e(t_1^-) - H_s(t_1^+)] \, dt_1 \\
&+ \int_{t_0}^{t_1^-} \left\{ (\frac{\partial H_e}{\partial x} + \frac{d\lambda^T}{dt}) \delta x + \frac{\partial H_e}{\partial u} \delta u \right\} dt \\
&+ \int_{t_1^+}^{t_f} \left\{ (\frac{\partial H_s}{\partial X} + \frac{d\lambda^T}{dt}) \delta X + \frac{\partial H_s}{\partial u} \delta u \right\} dt
\end{aligned} \tag{48}
$$

Since $x(t_1^-)$ and $X(t_1^+)$ are dependent as shown by Eq. (25), the differentials $dx(t_1^-)$ and $dX(t_1^+)$ are not independent each other.

$$
dX(t_1^+) = B \cdot dx(t_1^-) \tag{49}
$$

where B is a 4×4 square matrix derived from Eq. (25). For the details, see Appendix B.

Substituting this equation into Eq. (48) yields

$$
\delta \overline{J} = [\frac{\partial \Phi}{\partial X(t_f)} + 2 \sum_{i=1}^{3} P_i \psi_i \frac{\partial \psi_i}{\partial X(t_f)} - \lambda^T(t_f)]
$$

$$\delta X(t_f) + [\frac{\partial \Phi}{\partial t_f} + 2 \sum_{i=1}^{3} P_i \psi_i \frac{\partial \psi_i}{\partial t_f}$$

$$+ H_s(t_f) - \lambda^T(t_f) \cdot \frac{dX(t_f)}{dt}] dt_f$$

$$+ [2P_4 \psi_4 \frac{\partial \psi_4}{\partial x(t_1^-)} - \lambda^T(t_1^-)$$

$$+ \lambda^T(t_1^+) \cdot B] dx(t_1^-)$$

$$+ [2P_4 \psi_4 (\frac{\partial \psi_4}{\partial t_1} - \frac{\partial \psi_4}{\partial x(t_1^-)} \cdot \frac{dx(t_1^-)}{dt})$$

$$+ H_e(t_1^-) - H_s(t_1^+)] dt_1$$

$$+ \int_{t_0}^{t_1^-} \left\{ (\frac{\partial H_e}{\partial x} + \frac{d\lambda^T}{dt}) \delta x + \frac{\partial H_e}{\partial u} \delta u \right\} dt$$

$$+ \int_{t_1^+}^{t_f} \left\{ (\frac{\partial H_s}{\partial X} + \frac{d\lambda^T}{dt}) \delta X + \frac{\partial H_s}{\partial u} \delta u \right\} dt$$

$$(50)$$

Choosing the Lagrange multipliers $\lambda(t)$ to cuase the coefficients of δx and δX to vanish, the co-state equations are obtained.

$$\frac{d\lambda^T}{dt} = -\frac{\partial H_e}{\partial x} \quad , \quad t \in [t_0, t_1^-] \qquad (51)$$

$$\frac{d\lambda^T}{dt} = -\frac{\partial H_s}{\partial X} \quad , \quad t \in [t_1^+, t_f] \qquad (52)$$

with boundary conditions

$$\lambda^T(t_f) = \frac{\partial \Phi}{\partial X(t_f)} + 2 \sum_{i=1}^{3} P_i \psi_i \frac{\partial \psi_i}{\partial X(t_f)} \qquad (53)$$

$$\lambda^T(t_1^-) = \lambda^T(t_1^+) \cdot B + 2P_4 \psi_4 \frac{\partial \psi_4}{\partial x(t_1^-)} \qquad (54)$$

Eq.(50) becomes

$$\delta J = [\frac{\partial \Phi}{\partial t_f} + 2 \sum_{i=1}^{3} P_i \psi_i \frac{\partial \psi_i}{\partial t_f} + H_s(t_f)$$

$$- \lambda^T(t_f) \cdot \frac{dX(t_f)}{dt}] dt_f$$

$$+ [2P_4 \psi_4 (\frac{\partial \psi_4}{\partial t_1} - \frac{\partial \psi_4}{\partial x(t_1^-)} \cdot \frac{dx(t_1^-)}{dt})$$

$$+ H_e(t_1^-) - H_s(t_1^+)] dt_1$$

$$+ \int_{t_0}^{t_1^-} \left\{ \frac{\partial H_e}{\partial u} \delta u \right\} dt + \int_{t_1^+}^{t_f} \left\{ \frac{\partial H_s}{\partial u} \right\} dt \qquad (55)$$

For an extremum, δJ must be zero for arbitrary $\delta u(t)$, dt_f, and dt_1. This can only happen if

$$\frac{\partial H_e}{\partial u} = 0 \quad , \quad t \in [t_0, t_1^-] \qquad (56)$$

$$\frac{\partial H_s}{\partial u} = 0 \quad , \quad t \in [t_1^+, t_f] \qquad (57)$$

$$\frac{\partial \Phi}{\partial t_f} + 2 \sum_{i=1}^{3} P_i \psi_i \frac{\partial \psi_i}{\partial t_f} + H_s(t_f)$$

$$- \lambda^T(t_f) \cdot \frac{dX(t_f)}{dt} = 0 \qquad (58)$$

$$2P_4 \psi_4 (\frac{\partial \psi_4}{\partial t_1} - \frac{\partial \psi_4}{\partial x(t_1^-)} \cdot \frac{dx(t_1^-)}{dt})$$

$$+ H_e(t_1^-) - H_s(t_1^+) = 0 \qquad (59)$$

Now, Euler-Lagrange equations (51) ~ (54) and (56) ~ (59) are obtained. Eqs. (56) and (57) are known also as optimality condition, Eqs. (58) and (59) as transversality conditions at $t = t_f$ and $t = t_1$. Since a direct method is employed, transversality conditions are not used in solving the present problem numerically. It is clear at a glance of Eq. (54) that the Lagrange multipliers have a discontinuity at $t = t_1$.

For the present problem, co-state equations are as follows:

for $t \in [t_0, t_1^-]$,

$$\frac{d\lambda_1}{dt} = -\left\{ -\lambda_2 x_4/x_1^2 + \lambda_3(-x_4^2/x_1^2 + 2/x_1^3) \right.$$

$$\left. + \lambda_4 x_3 x_4/x_1^2 \right\} \tag{60}$$

$$\frac{d\lambda_2}{dt} = -\left\{ 0 \right\} \tag{61}$$

$$\frac{d\lambda_3}{dt} = -\left\{ \lambda_1 - \lambda_4 x_4/x_1 \right\} \tag{62}$$

$$\frac{d\lambda_4}{dt} = -\left\{ \lambda_2/x_1 + 2\lambda_3 x_4/x_1 - \lambda_4 x_3/x_1 \right\} \tag{63}$$

for $t \in [t_1^+, t_f]$,

$$\frac{d\lambda_1}{dt} = -A\left\{ -\lambda_2 X_4/X_1^2 \right.$$

$$+ \lambda_3(-X_4^2/X_1^2 + 2/x_1^3)$$

$$\left. + \lambda_4 X_3 X_4/X_1^2 \right\} \tag{64}$$

$$\frac{d\lambda_2}{dt} = -A\left\{ 0 \right\} \tag{65}$$

$$\frac{d\lambda_3}{dt} = -A\left\{ \lambda_1 - \lambda_4 X_4/X_1 \right\} \tag{66}$$

$$\frac{d\lambda_4}{dt} = -A\left\{ \lambda_2/X_1 \right.$$

$$\left. + 2\lambda_3 X_4/X_1 - \lambda_4 X_3/X_1 \right\} \tag{67}$$

The above equations are written compactly in vectoral form.

$$\frac{d\lambda}{dt} = g[x, \lambda] \quad , \quad t \in [t_0, t_1^-] \tag{68}$$

$$\frac{d\lambda}{dt} = G[X, \lambda] \quad , \quad t \in [t_1^+, t_f] \tag{69}$$

where g and G are the column vectors whose elements are the right-hand sides of Eqs.(60) $\sim$ (63), and of Eqs. (64) $\sim$ (67) respectively.

Boundary conditions are

$$\lambda^T(t_f) = 2(P_1\psi_1, 0, P_2\psi_2, P_3\psi_3) \tag{70}$$

$$\lambda^T(t_1^-) = \lambda^T(t_1^+) \cdot B + 4P_4\psi_4(1/x_1^2(t_1^-),$$

$$0, x_3(t_1^-), x_4(t_1^-)) \tag{71}$$

Well-defind three-point boundary-value problem is:

Equations of motion (23) and (24) with initial conditions (27) and transformations (25)

Co-state equations (68) and (69) with transversality conditions (70) and (71)

3.3. LONG'S TRANSFORMATION[19]

Since the present problem is a free-final-time problem, both t_1 and t_f ($= t_1 + t_2$) must be determined through the searches. The following parameterization on t due to Long is employed to alleviate some of the difficuties involved with variable t_1 and t_f, especially extrapolation problems. For the conventional free-final-time problem, Long introduced the new independent variable s defined by

$$t - t_0 = t_f \cdot s \tag{72}$$

Assuming $t_0 = 0$ without loss of generality yields

$$t = t_f \cdot s \tag{73}$$

s ranges from 0 to 1 when t ranges from 0 to variable t_f in each search. Long

extended the method to the multi-point boundary-value (MPBV) problems.

