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On the Convergence of the Finite Element Solution of a Nonlinear Crack Type Problem in Finite Elasticity*

by

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SUMMARY

In this paper we consider the crack type problem in a hyperelastic body. The material is assumed to be isotropic, incompressible, and to have a special nonlinear constitutive equation under large deformation. This body is subjected to shear deformation under the action of body force. The problem leads to the nonlinear boundary value problem. The analytical solution of this problem exhibits the singularity at the crack tip. We investigate the convergence rate of the Galerkin approximation solution by introducing the weighted Sobolev space in functional analysis technique. The results show the convergence rate $\| u - U_h \|_{W_2^1} \sim h^{\frac{1}{4}}$ for the singularity grad $u \sim r^{-\frac{1}{4}}$

要　　約

高伸長弾性体中のクラック型の問題を考える。材料は等方性，非圧縮性で，大変形下で非線形構成関係を有する。この物体が体積力のもとでせん断変形をうける時，非線形境界値問題が導びかれる。この問題はクラック状の先端部で，特異性のある解析解をもつ。この問題のガラーキン近似有限要素解の収束性につき，函数解析手法における重みつきソボレフ空間を導入して調べた結果，grad $u \sim r^{-\frac{1}{4}}$ の特異性に対し，収束率 $\| u - U_h \|_{W_2^1} \sim h^{\frac{1}{4}}$ を得た。

§0. Introduction

Fracture under large strain becomes important engineering problem. The investigations are required in many industrial fields, such as fracture of rubber based materials in aerospace field, creep fracture in atomic or chemical plant field.

On the other hand, owing to the rapid development of large scale digital computer, the finite element method becomes very effective analytical tool for many continuum mechanics boundary value problems, such as solid and fluid mechanics, heat conduction, and electro-magnetic field problems.[1][2] Naturally this method has been applied to the fracture mechanics.

The origin of fracture mechanics comes from Griffith's investigation of energy release rate in brittle materials. Within this couple of decades, application of fracture mechanics to engineering problems becomes very popular.[7] In the viewpoint of strength of materials, failure of the structural member is described by the stress or strain in the material. In fracture mechanics, on the other hand, the failure of the member is associated with the special parameters determined by the stress or strain distribution field near the crack, which is assumed to be involved in the member. Thus, the concept of the fracture mechanics is based on the modern reliability engineering idea.

In fracture mechanics, estimation of those

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fracture mechanics parameters under the specified load condition is important. Usually the gradient of the unknown of boundary value problem is necessary in order to calculate such parameters. But it is well known that there exists singularity at the crack tip, and finite element solution near crack tip is disturbed by this singularity. This fact is deeply investigated from experiment and theoretical analysis.[8] Moreover, many engineers succeeded in recovering the accuracy of finite element solution near crack tip.[9] Most of these investigations have been forcused to the linear problem. What have been done in the nonlinear crack problem?

In contrast to the linear problem, few knowledges were accumulated in nonlinear crack problem, especially under large strain condition. Theoretical study of finite element method under large strain crack problem, such as error estimate, is under holizon.*

After Rice-Rosen[10] and Hutchinson[11] investigated the singularity analysis for elasto-plastic materials in infinitesimal condition, Wong-Shield,[15] Knowles−Sternberg[12][13] and Knowles[14] studied about the crack singularity of rubber-like materials under large strain. On the other hand, the application of monotone operator theory[19]~[23] to the nonlinear finite elasticity problem[2]~[6] is undertaken, for instance, by Oden and Wellford.[16]

Now the background for the study of numerical analysis for our problem has been arranged. In this report, we'll first study the crack singularity of rubber-like materials under a special loading condition. After that we'll investigate the convergence rate of finite element Galerkin approximate solution of the problem, applying monotone operator theory with the aid of weighted Sobolev space[18][26] and finite element interpolation theory.[24] These investigations will be very helpful to the through understanding of finite element method for crack-type problem.

*Recently some basic numerical investigation is executed by Babuska.[17]

§1. Formulation of anti-plane crack problem in finite elasticity theory

Let us consider the isotropic, homogeneous, incompressible elastic body which occupies region Ω_r in the unstressed state. After deformation this body moves to the position. [Fig. 1]

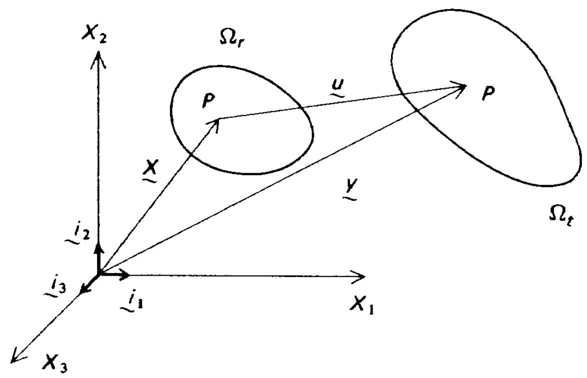

Fig. 1 Deformation of the body

$$\underset{\sim}{y} \equiv \underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(x) \quad \underset{\sim}{x} \in \Omega_r \tag{1}$$

where $\underset{\sim}{x}$ is the position vector in Ω_r, $\underset{\sim}{y}$ the position vector in the deformation image Ω_t of Ω_r, and $\underset{\sim}{u}$ is the displacement vector. The deformation gradient tensor at $\underset{\sim}{x}$ is

$$\underset{\sim}{F}(\underset{\sim}{x}) = \nabla \underset{\sim}{y}(\underset{\sim}{x}) \tag{2}$$

the geometrical response of the deformation of the body is described by this deformation tensor

$$d\underset{\sim}{y} = \underset{\sim}{F}(\underset{\sim}{x}) \; d\underset{\sim}{x} \tag{3}$$