For the present problem, the parameterization is given by

$$t = \begin{cases} t_1 \cdot s & , \\ t_1 + (t_f - t_1)(s-1) & , \end{cases}$$

$$t \in [0, \ t_1^-] \tag{74}$$

$$t \in [t_1^+, \ t_f] \tag{75}$$

s ranges from 0 to 1 and from 1 to 2 when t from 0 to t_1^- and from t_1^+ to t_f. The relation between the derivatives with respect to t and s is

$$\frac{d(\ )}{dt} = \frac{1}{t_1} \cdot \frac{d(\ )}{ds} = \frac{1}{t_1}(\ \dot{}\ ) \quad ,$$

$$t \in [0, t_1^-] \quad s \in [0, 1^-] \tag{76}$$

$$\frac{d(\ )}{dt} = \frac{1}{t_f - t_1} \cdot \frac{d(\ )}{ds} = \frac{1}{t_f - t_1}(\ \dot{}\ ),$$

$$t \in [t_1^+, t_f] \quad s \in [1^+, 2] \tag{77}$$

Substituting the above equations into the previously derived equations yields:

Equations of motion;
    for $s \in [0, 1^-]$,

$$\dot{x}_1 = t_1 \left\{ x_3 \right\} \tag{78}$$

$$\dot{x}_2 = t_1 \cdot \left\{ x_4 / x_1 \right\} \tag{79}$$

$$\dot{x}_3 = t_1 \cdot \left\{ x_4^2 / x_1 - 1/x_1^2 \right.$$

$$\left. + \hat{T}_e \sin u / (1 - \hat{m}_{ce} \cdot t_1 \cdot s) \right\} \tag{80}$$

$$\dot{x}_4\ ^\bullet = t_1 \cdot \left\{ -x_3 x_4 / x_1 \right.$$

$$\left. + \hat{T}_e \cos u / (1 - \hat{m}_{ce} \cdot t_1 \cdot s) \right\} \tag{81}$$

in vectoral form,

$$\dot{x}(s) = t_1 \cdot f[x(s), u(s), s, t_1] \tag{82}$$

for $s \in [1^+, 2]$

$$\dot{X}_1 = (t_f - t_1) \cdot A \cdot \left\{ X_3 \right\} \tag{83}$$

$$\dot{X}_2 = (t_f - t_1) \cdot A \cdot \left\{ X_4 / X_1 \right\} \tag{84}$$

$$\dot{X}_3 = (t_f - t_1) \cdot A \cdot [X_4^2 / X_1 - 1/X_1^2$$

$$+ \hat{T}_s \sin u / \{1 - \hat{m}ce \cdot t_1$$

$$- \hat{m}ce \cdot (t_f - t_1) \cdot (s-1)\}] \tag{85}$$

$$\dot{X}_4 = (t_f - t_1) \cdot A \cdot [-X_3 X_4 / X_1$$

$$+ \hat{T}_s \cos u / \{1 - \hat{m}_{ce} \cdot t_1 - \hat{m}_{ce}$$

$$\cdot (t_f - t_1) \cdot (s-1)\}] \tag{86}$$

in vectoral form,

$$\dot{X}(s) = (t_f - t_1) \cdot F[X(s), u(s), s, t_f, t_1] \tag{87}$$

Co-state equations;
    for $s \in [0, 1^-]$

$$\dot{\lambda}(s) = t_1 \cdot g[x(s), \lambda(s)] \tag{88}$$

for $s \in [1^+, 2]$,

$$\dot{\lambda}(s) = (t_f - t_1) \cdot G[X(s), \lambda(s)] \tag{89}$$

Boundary conditions;

$$\psi_1 = X_1(2) - X_{1f} = 0 \tag{90}$$

$$\psi_2 = X_3(2) - X_{3f} = 0 \tag{91}$$

$$\psi_3 = X_4(2) - X_{4f} = 0 \tag{92}$$

$$\psi_4 = x_3^2(1^-) + x_4^2(1^-) - 2/x_1(1^-)$$

$$- E_e = 0 \qquad (93)$$

Transversality conditions;

$$\lambda^T(2) = 2(P_1\psi_1, 0, P_2\psi_2, P_3\psi_3) \qquad (94)$$

$$\lambda^T(1^-) = \lambda^r(1^+) \cdot B + 4P_4\psi_4(1/x_1^2(1^-),$$

$$0, x_3(1^-), x_4(1^-)) \qquad (95)$$

J, $\overline{J}$ and H_e, H_s are also rewritten as

$$J = \Phi[X(2), t_f] + \int_0^{1^-} t_1$$

$$\cdot L_e[x(s), u(s), s, t_1] ds + \int_{1^+}^{2}(t_f - t_1)$$

$$\cdot L_s[X(s), u(s), s, t_f, t_1] ds \qquad (96)$$

$$\overline{J} = \Phi[X(2), t_f] + \sum_{i=1}^{3} P_i\psi_i^2 [X(2), t_f]$$

$$+ P_4\psi_4^2 [x(1), t_1] + \int_0^{1^-} t_1$$

$$\cdot L_e[x(s), u(s), s, t_1] ds + \int_{1^+}^{2}(t_f - t_1)$$

$$\cdot L_s[X(s), u(s), s, t_f, t_1] ds \qquad (97)$$

$$H_e[x(s), u(s), s, t_1]$$

$$= t_1 \cdot \left\{ L_e[x(s), u(s), s, t_1] + \lambda^T(s) \right.$$

$$\left. \cdot f[x(s), u(s), s, t_1] \right\} \qquad (98)$$

$$= t_1 \cdot H_e[x(t), u(t), t] \qquad (99)$$

$$H_s[X(s), u(s), s, t_f, t_1]$$

$$= (t_f - t_1) \cdot \left\{ L_s[X(s), u(s), s, t_f, t_1] \right.$$

$$\left. + \lambda^T(s) \cdot F[X(s), u(s), s, t_f, t_1] \right\} (100)$$

$$= (t_f - t_1) \cdot H_s[X(t), u(t), t] \qquad (101)$$

4. THREE-DIMENSIONAL SEARCH PROCEDURE

The augmented performance index J is a function of three parameters, i.e., Δt_f, Δt_1, and α, the corrections of t_f and t_1, and the control (steering angle) correction length along the search direction respectively.

$$\overline{J} \equiv \overline{J}[\alpha, \Delta t_1, \Delta t_f] \qquad (102)$$

In Ref. 16, Powers and Shieh developed the two-dimensional search procedure (2-DSP) to approximate the minimum of a function of two parameters, i.e., $\overline{J}$ $[\alpha, \Delta t_f]$. The 2-DSP with conjugate gradient method employing the penalty functions was applied to a minimum-time interplanetary orbit transfer with three terminal equality constraints while improving considerably the convergence rate.

Authors[3] have also applied the 2-DSP to a minimum-time Earth escape problem with one terminal equality constraint while getting the good convergence.

To solve the present problem, authors have developed a three-dimensional search procedure (3-DSP) with Δt_f, Δt_1, and α as three parameters. The details are shown hereafter.

At the n-th search in the N-th iteration, the three-dimensional search is performed with

$$u^{(N,n)}(s) = u^{(N)}(s) - \alpha^{(n)}p^{(N)}(s) ,$$

$$s \in [0, 1^-] \quad s \in [1^+, 2] \qquad (103)$$

$$t_1^{(N,n)} = t_1^{(N)} + \Delta t_1^{(n)} \qquad (104)$$

$$t_f^{(N,n)} = t_f^{(N)} + \Delta t_f^{(n)} \qquad (105)$$

where $u^{(N)}(s)$, $t_1^{(N)}$, $t_f^{(N)}$ are the results of the N-1 -th iteration and $p^{(N)}(s)$ is a control search direction (gradient direction) in the N-th iteration. $u^{(N)}(s)$, $t_1^{(N)}$, and $t_f^{(N)}$ are then updated for the N+1 -th iteration using the $\alpha^{(N)}, \Delta t_1^{(N)}$, and $\Delta t_f^{(N)}$ which give the least J in the N-th iteration.

$$u^{(N+1)}(s) = u^{(N)}(s) - \alpha^{(N)} p^{(N)}(s) \quad (106)$$

$$t_1^{(N+1)} = t_1^{(N)} + \Delta t_1^{(N)} \quad (107)$$

$$t_f^{(N+1)} = t_f^{(N)} + \Delta t_f^{(N)} \quad (108)$$

It is necessary to assume at least a quadratic function in parameters for a surface-fitting.

$$\begin{aligned}
\overline{J} [\alpha, \Delta t_1, \Delta t_f] \equiv\ & C_1 + C_2 \alpha + C_3 \Delta t_1 \\
& + C_4 \Delta t_f + C_5 \alpha \cdot \Delta t_1 + C_6 \Delta t_1 \cdot \Delta t_f \\
& + C_7 \alpha \cdot \Delta t_f + C_8 \alpha^2 + C_9 \Delta t_1^2 \\
& + C_{10} \Delta t_f^2
\end{aligned} \quad (109)$$

where $C_1 \sim C_{10}$ are the coefficients to be determined through the surface-fitting. Ten data points concerning J are at least necessary in order to determine them. However, four data points are readily available after the first calculation of J.

Clearly,

$$\overline{J}^{(0)} \equiv \overline{J}[0, 0, 0] = C_1 \quad (110)$$

Taking the partial differentials of Eq. (109) with respect to α, Δt_1, and Δt_f respectively yields

$$\frac{\partial \overline{J}}{\partial \alpha} \equiv \overline{J}_\alpha [\alpha, \Delta t_1, \Delta t_f]$$

$$= C_2 + C_5 \Delta t_1 + C_7 \Delta t_f + 2 C_8 \alpha \quad (111)$$

$$\frac{\partial \overline{J}}{\partial t_1} \equiv \overline{J}_{t_1} [\alpha, \Delta t_1, \Delta t_f]$$

$$= C_3 + C_5 \alpha + C_6 \Delta t_f + 2 C_9 \Delta t_1 \quad (112)$$

$$\frac{\partial \overline{J}}{\partial t_f} \equiv \overline{J}_{t_f} [\alpha, \Delta t_1, \Delta t_f]$$

$$= C_4 + C_6 \Delta t_1 + C_7 \alpha + 2 C_{10} \Delta t_f \quad (113)$$

Clearly,

$$\overline{J}_\alpha^{(0)} \equiv \overline{J}_\alpha [0, 0, 0] = C_2 \quad (114)$$

$$\overline{J}_{t_1}^{(0)} \equiv \overline{J}_{t_1} [0, 0, 0] = C_3 \quad (115)$$

$$\overline{J}_{t_f}^{(0)} \equiv \overline{J}_{t_f} [0, 0, 0] = C_4 \quad (116)$$

J_α, J_{t_1}, and J_{t_f} must be derived in analytical form to obtain C_2, C_3, and C_4 numerically using Eqs. (114)~(116).

J_α and J_{t_f} can be derived in the similar manner in Ref. 16, where Powers and Shieh developed the 2-DSP.

Using Eqs.(97)~(101) yields

$$\begin{aligned}
\frac{\partial \overline{J}}{\partial \alpha} \equiv\ & \frac{\partial}{\partial \alpha} (\Phi) + \frac{\partial}{\partial \alpha} \left(\sum_{i=1}^{3} P_i \psi_i^2 \right) + \frac{\partial}{\partial \alpha} (P_4 \psi_4^2) \\
& + \frac{\partial}{\partial \alpha} \int_0^{1^-} \left\{ t_1 L_e + \lambda^T (t_1 f - \dot{x}) \right. \ ds \\
& + \frac{\partial}{\partial \alpha} \int_{1^+}^{2} [(t_f - t_1) L_s \\
& + \lambda^T \left\{ (t_f - t_1) F - \dot{X} \right\}] ds \\
= \int_0^{1^-} & t_1 \left\{ \frac{\partial L_e}{\partial u} \frac{\partial u}{\partial \alpha} + \lambda^T \frac{\partial f}{\partial u} \frac{\partial u}{\partial \alpha} \right\} ds \\
& + \int_{1^+}^{2} (t_f - t_1) \left\{ \frac{\partial L_s}{\partial u} \frac{\partial u}{\partial \alpha} + \lambda^T \frac{\partial F}{\partial u} \frac{\partial u}{\partial \alpha} \right\} ds
\end{aligned}$$

$$= t_1 \int_0^{1^-} \left\{ (\frac{\partial L_e}{\partial u} + \lambda^T \frac{\partial f}{\partial u}) \frac{\partial u}{\partial \alpha} \right\} ds$$

$$+ (t_f - t_1) \int_{1^+}^2 \left\{ (\frac{\partial L_s}{\partial u} + \lambda^T \frac{\partial F}{\partial u}) \frac{\partial u}{\partial \alpha} \right\} ds$$

$$= \int_0^{1^-} (\frac{\partial H_e[s]}{\partial u} \frac{\partial u}{\partial \alpha}) ds$$

$$+ \int_{1^+}^2 (\frac{\partial H_s[s]}{\partial u} \frac{\partial u}{\partial \alpha}) ds \qquad (117)$$

Substituting Eq.(106) into this equation yields

$$\frac{\partial \bar{J}}{\partial \alpha} = -\int_0^{1^-} \frac{\partial H_e[s]}{\partial u} p(s) ds - \int_{1^+}^2 \frac{\partial H_s[s]}{\partial u} p(s) ds$$

$$(118)$$

Similarly,

$$\frac{\partial \bar{J}}{\partial t_f} \equiv \frac{\partial}{\partial t_f}(\Phi) + \frac{\partial}{\partial t_f}(\sum_{i=1}^3 P_i \psi_i^2) + \frac{\partial}{\partial t_f}(P_4 \psi_4^2)$$

$$+ \frac{\partial}{\partial t_f} \int_0^{1^-} \left\{ t_1 L_e + \lambda^T (t_1 f - \dot{x}) \right\} ds$$

$$+ \frac{\partial}{\partial t_f} \int_{1^+}^2 [(t_f - t_1) L_s + \lambda^T$$

$$\left\{ (t_f - t_1) F - \dot{X} \right\}] ds$$

$$= \frac{\partial}{\partial t_f}(\Phi) + \frac{\partial}{\partial t_f}(\sum_{i=1}^3 P_i \psi_i^2) + \left\{ \frac{\partial \Phi}{\partial X(2)} \right.$$

$$+ \frac{\partial}{\partial X(2)}(\sum_{i=1}^3 P_i \psi_i^2) \right\} \cdot \frac{dX}{dt} \Big|_{t_f} + L_s[t_f]$$

Substituting Eq.(53) into this equation yields

$$\frac{\partial \bar{J}}{\partial t_f} = \frac{\partial}{\partial t_f}(\Phi) + \frac{\partial}{\partial t_f}(\sum_{i=1}^3 P_i \psi_i^2)$$

$$+ \lambda^T(t_f) F(t_f) + L_s[t_f]$$

$$= \frac{\partial}{\partial t_f}(\Phi) + \frac{\partial}{\partial t_f}(\sum_{i=1}^3 P_i \psi_i^2) + H_s[t_f]$$

$$(119)$$

$$= \frac{\partial}{\partial t_f}(\Phi) + \frac{\partial}{\partial t_f}(\sum_{i=1}^3 P_i \psi_i^2) + \frac{1}{t_f - t_1} H_s[2]$$

$$(120)$$

$\bar{J}_{t_1}$ is derived as follows:

$$\frac{\partial \bar{J}}{\partial t_1} \equiv \frac{\partial}{\partial t_1}(\Phi) + \frac{\partial}{\partial t_1}(\sum_{i=1}^3 P_i \psi_i^2) + \frac{\partial}{\partial t_1}(P_4 \psi_4^2)$$

$$+ \frac{\partial}{\partial t_1} \int_0^{1^-} \left\{ t_1 L_e + \lambda^T(t_1 f - \dot{x}) \right\} ds$$

$$+ \frac{\partial}{\partial t_1} \int_{1^+}^2 [(t_f - t_1) L_s$$

$$+ \lambda^T \left\{ (t_f - t_1) F - \dot{X} \right\}] ds$$

$$= \frac{\partial(P_4 \psi_4)}{\partial t_1} + \frac{\partial(P_4 \psi_4)}{\partial x(1^-)} \cdot \frac{dx}{dt} \Big|_{t_1}$$

$$+ \frac{\partial}{\partial t_1} \int_0^{t_1} [L_e[t] + \lambda^T(t) \left\{ f(t) - \frac{dx}{dt} \right\}] dt$$

$$+ \frac{\partial}{\partial t_1} \int_{t_1}^{t_f} [L_s[t]$$

$$+ \lambda^T(t) \left\{ F(t) - \frac{dX}{dt} \right\}] dt$$

$$= \frac{\partial(P_4 \psi_4)}{\partial t_1} + \frac{\partial(P_4 \psi_4)}{\partial x(1^-)} \cdot \frac{1}{t_1} \dot{x}(1^-)$$

$$+ L_e[t_1^-] - L_s[t_1^+]$$

$$= \frac{\partial(P_4 \psi_4)}{\partial t_1} + \frac{\partial(P_4 \psi_4)}{\partial x(1^-)} \cdot \frac{1}{t_1} \dot{x}(1^-)$$

$$+ L_e[1^-] - L_s[1^+] \qquad (121)$$

For the present problem, Eqs. (118), (121), and (120) are reduced to

$$\bar{J}_\alpha = -\int_0^{1^-} \left\{ \frac{\partial H_e[s]}{\partial u} \right\}^2 ds - \int_{1^+}^2 \left\{ \frac{\partial H_s[s]}{\partial u} \right\}^2 ds$$

$$(122)$$

$$\overline{J}_{t_1} = \frac{4P_4\psi_4}{t_1}[\dot{x}_1/x_1^2 + x_3\dot{x}_3 + x_4\dot{x}_4]_{s=1^-}$$

(123)

$$\overline{J}_{t_f} = 1 + \frac{1}{t_f - t_1}H_s[2]$$

(124)

Besides the information supplied by Eqs. (110), (122), (123), and (124), six more data points concerning $\overline{J}$ are necessary to determine the remaining six coefficients in Eq. (109).

three funciton evaluation at first step

Evaluate

$$\overline{J}^{(1)} \equiv \overline{J}[\alpha^{(1)}, 0, 0]$$

$$= C_1 + C_2\alpha^{(1)} + C_8\alpha^{(1)^2}$$

(125)

$$\overline{J}^{(2)} \equiv \overline{J}[0, \Delta t_1^{(2)}, 0]$$

$$= C_1 + C_3\Delta t_1^{(2)} + C_9\Delta t_1^{(2)^2}$$

(126)

$$\overline{J}^{(3)} \equiv \overline{J}[0, 0, \Delta t_f^{(3)}]$$

$$= C_1 + C_4\Delta t_f^{(3)} + C_{10}\Delta t_f^{(3)^2}$$

(127)

where

$$\alpha^{(1)} = 2 \times (t_f^{(0)} - \overline{J}^{(0)}) \cdot \overline{J}_\alpha^{(0)}$$

(128)

$$\Delta t_1^{(2)} = -0.01 \times t_1^{(0)} \cdot \text{sgn}[\overline{J}_{t_1}^{(0)}]$$

(129)

$$\Delta t_f^{(3)} = -0.01 \times t_f^{(0)} \cdot \text{sgn}[\overline{J}_{t_f}^{(0)}]$$

(130)

at the first iteration referring to the 2-DSP developed in Ref. 16, and

$$\alpha^{(1)} = \alpha^{(N-1)}$$

(131)

$$\Delta t_1^{(2)} = \Delta t_1^{(N-1)}$$

(132)

$$\Delta t_f^{(3)} = \Delta t_f^{(N-1)}$$

(133)

at the N-th iteration.

Calculate $\overline{J}_t^{(j)}$ and $\overline{J}_{t_f}^{(j)}$ j=1~3 using Eqs. (123) and (124).

$C_5 \sim C_7$ are obtained using Eqs.(112) and (113).

$$C_5 = \frac{\overline{J}_{t_1}^{(1)}[\alpha^{(1)}, 0, 0] - C_3}{\alpha^{(1)}}$$

$$= \frac{\overline{J}_{t_1}^{(1)}[\alpha^{(1)}, 0, 0] - \overline{J}_{t_1}^{(0)}[0, 0, 0]}{\alpha^{(1)}}$$

(134)

$$C_6 = \frac{\overline{J}_{t_f}^{(2)}[0, \Delta t_1^{(2)}, 0] - C_4}{\Delta t_1^{(2)}}$$

$$= \frac{\overline{J}_{t_f}^{(2)}[0, \Delta t_1^{(2)}, 0] - \overline{J}_{t_f}^{(0)}[0, 0, 0]}{\Delta t_1^{(2)}}$$

(135)

$$C_7 = \frac{\overline{J}_{t_f}^{(1)}[\alpha^{(1)}, 0, 0] - C_4}{\alpha^{(1)}}$$

$$= \frac{\overline{J}_{t_f}^{(1)}[\alpha^{(1)}, 0, 0] - \overline{J}_{t_f}^{(0)}[0, 0, 0]}{\alpha^{(1)}}$$

(136)

$C_8 \sim C_{10}$ are obtained using Eqs.(125) ~ (127).

$$C_8 = \frac{\overline{J}^{(1)}[\alpha^{(1)}, 0, 0] - (C_1 + C_2\alpha^{(1)})}{\alpha^{(1)^2}}$$

$$= \frac{\overline{J}^{(1)}[\alpha^{(1)}, 0, 0] - \{\overline{J}^{(0)}[0, 0, 0]}{\alpha^{(1)^2}}$$

$$\left. + \alpha^{(1)}\overline{J}_\alpha^{(0)}[0, 0, 0]\right\}$$

(137)

$$C_9 = \frac{\overline{J}^{(2)}[0, \Delta t_1^{(2)}, 0] - (C_1 + C_3\Delta t_1^{(2)})}{\Delta t_1^{(2)^2}}$$

$$= \frac{\overline{J}^{(2)}[0, \Delta t_1^{(2)}, 0] - \{\overline{J}^{(0)}[0, 0, 0]}{\Delta t_1^{(2)^2}}$$

$$\left. + \Delta t_1^{(2)}\overline{J}_{t_1}^{(0)}[0, 0, 0]\right\}$$

(138)

$$C_{10} = \frac{\overline{J}^{(3)}[0, 0, \Delta t_f^{(3)}] - (C_1 + C_4 \Delta t_f^{(3)})}{\Delta t_f^{(3)^2}}$$

$$= \frac{\overline{J}^{(3)}[0, 0, \Delta t_f^{(3)}] - \left\{ \overline{J}^{(0)}[0, 0, 0] \right.}{\Delta t_f^{(3)^2}}$$

$$\left. + \Delta t_f^{(3)} \overline{J}_{t_f}^{(0)}[0, 0, 0] \right\}$$

(139)

surface-fitting with $C_1 \sim C_{10}$

Substituting $\overline{J}_\alpha = 0$, $\overline{J}_{t_1} = 0$, and $\overline{J}_{t_f} = 0$ into Eqs. (111)$\sim$(113), the following simultaneous linear equations are obtained.

$$\begin{pmatrix} 2C_8, & C_5, & C_7 \\ C_5, & 2C_9, & C_6 \\ C_7, & C_6, & 2C_{10} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \Delta t_1 \\ \Delta t_f \end{pmatrix} = \begin{pmatrix} -C_2 \\ -C_3 \\ -C_4 \end{pmatrix}$$

(140)

Evaluate four functions using the solutions α, Δt_1, and Δt_f of the above equations.

$$\overline{J}^{(4)} \equiv \overline{J}[\alpha, \Delta t_1, 0] = \overline{J}[\alpha^{(4)}, \Delta t_1^{(4)}, 0]$$

(141)

$$\overline{J}^{(5)} \equiv \overline{J}[\alpha, 0, \Delta t_f] = \overline{J}[\alpha^{(5)}, 0, \Delta t_f^{(5)}]$$

(142)

$$\overline{J}^{(6)} \equiv \overline{J}[0, \Delta t_1, \Delta t_f]$$

$$= \overline{J}[0, \Delta t_1^{(6)}, \Delta t_f^{(6)}]$$

(143)

$$\overline{J}^{(7)} \equiv \overline{J}[\alpha, \Delta t_1, \Delta t_f]$$

$$= \overline{J}[\alpha^{(7)}, \Delta t_1^{(7)}, \Delta t_f^{(7)}]$$

(144)

Calculate $\overline{J}_{t_1}^{(j)}$ and $\overline{J}_{t_f}^{(j)}$ $j = 4 \sim 7$.

Seven function evaluations so far have brought twenty-five data points concern-

ing $\overline{J}$, i.e., $\overline{J}^{(j)}$, $\overline{J}_{t_1}^{(j)}$, $\overline{J}_{t_f}^{(j)}$ $j = 0 \sim 7$ and $\overline{J}_\alpha^{(0)}$.

surface-fitting with $\overline{J}^{(0)}(=C_1)$, $\overline{J}_\alpha^{(0)}(=C_2)$, $\overline{J}^{(1)} \sim \overline{J}^{(6)}$, $\overline{J}_{t_1}^{(j)}$, $\overline{J}_{t_f}^{(j)}$ to obtain $C_3 \sim C_{10}$

Jacobian matrix

$$J \equiv \begin{pmatrix} \dfrac{\partial^2 \overline{J}}{\partial \alpha^2}, & \dfrac{\partial^2 \overline{J}}{\partial \alpha \partial t_1}, & \dfrac{\partial^2 \overline{J}}{\partial \alpha \partial t_f} \\[2mm] \dfrac{\partial^2 \overline{J}}{\partial t_1 \partial \alpha}, & \dfrac{\partial^2 \overline{J}}{\partial t_1^2}, & \dfrac{\partial^2 \overline{J}}{\partial t_1 \partial t_f} \\[2mm] \dfrac{\partial^2 \overline{J}}{\partial t_f \partial \alpha}, & \dfrac{\partial^2 \overline{J}}{\partial t_f \partial t_1}, & \dfrac{\partial^2 \overline{J}}{\partial t_f^2} \end{pmatrix}$$

$$= \begin{pmatrix} 2C_8, & C_5, & C_7 \\ C_5, & 2C_9, & C_6 \\ C_7, & C_6, & 2C_{10} \end{pmatrix}$$

(145)

is defined from Eq.(109).

If J is positive definite[20], obtain new α, Δt_1, and Δt_f by solving* Eq. (140). If not, repeat the procedure until $j=7$.

If J is not positive definite for all $j(0 \sim 7)$; surface-fitting with $\overline{J}^{(0)}(= C_1)$, $\overline{J}^{(0)} \sim \overline{J}^{(7)}$, $\overline{J}_{t_1}^{(j)}$, $\overline{J}_{t_f}^{(j)}$ to obtain $C_2 \sim C_{10}$

If not good for all $j(0 \sim 7)$, α, Δt_1, and Δt_f are newly defined as follows:

$$\alpha = \alpha^{(1)}/2, \quad \Delta t_1 = \Delta t_1^{(2)}/2, \quad \Delta t_f = \Delta t_f^{(3)}/2$$

(146)

Evaluate three functions

$$\overline{J}^{(8)} \equiv \overline{J}[\alpha, \Delta t_1, 0] = \overline{J}[\alpha^{(8)}, \Delta t_1^{(8)}, 0]$$

(147)

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* Strictly speaking, this condition was necessary also when Eq. (140) was solved in order to compute $\overline{J}^{(4)} \sim \overline{J}^{(7)}$. However, authors did solve Eq. (140) only under the condition of $|J| \neq 0$. If $|J| = 0$, $\alpha, \Delta t_1$, and Δt_f are defined by Eq. (146).

$$\overline{J}^{(9)} \equiv \overline{J}[\alpha, 0, \Delta t_f] = \overline{J}[\alpha^{(9)}, 0, \Delta t_f^{(9)}]$$

(148)

$$\overline{J}^{(10)} \equiv \overline{J}[0, \Delta t_1, \Delta t_f]$$

$$= \overline{J}[0, \Delta t_1^{(10)}, \Delta t_f^{(10)}]$$

(149)

$\overline{J}_{t_1}$ and $\overline{J}_{t_f}$ are not calculated hereafter.
surface-fitting with the smallest ten $\overline{J}s$
among the eleven $\overline{J}s$ of $\overline{J}^{(0)} \sim \overline{J}^{(10)}$ to
obtain $C_1 \sim C_{10}$

If good, evaluate

$$\overline{J}^{(11)} \equiv \overline{J}[\alpha, \Delta t_1, \Delta t_f]$$

$$= \overline{J}[\alpha^{(11)}, \Delta t_1^{(11)}, \Delta t_f^{(11)}]$$

(150)

If not good, evaluate $\overline{J}^{(11)}$ using the previous α, Δt_1, and Δt_f.
surface-fitting with the smallest ten $\overline{J}s$
among the twelve $\overline{J}s$ of $\overline{J}^{(0)} \sim \overline{J}^{(11)}$ to
obtain $C_1 \sim C_{10}$

Continue the searches until the specified maximum number of searches. α, Δt_1, and Δt_f corresponding to the smallest $\overline{J}$ are determined as $\alpha^{(N)}$, $\Delta t_1^{(N)}$, $\Delta t_f^{(N)}$. u, t_1, and t_f are corrected by Eqs. (106)~(108) for the N+1 -th iteration.

The procedure described so far is repeated compactly below.

1) $C_1 = \overline{J}^{(0)}$, $C_2 = \overline{J}_\alpha^{(0)}$; fit Eq. (109) with $\overline{J}^{(1)} \sim \overline{J}^{(6)}$, $\overline{J}_{t_1}^{(j)}$, $\overline{J}_{t_f}^{(j)}$ for j=0; if good, go to 4); if not, repeat 1) for larger j to 7.

2) Replace $\overline{J}_\alpha^{(0)}$ by $\overline{J}^{(7)}$; fit Eq. (109); if good, go to 4); if not, repeat 2) for larger j to 7.

3) $\alpha = \alpha^{(1)}/2$, $\Delta t_1 = \Delta t_1^{(2)}/2$, $\Delta t_f = \Delta t_f^{(3)}/2$.

4) Evaluate $\overline{J}^{(8)} = \overline{J}[\alpha, \Delta t_1, 0]$, $\overline{J}^{(9)} = \overline{J}[\alpha, 0, \Delta t_f]$, $\overline{J}^{(10)} = \overline{J}[0, \Delta t_1, \Delta t_f]$; select the ten points with the smallest costs among $\overline{J}^{(0)} \sim \overline{J}^{(10)}$; fit Eq. (109); if good, go to 5); if not, evaluate $\overline{J}^{(11)} = \overline{J}[\alpha, \Delta t_1, \Delta t_f]$; go to 6).
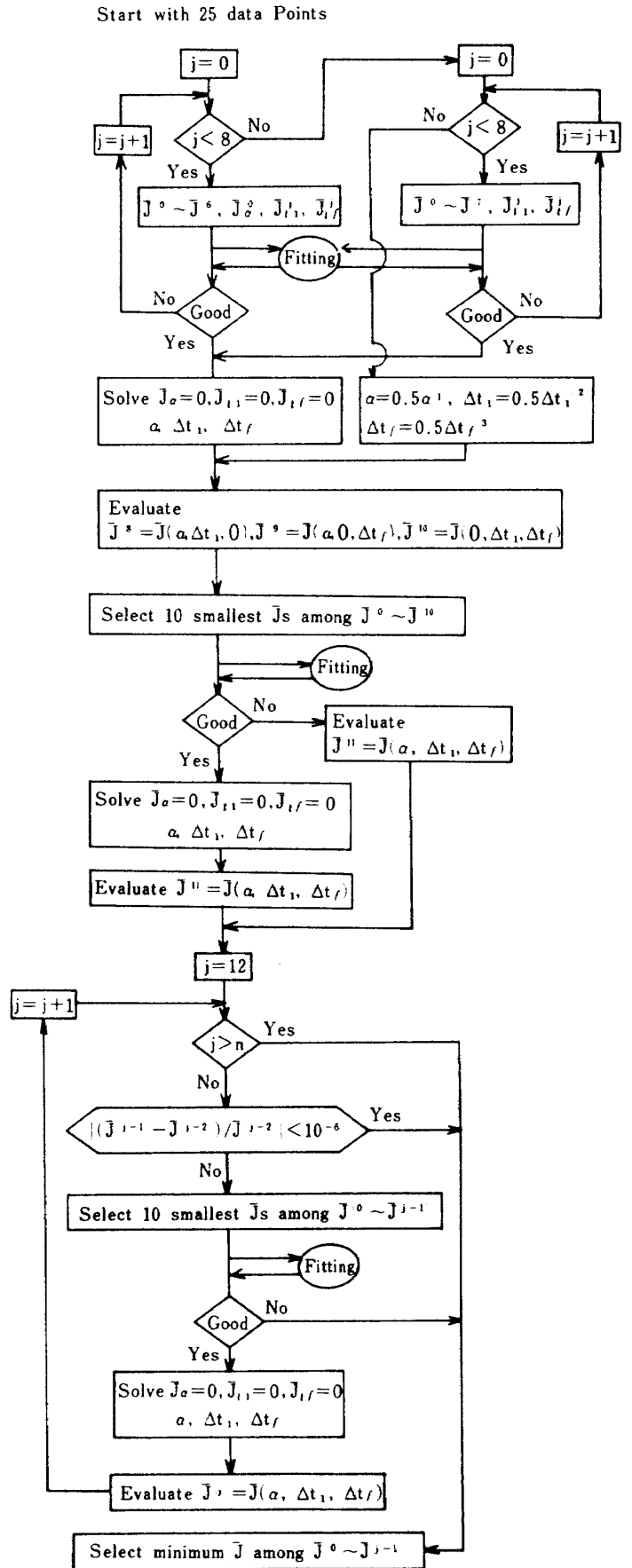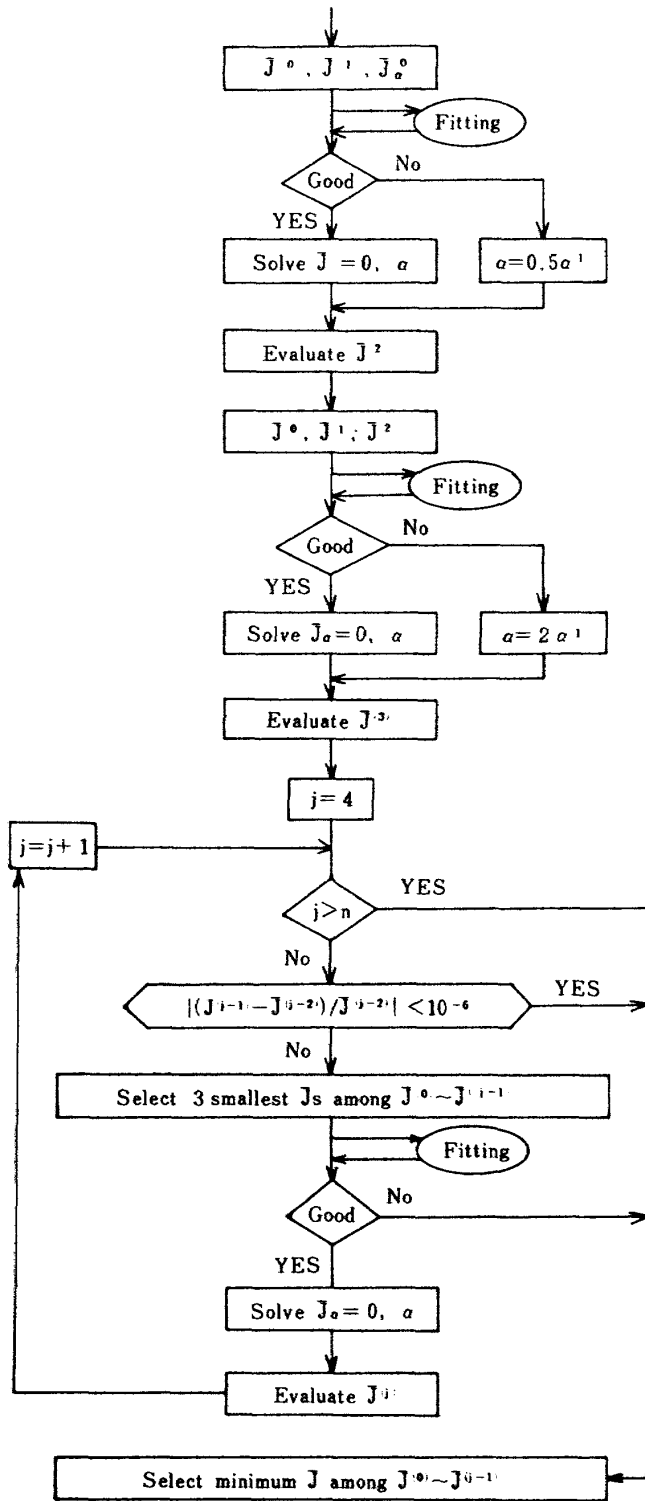

Figure 2. Flow Chart of 3-DSP

Figure 3. Flow Chart of 1-DSP

5) Evaluate $\bar{J}^{(11)}=\bar{J}[\alpha,\Delta t_1,\Delta t_f]$.

6) Select the ten points with the smallest costs among j $\bar{J}$s (j$\geq$12); fit Eq. (109); if not good or $|(\bar{J}^{(j-1)}-\bar{J}^{(j-2)})/\bar{J}^{(j-2)}|<$ 10^{-6}, go to 7); otherwise evaluate $\bar{J}^{(j)}$; repeat 6) for larger j to the specified

maximum number of searches.

7) Select the point with the smallest cost and stop → next iteration.

The flow chart of 3-DSP is shown in Figure 2.

As to be mentioned later in Chapter 5, a conventional one-dimensional search procedure (1-DSP) is introduced with $\Delta t_1 =0$ and $\Delta t_f=0$ in order to improve the convergence characteristics. 3-DSP and 1-DSP are used in series. As for 1-DSP, the flow chart is just shown in Figure 3.

5. SIMULATION RESULTS

The three-dimensional search procedure (3-DSP) with the gradient method is applied to the minimum-time low-thrust orbit transfer from the geosynchronous orbit to the heliocentric Mars orbit.

The simulation program is coded in FORTRAN and Runge-Kutta-Gill method is employed for integration. The computation is carried out in double precision using the digital computer FACOM M-380.

Parameters used in the simulations are shown in Table 1. $\hat{T}_s$ and $\hat{m}_{cs}$ correspond to those in Refs. 2 and 16, where minimal time $t_f^*=3.319$ in heliocentric time unit t_s is obtained. Authors have obtained minimal time $t_f^*=209.49$ in geocentric time unit t_e using $\hat{T}_e$ and $\hat{m}_{ce}$ in Table 1 with 2-DSP in a manner similar to that in Ref. 3. The sum of t_1^* and t_2^* is 1425.54 in t_e. E_e is chosen through a number of simulations in order to assure that $E(t_1^-)$ becomes positive.

An angle ζ between $v(t_1^-)$ and Earth's orbital velocity v_e is introduced as an additional parameter which determines the magnitude of $v(t_1^+)$. Since Mars is one

$\hat{T}_s = 0.1405$, $\hat{m}_{cs} = 0.0749$

(Ref. 2, 16)

$\hat{T}_e = 3.713 \times 10^{-3}$, $\hat{m}_{ce} = 2.044 \times 10^{-4}$

$E_e = 5 \times 10^{-3}$

Table 1. Parameters Used in
the Simulations

of the outer planets, it may be predicted that the larger magnitude of $v(t_1^+)$ leads to the shorter flight time.

The simulations are started, but have shown that it is difficult in many cases to get a good convergence using only 3-DSP. Taking account of the fact that the magnitude of α is extremely small compared with Δt_1 and Δt_f, a conventional one-dimensional search procedure (1-DSP) is introduced with $\Delta t_1 = 0$ and $\Delta t_f = 0$. 3-DSP and 1-DSP are used in series.

Then, the simulations are started again, and a number of combinations have been tested of search procedures and penalty functions. The resultant data are presented in Table 2. The criterion of the convergence ϵ is a square root of the integral of $\left(\dfrac{\partial H}{\partial u}\right)^2$, which should be zero if the optimality condition is satisfied. $n(m)$ means n iterations with m-DSP, and an arrow the change of the penalty functions.

3-DSP demonstrates its effectiveness especially when used successively after the

ζ	ε	Iterations	t_f	t_1 / t_2	$\Delta E(t_1)$ / ΔX_{1f}	ΔX_{3f} / ΔX_{4f}
0°	7.37	15 3(1)+3(2) →2(1)+3(3)→4(3)	1388.10	217.3 / 3.195	7.33×10⁻⁵ / -1.05×10⁻⁴	-2.87×10⁻⁴ / -7.37×10⁻⁴
	11.4	12 5(3)→4(3)→3(3)	1378.71	216.8 / 3.170	4.75×10⁻⁵ / 5.53×10⁻⁴	4.24×10⁻⁵ / 1.45×10⁻³
30°	0.480	10 3(1)+3(3)→2(1)+2(3)	1364.75	215.6 / 3.136	1.25×10⁻⁴ / 2.99×10⁻⁴	-3.33×10⁻⁴ / 3.26×10⁻⁴
	3.87	12 5(3)→4(3)→3(3)	1366.44	215.6 / 3.140	6.48×10⁻⁵ / 2.81×10⁻⁴	-1.22×10⁻⁴ / 6.42×10⁻⁴
60°	0.617	8 3(1)→3(3)→2(1)	1372.52	215.2 / 3.158	2.08×10⁻⁴ / 2.10×10⁻⁴	-2.60×10⁻⁴ / 1.55×10⁻⁴
	10.6	12 5(3)→4(3)→3(3)	1366.75	215.2 / 3.142	1.90×10⁻⁵ / 5.05×10⁻⁴	9.59×10⁻⁵ / 1.22×10⁻³
90°	0.796	7 5(3)→2(3)	1395.87	215.5 / 3.221	8.36×10⁻⁵ / 2.96×10⁻⁴	-4.16×10⁻⁴ / 3.92×10⁻⁴
	0.898	10 3(1)+3(3)→2(1)+2(3)	1400.64	214.9 / 3.236	1.40×10⁻⁴ / 3.34×10⁻⁴	-3.24×10⁻⁴ / 4.28×10⁻⁴

$t_1^* = 209.49$, $t_2^* = 3.319$, $t_1^* + t_2^* = 1425.54$

P_1 : $1 \times 10^3 \to 2 \times 10^3 \to 4 \times 10^3$

$P_2 \sim P_4$: $2 \times 10^3 \to 5 \times 10^3 \to 1 \times 10^4$

Table 2. Simulation Results

use of 1-DSP in the first two or three iterations.

t_1 is longer than t_1^*, but t_2 is shorter than t_2^*. The resultant total mission time t_f is shorter than the sum of t_1^* and t_2^* even in the case of bad convergence. It is very interesting that the minimal time for the total flight is shorter than the sum of the minimal times for the partitioned flights, i.e., the escape portion and the interplanetary portion. In the case of $\zeta=30°$, t_f is as short as 4.2 percents.

The control histories of that case are shown in Figure 4 and 5. The steering angle shown in Figure 4 is β, and in Figure 5 u, which appears in the equations of motion. β and u are defined in Figure 1, and measured from the tangential direction, i.e., the direction of velocity vector, and from the local horizontal direction respectively. β is chosen for the purpose

of comparison with the optimal escape problem, where β is usually depicted as the steering angle in the figures.

It is clear that the control history in the escape portion is quite different from that in an optimal escape problem. In the optimal escape problem, the thrust vector oscillates around the velocity vector. The amplitude increases gradually, but begins to decrease, and reaches to zero at the escape point. $\dfrac{d\beta}{dt}$ also vanishes at the same point. This is a well-known characteristic of the control history for the problem[3]. However, in the present problem, β oscillates in the negative range in the first portion, then increases gradually to the positive range, and reaches finally up to $30°$. The required time t_1 is a little longer than t_1^*.

Figure 5 shows the steering angle u in the interplanetary portion. The real line
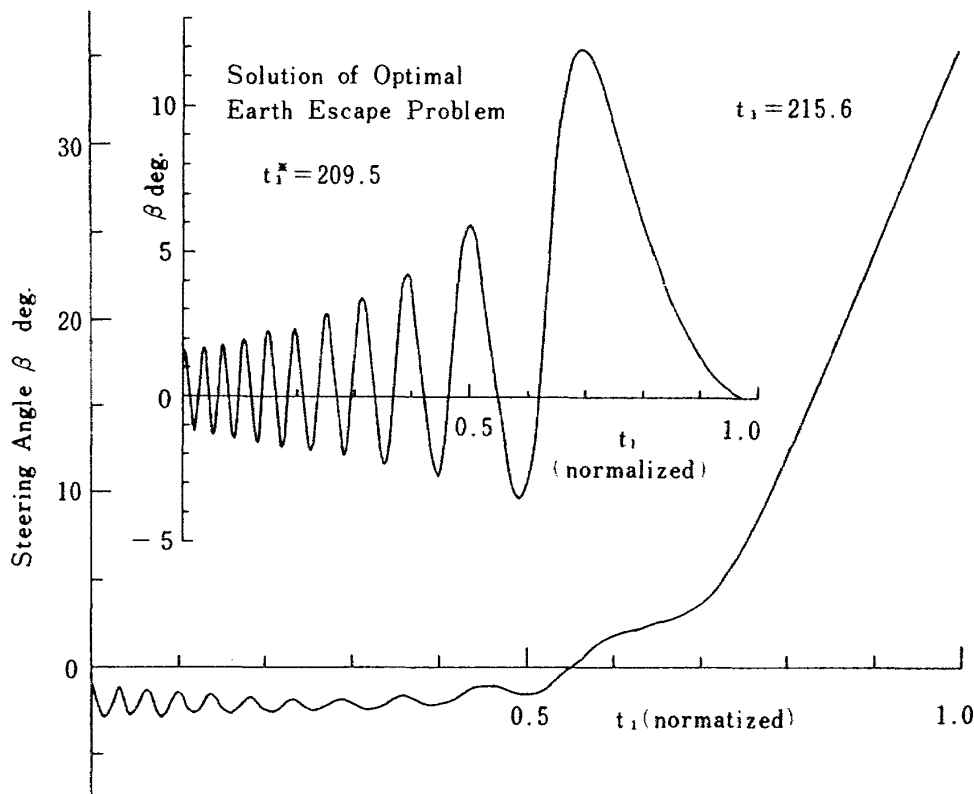

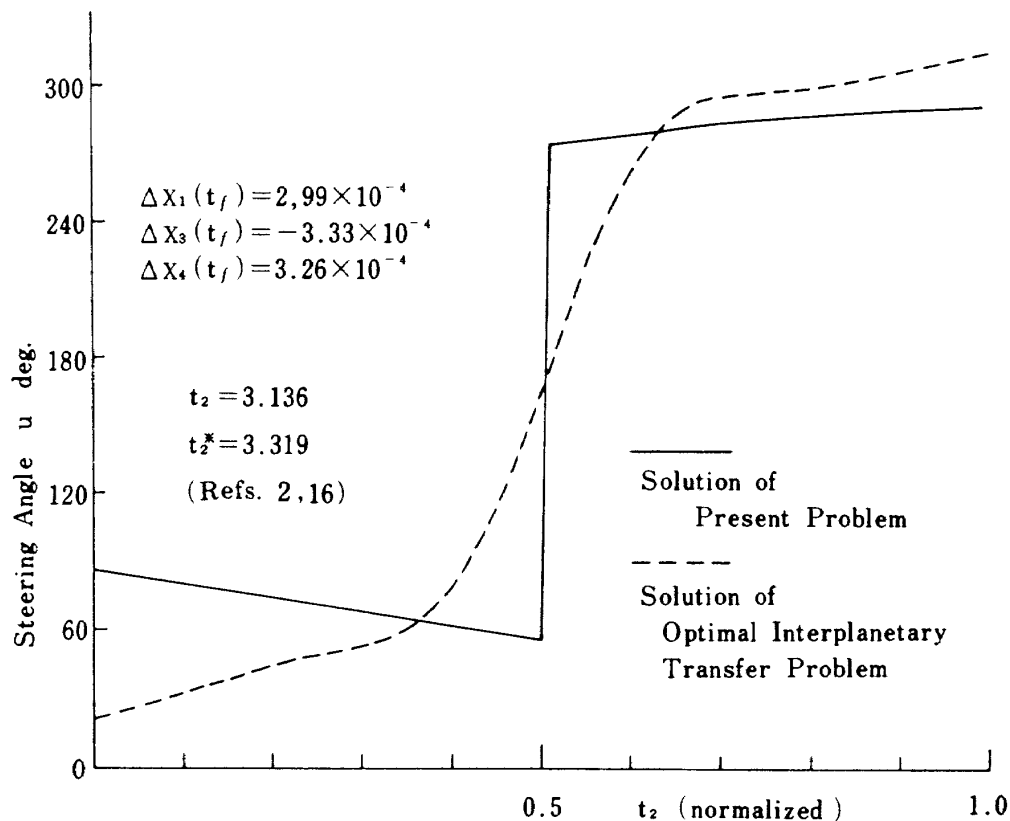Figure 4. Steering History for the Earth Escape Portion

Figure 5. Steering History for the Interplanetary Flight Portion

shows a solution for the present problem, and the broken line for the optimal interplanetary transfer problem. In this portion, the control histories are rather similar to each other. The required time t_2 is a little shorter than t_2^*.

The trajectories are shown in Figure 6 and 7. Figure 6 shows that the solution for the present problem follows outside the trajectory of the optimal escape problem almost all the period, and enters the inside at the final portion. Generally speaking, Figure 6 and 7 show that the trajectories are rather similar in both portions.

The resultant total mission time $t_f(=t_1+t_2)$, 1364.75 is shorter by 4.2 percents than the sum of t_1^* and t_2^*, 1425.54.
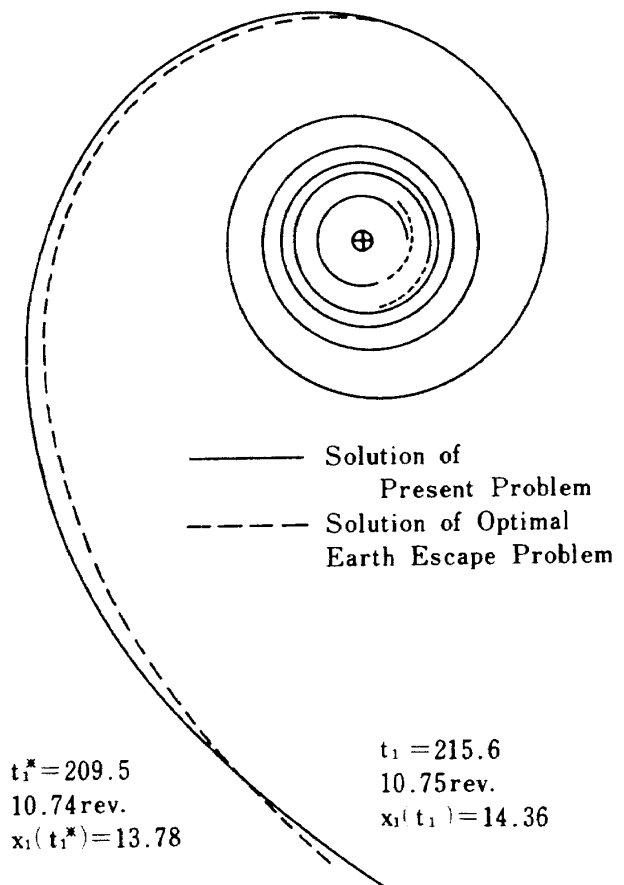

Figure 6. Trajectory for the Earth
Escape Portion

—————— Solution on of Present Problem

— — — — Solution of Optimal Interplanetary
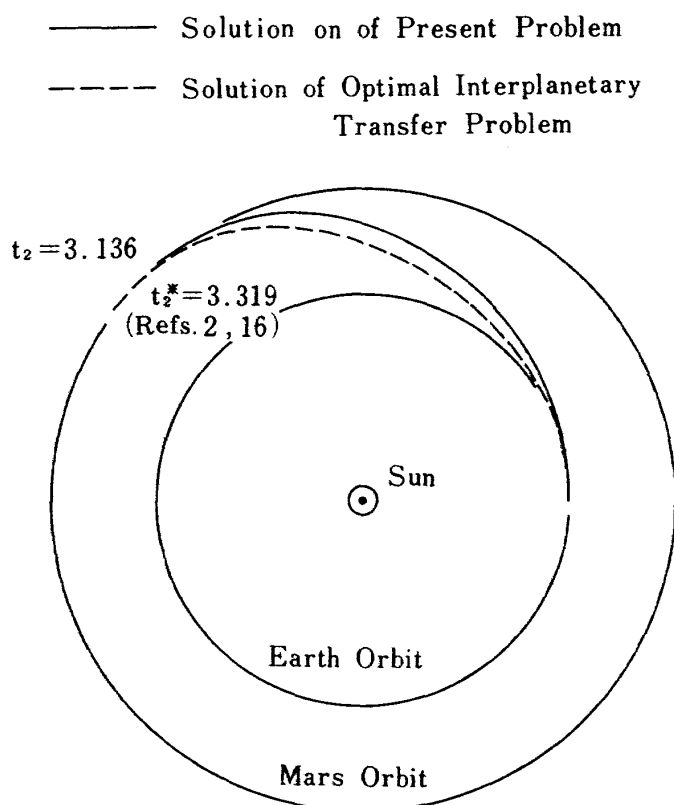Transfer Problem

Figure 7. Trajectory for the Interplanetary
Flight Portion

6. CONCLUDING REMARKS

A three-dimensional search procedure (3-DSP) is developed. 3-DSP with the gradient method employing the penalty functions is applied to the minimum-time low-thrust orbit transfer from the geosynchronous orbit to the heliocentric Mars orbit including Earth escape spiral trajectory.

As far as authors know, such a complicated optimal problem (three-point boundary-value problem) has not been solved numerically in the field of low thrust orbit transfers.

The obtained total mission-time $t_f(=t_1+t_2)$ is slightly shorter than the sum of t_1^* and t_2^*, minimum-time solutions for Earth escape problem and for Earth-Mars transfer problem respectively. It is very interesting that the minimal time for the total flight is shorter than the sum of the minimal times for the partitioned flights.

The control history in the escape portion is quite different from that in an optimal escape problem, but in the interplanetary portion it is similar to that in an optimal interplanetary transfer problem. The trajectories are rather similar in both portions.

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APPENDIX A TRANSFORMATIONS OF $x(t_1^-)$ INTO $X(t_1^+)$

Both t_1^- and t_1^+ will be omitted for simplicity in this section. The case shown in Figure A-1 $(0 < \theta_e < \pi/2, x_2 < \theta_e)$ is considered at first.

In the Figure A-1;

O_G origin of the geocentric coordinate system (center of mass of the Earth)

O_H origin of the heliocentric coordinate system (center of mass of the sun)

Q position of the spacecraft

θ_e transfer angle of the Earth in the heliocentric coordinate system

θ_s angle between the Earth and the spacecraft seen from the sun

θ_Q angle between the Earth and the sun seen from the spacecraft

In the triangle $QO_G O_H$, Law of cosines gives

$$
\begin{aligned}
(X_1 R_{AU})^2 &= R_{AU}^2 + (x_1 r_0)^2 - 2x_1 r_0 R_{AU} \cdot \cos(x_2 - \theta_e + \pi) \\
&= R_{AU}^2 + x_1^2 r_0^2 + 2x_1 r_0 R_{AU} \cdot \cos(\theta_e - x_2) \tag{A-1}
\end{aligned}
$$

$$
(x_1 r_0)^2 = R_{AU}^2 + (X_1 R_{AU})^2 - 2X_1 R_{AU}^2 \cos\theta_s \tag{A-2}
$$

$$
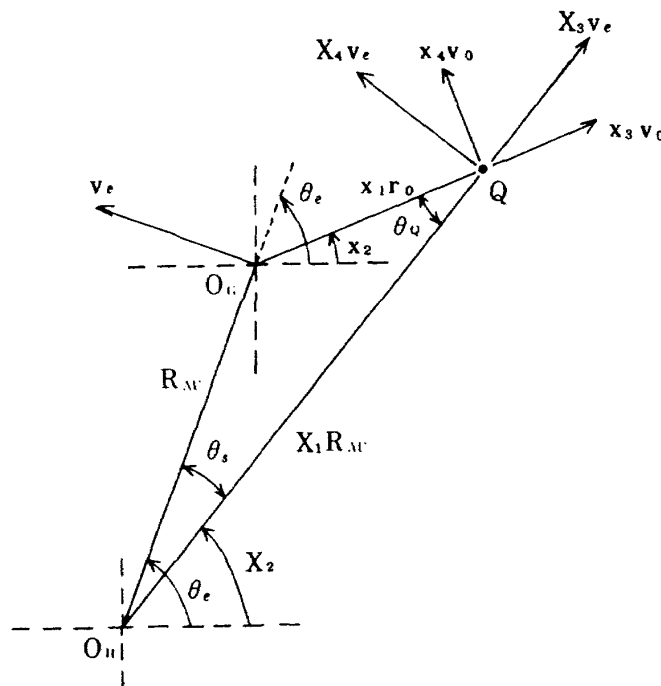R_{AU}^2 = (x_1 r_0)^2 + (X_1 R_{AU})^2 - 2x_1 X_1 r_0 R_{AU} \cos\theta_Q \tag{A-3}
$$

Figure A-1. Relative Loacations of Sun, Earth, and Spacecraft
at $t = t_1$ $(0 < \theta_e < \pi / 2, x_2 < \theta_e)$

Law of sines gives

$$\frac{r_1 r_0}{\sin \theta_s} = \frac{R_{AU}}{\sin \theta_Q} = \frac{X_1 R_{AU}}{\sin (x_2 - \theta_e + \pi)} = \frac{X_1 R_{AU}}{\sin (\theta_e - x_2)} \qquad \text{(A-4)}$$

Dividing the both sides of Eq.(A-1) by R_{AU}^2 yields

$$X_1^2 = 1 + x_1^2 \frac{r_0^2}{R_{AU}^2} + 2x_1 \frac{r_0}{R_{AU}} \cos (\theta_e - x_2)$$

X_1 is derived;

$$X_1 = \sqrt{1 + R^2 x_1^2 + 2Rx_1 \cos (\theta_e - x_2)} \qquad \text{(A-5)}$$

where $R = r_0/R_{AU}$.

From Eqs.(A-2) and (A-4),

$$\cos \theta_s = \frac{R_{AU}^2 + (X_1 R_{AU})^2 - (x_1 r_0)^2}{2X_1 R_{AU}^2} = \frac{1 + X_1^2 - R^2 x_1^2}{2X_1} \qquad \text{(A-6)}$$

$$\cos \theta_s = \frac{x_1 r_0 \sin (x_2 - \theta_e + \pi)}{X_1 R_{AU}} = \frac{Rx_1 \sin (\theta_e - x_2)}{X_1} \qquad \text{(A-7)}$$

Then,

$$\tan \theta_s \equiv \frac{\sin \theta_s}{\cos \theta_s} = \frac{2Rx_1 \sin (\theta_e - x_2)}{1 + X_1^2 - R^2 x_1^2} \qquad \text{(A-8)}$$

X_2 is derived;

$$X_2 = \theta_e - \theta_s = \theta_e - \tan^{-1} \left\{ \frac{2Rx_1 \sin (\theta_e - x_2)}{1 + X_1^2 - R^2 x_1^2} \right\} \qquad \text{(A-9)}$$

From Eqs.(A-3) and (A-4),

$$\cos \theta_Q = \frac{(x_1 r_0)^2 + (X_1 R_{AU})^2 - R_{AU}^2}{2x_1 X_1 r_0 R_{AU}} = \frac{X_1^2 + R^2 x_1^2 - 1}{2Rx_1 X_1} \qquad \text{(A-10)}$$

$$\sin \theta_Q = \frac{R_{AU} \sin (\theta_e - x_2)}{X_1 R_{AU}} = \frac{\sin (\theta_e - x_2)}{X_1} \qquad \text{(A-11)}$$

From the Figure A-1,

$$X_3 v_e = -v_e \sin \theta_s + x_3 v_0 \cos \theta_Q + x_4 v_0 \sin \theta_Q \qquad \text{(A-12)}$$

$$X_4 v_e = v_e \cos \theta_s - x_3 v_0 \sin \theta_Q + x_4 v_0 \cos \theta_Q \qquad \text{(A-13)}$$

Then, X_3 and X_4 are derived;

$$X_3 = -\sin \theta_s + V(x_3 \cos \theta_Q + x_4 \sin \theta_Q)$$

$$= -\frac{Rx_1 \sin (\theta_e - x_2)}{X_1} + V\left\{ \frac{x_3(X_1^2 + R^2 x_1^2 - 1)}{2Rx_1 X_1} \right.$$

$$\left. + \frac{x_4 \sin (\theta_e - x_2)}{X_1} \right\} \qquad \text{(A-14)}$$

$$X_4 = \cos \theta_s + V(-x_3 \sin \theta_Q + x_4 \cos \theta_Q)$$

$$= \frac{X_1^2 - R^2 x_1^2 + 1}{2X_1} + V\left\{ -\frac{x_3 \sin (\theta_e - x_2)}{X_1} \right.$$

$$\left. + \frac{x_4(X_1^2 + R^2 x_1^2 - 1)}{2Rx_1 X_1} \right\} \qquad \text{(A-15)}$$

where $V = v_0 / v_e$.

For the other cases, e.g., $(0 < \theta_e < \pi/2, x_2 > \theta_e)$, the same results are obtained. Therefore, the transformations are given by Eqs.(A-5), (A-9), (A-14), and (A-15).

APPENDIX B   DERIVATION OF MATRIX B; $dX(t_1^+) = B \cdot dx(t_1^-)$

Both t_1^- and t_1^+ will be omitted for simplicity in this section. From the result of Appendix A, X is described as follows:

$$X_1 = X_1 [x_1, x_2] \qquad \text{(B-1)}$$

$$X_2 = X_2 [x_1, x_2, X_1] \qquad \text{(B-2)}$$

$$X_3 = X_3 [x, X_1] \qquad \text{(B-3)}$$

$$X_4 = X_4 [x, X_1] \qquad \text{(B-4)}$$

Taking the total differentials of the above equations yields

$$dX_1 = \frac{\partial X_1 [x_1, x_2]}{\partial x_1} dx_1 + \frac{\partial X_1 [x_1, x_2]}{\partial x_2} dx_2 = B_{11} dx_1 + B_{12} dx_2 \qquad \text{(B-5)}$$

$$dX_2 = \frac{\partial X_2 [x_1, x_2, X_1]}{\partial x_1} dx_1 + \frac{\partial X_2 [x_1, x_2, X_1]}{\partial x_2} dx_2$$

$$+ \frac{\partial X_2 [x_1, x_2, X_1]}{\partial X_1} dX_1 \qquad \text{(B-6)}$$

$$dX_3 = \frac{\partial X_3 [x, X_1]}{\partial x_1} dx_1 + \frac{\partial X_3 [x, X_1]}{\partial x_2} dx_2 + \frac{\partial X_3 [x, X_1]}{\partial x_3} dx_3$$

$$+ \frac{\partial X_3 [x, X_1]}{\partial x_4} dx_4 + \frac{\partial X_3 [x, X_1]}{\partial X_1} dX_1 \qquad (B-7)$$

$$dX_4 = \frac{\partial X_4 [x, X_1]}{\partial x_1} dx_1 + \frac{\partial X_4 [x, X_1]}{\partial x_2} dx_2 + \frac{\partial X_4 [x, X_1]}{\partial x_3} dx_3$$

$$+ \frac{\partial X_4 [x, X_1]}{\partial x_4} dx_4 + \frac{\partial X_4 [x, X_1]}{\partial X_1} dX_1 \qquad (B-8)$$

Substituting Eq.(B-5) into Eqs.(B-6) ~ (B-8) yields

$$dX_2 = \left\{ \frac{\partial X_2 [x_1, x_2, X_1]}{\partial x_1} + \frac{\partial X_2 [x_1, x_2, X_1]}{\partial X_1} \cdot B_{11} \right\} dx_1$$

$$+ \left\{ \frac{\partial X_2 [x_1, x_2, X_1]}{\partial x_2} + \frac{\partial X_2 [x_1, x_2, X_1]}{\partial X_1} \cdot B_{12} \right\} dx_2$$

$$= B_{21} dx_1 + B_{22} dx_2 \qquad (B-9)$$

$$dX_3 = \left\{ \frac{\partial X_3 [x, X_1]}{\partial x_1} + \frac{\partial X_3 [x, X_1]}{\partial X_1} \cdot B_{11} \right\} dx_1$$

$$+ \left\{ \frac{\partial X_3 [x, X_1]}{\partial x_2} + \frac{\partial X_3 [x, X_1]}{\partial x_1} \cdot B_{12} \right\} dx_2$$

$$+ \frac{\partial X_3 [x, X_1]}{\partial x_3} dx_3 + \frac{\partial X_3 [x, X_1]}{\partial x_4} dx_4$$

$$= B_{31} dx_1 + B_{32} dx_2 + B_{33} dx_3 + B_{34} dx_4 \qquad (B-10)$$

$$dX_4 = \left\{ \frac{\partial X_4 [x, X_1]}{\partial x_1} + \frac{\partial X_4 [x, X_1]}{\partial X_1} \cdot B_{11} \right\} dx_1$$

$$+ \left\{ \frac{\partial X_4 [x, X_1]}{\partial x_2} + \frac{\partial X_4 [x, X_1]}{\partial X_1} \cdot B_{12} \right\} dx_2$$

$$+ \frac{\partial X_4 [x, X_1]}{\partial x_3} dx_3 + \frac{\partial X_4 [x, X_1]}{\partial x_4} dx_4$$

$$= B_{41} dx_1 + B_{42} dx_2 + B_{43} dx_3 + B_{44} dx_4 \qquad (B-11)$$

Defining 4 x 4 square matrix

$$
B = \begin{pmatrix} B_{11}, & B_{12}, & 0, & 0 \\ B_{21}, & B_{22}, & 0, & 0 \\ B_{31}, & B_{32}, & B_{33}, & B_{34} \\ B_{41}, & B_{42}, & B_{43}, & B_{44} \end{pmatrix}, \tag{B-12}
$$

Eqs. (B-5), (B-9), (B-10), and (B-11) are described in vectoral form.

$$
dX = B \cdot dx \tag{B-13}
$$

The elements of B are obtained using Eqs.(A-5), (A-9), (A-14), and (A-15). Comparison of the terms involving x_3 and x_4 in Eqs.(A-14) and (A-15) shows clearly that

$$
B_{43} = -B_{34} \tag{B-14}
$$

$$
B_{44} = B_{33} \tag{B-15}
$$

Eq. (A-5) is;

$$
X_1 = \sqrt{1 + R^2 x_1^2 + 2Rx_1 \cos(\theta_e - x_2)} \tag{B-16}
$$

$$
B_{11} \equiv \frac{\partial X_1[x_1, x_2]}{\partial x_1} = \frac{R^2 x_1 + R \cos(\theta_e - x_2)}{\sqrt{1 + R^2 x_1^2 + 2Rx_1 \cos(\theta_e - x_2)}}
$$

$$
= \frac{R^2 x_1 + R \cos(\theta_e - x_2)}{X_1} \tag{B-17}
$$

$$
B_{12} \equiv \frac{\partial X_1[x_1, x_2]}{\partial x_2} = \frac{Rx_1 \sin(\theta_e - x_2)}{\sqrt{1 + R^2 x_1^2 + 2Rx_1 \cos(\theta_e - x_2)}}
$$

$$
= \frac{Rx_1 \sin(\theta_e - x_2)}{X_1} \tag{B-18}
$$

Eq.(A-9) is;

$$
X_2 = \theta_e - \tan^{-1}\left\{\frac{2Rx_1 \sin(\theta_e - x_2)}{1 + X_1^2 - R^2 x_1^2}\right\} \tag{B-19}
$$

Then,

$$
\tan(\theta_e - X_2) = \frac{2Rx_1 \sin(\theta_e - x_2)}{1 + X_1^2 - R^2 x_1^2}
$$

Taking the total differential of the above equation yields

$$-\frac{dX_2}{\cos^2(\theta_e - x_2)} = \frac{2R\sin(\theta_e - x_2)dx_1 - 2Rx_1\cos(\theta_e - x_2)dx_2}{1 + X_1^2 - R^2x_1^2}$$

$$+\frac{2Rx_1\sin(\theta_e - x_2)(2R^2x_1dx_1 - 2X_1dX_1)}{(1 + X_1^2 - R^2x_1^2)^2}$$

$$= \frac{2R\sin(\theta_e - x_2)(X_1^2 + R^2x_1^2 + 1)}{(1 + X_1^2 - R^2x_1^2)^2}dx_1$$

$$-\frac{2Rx_1\cos(\theta_e - x_2)}{1 + X_1^2 - R^2x_1^2}dx_2 - \frac{4Rx_1X_1\sin(\theta_e - x_2)}{(1 + X_1^2 - R^2x_1^2)^2}dX_1$$

Therefore,

$$B_{21} = -\cos^2(\theta_e - X_2)\left\{\frac{2X_1(X_1^2 + R^2x_1^2 + 1)B_{12}}{x_1(1 + X_1^2 - R^2x_1^2)^2} - \frac{4X_1^2B_{12}}{(1 + X_1^2 - R^2x_1^2)^2}B_{11}\right\}$$

$$\text{(B-20)}$$

$$B_{22} = -\cos^2(\theta_e - X_2)\left\{-\frac{2x_1(X_1B_{11} - R^2x_1)}{1 + X_1^2 - R^2x_1^2} - \frac{4X_1^2B_{12}}{(1 + X_1^2 - R^2x_1^2)^2}B_{12}\right\}$$

$$\text{(B-21)}$$

Eq. (A-14) is;

$$X_3 = -\frac{x_1R\sin(\theta_e - x_2)}{X_1} + \frac{Vx_3(R^2x_1^2 + X_1^2 - 1)}{2Rx_1X_1} + \frac{Vx_4\sin(\theta_e - x_2)}{X_1} \quad \text{(B-22)}$$

Taking the total differential of Eq.(B-22) yields

$$dX_3 = -\frac{R\sin(\theta_e - x_2)}{X_1}dx_1 + \frac{Rx_1\cos(\theta_e - x_2)}{X_1}dx_2 + \frac{Rx_1\sin(\theta_e - x_2)}{X_1^2}dX_1$$

$$+\frac{V\left\{2R^2x_1x_3dx_1 + (X_1^2 + R^2x_1^2 - 1)dx_3 + 2x_3X_1dX_1\right\}}{2Rx_1X_1}$$

$$-\frac{Vx_3(X_1^2 + R^2x_1^2 - 1)(X_1dx_1 + x_1dX_1)}{2Rx_1^2X_1^2}$$

$$-\frac{Vx_4\cos(\theta_e - x_2)}{X_1}dx_2 + \frac{V\sin(\theta_e - x_2)}{X_1}dx_4 - \frac{Vx_4\sin(\theta_e - x_2)}{X_1^2}dX_1$$

Therefore,

$$B_{31} = -\frac{B_{12}}{x_1} + \frac{Vx_3(R^2x_1^2 - X_1^2 + 1)}{2Rx_1^2X_1}$$

$$+ \left\{ \frac{(x_1R - Vx_4)B_{12}}{Rx_1X_1} + \frac{Vx_3(X_1^2 - R^2x_1^2 + 1)}{2Rx_1X_1^2} \right\} B_{11} \qquad \text{(B-23)}$$

$$B_{32} = \frac{(B_{11}X_1 - R^2x_1)(Rx_1 - Vx_4)}{RX_1}$$

$$+ \left\{ \frac{(x_1R - Vx_4)B_{12}}{Rx_1X_1} + \frac{Vx_3(X_1^2 - R^2x_1^2 + 1)}{2Rx_1X_1^2} \right\} B_{12} \qquad \text{(B-24)}$$

$$B_{33} = \frac{V(X_1^2 + R^2x_1^2 - 1)}{2Rx_1X_1} \qquad \text{(B-25)}$$

$$B_{34} = \frac{VB_{12}}{Rx_1} \qquad \text{(B-26)}$$

Eq.(A-15) is;

$$X_4 = \frac{X_1^2 + 1 - R^2x_1^2}{2X_1} - \frac{Vx_3 \sin(\theta_e - x_2)}{X_1} + \frac{Vx_4(X_1^2 + R^2x_1^2 - 1)}{2Rx_1X_1}$$

$$\text{(B-27)}$$

Taking the total differential of Eq.(B-27) yields

$$dX_4 = \frac{-R^2x_1dx_1 + X_1dX_1}{X_1} - \frac{X_1^2 + 1 - R^2x_1^2}{2X_1^2}dX_1$$

$$+ \frac{Vx_3 \cos(\theta_e - x_2)}{X_1}dx_2 - \frac{V \sin(\theta_e - x_2)}{X_1}dx_3$$

$$+ \frac{Vx_3 \sin(\theta_e - x_2)}{X_1^2}dX_1$$

$$+ \frac{V\left\{ 2R^2x_1x_4dx_1 + (X_1^2 + R^2x_1^2 - 1)dx_4 + 2x_4X_1dX_1 \right\}}{2Rx_1X_1}$$

$$- \frac{Vx_4(X_1^2 + R^2x_1^2 - 1)(X_1dx_1 + x_1dX_1)}{2Rx_1^2X_1^2}$$

Therefore,

$$B_{41} = \frac{R(-Rx_1 + Vx_4)}{X_1} - \frac{Vx_4(X_1^2 + R^2 x_1^2 - 1)}{2Rx_1^2 X_1}$$

$$+ \left\{ \frac{X_1^2 + R^2 x_1^2 - 1}{2X_1^2} + \frac{Vx_3 B_{12}}{Rx_1 X_1} + \frac{Vx_4(X_1^2 - R^2 x_1^2 + 1)}{2Rx_1 X_1^2} \right\} B_{11} \qquad \text{(B-28)}$$

$$B_{42} = \frac{Vx_3(X_1 B_{11} - R^2 x_1)}{RX_1}$$

$$+ \left\{ \frac{X_1^2 + R^2 x_1^2 - 1}{2X_1^2} + \frac{Vx_3 B_{12}}{Rx_1 X_1} + \frac{Vx_4(X_1^2 - R^2 x_1^2 + 1)}{2Rx_1 X_1^2} \right\} B_{12} \qquad \text{(B-29)}$$

$$B_{43} = - \frac{VB_{12}}{Rx_1} = -B_{34} \qquad \text{(B-30)}$$

$$B_{44} = \frac{V(X_1^2 + R^2 x_1^2 - 1)}{2Rx_1 X_1} = B_{33} \qquad \text{(B-31)}$$

Now, all elements of B have been obtained.