For the physically admissible deformation, determinant of the deformation gradient tensor

$$J \equiv \det \underset{\sim}{F} \tag{4}$$

should be positive. In addition, incompressibility requires that

$$J = 1 \qquad \forall \; x \in \Omega_r \tag{5}$$

Let

$$\underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T \tag{6}$$

be the left Cauchy−Green tensor[18] for the deformation. Then the three elementary principal invariants of $\underset{\sim}{G}$ are

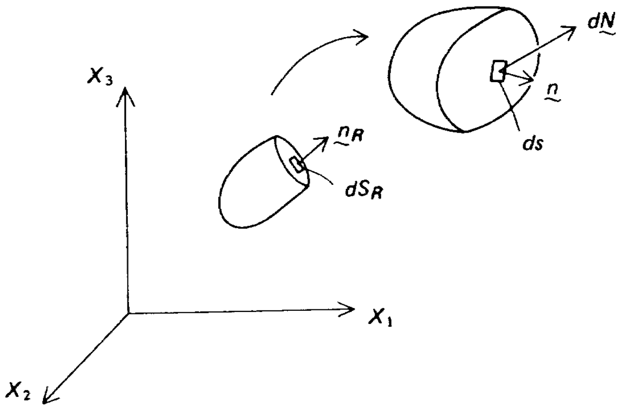

Fig. 2 Surface and force

$$\left.\begin{array}{l} I_1 = T_r G \quad I_2 = \frac{1}{2}[(T_r G)^2 - T_r(G)^2] \\ I_3 = \det G = J^2 = 1 \end{array}\right\} \qquad (7)$$

The strain energy density function can be written as $W = W(I_1, I_2)$ under the incompressible condition.

Let us define dN as the surface force vector on the infinitesimal surface element ds which has the direction n after deformation. The stress vector after deformation is defined as

$$t = \frac{dN}{ds} \qquad (8)$$

We introduce the stress vector which is defined per unit area before deformation as

$$t_r = \frac{dN}{ds_r} \qquad (9)$$

where suffix r denotes reference state position. Obviously

$$t\, ds = t_r\, ds_r \qquad (10)$$

Next relation is used afterwards

$$n\, ds = (F^T)^{-1}\, n_r\, ds_r \qquad (11)$$

Let us introduce Cauchy's stress tensor T as

$$t = T n \qquad (12)$$

Similarly Piola – Kirchhoff's stress tensor Σ as

$$t_r = \Sigma\, n_r \qquad (13)$$

then from Eq. (10) and Eq. (11)

$$\Sigma = T(F^T)^{-1} \qquad (14)$$

The body in static equilibrium state obeys Euler's rule[4]

$$\int_{\partial\Omega_t} t\, ds + \int_{\Omega_t} b\, \rho dv = 0 \qquad (15)$$

where b is the body force density vector per unit mass. Using Eqs. (10), (13) and the principle of mass conservation, $\rho dv = \rho_r dv_r$, and applying Green–Gausse formula to Eq. (15)

$$\int_{\Omega_r} (\mathrm{div}\, \Sigma + b\, \rho_r\, dv_r) = 0 \qquad (16)$$

Now we get the Cauchy's principle of motion

$$\mathrm{div}\, \Sigma = f \quad \text{in } \Omega_r \qquad (17)$$

where $f = -b\rho_r$ and assumed to be independent of X_3. Cauchy's stress tensor for incompressible elastic body is assumed to be ([3] §86)

$$T = 2\left[\frac{\partial W}{\partial I_1} G + \frac{\partial W}{\partial I_2}(I_1 1 - G)\, G\right] - P 1 \qquad (18)$$

where 1 is the unit tensor, scaler P is static pressure component, Eq. (14) and Eq. (16) leads to

$$\Sigma = 2\left[\frac{\partial W}{\partial I_1} F + \frac{\partial W}{\partial I_2}(I_1 1 - G) F\right] - p\,(F^{-1}) \qquad (19)$$

In case that energy density W is described by I_1 alone, Eq. (19) is simplified as

$$\Sigma = 2W'(I_1) F - p(F^T)^{-1} \qquad (20)$$

where W' denotes $\partial W / \partial I_1$.

Let us consider the cylindrical body which has a uniform cross section in X_3 direction. This body has a crack in the plane perpendicular to $X_1 - X_2$ plane. The crack is in $X_2 = 0$ plane and the crack tip is located at $X_1 = 0$. (Fig. 3) Now we will consider the antiplane shear deformation

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 + u(x_1, x_2) \qquad (21)$$

We will assume that there acts no body forces in X_1 and X_2 direction. In addition $u = 0$ for all boundary, including crack surface. In many engineering problem, the crack surface is assumed free from external forces. Here we consider the above boundary condition for the numerical analysis purpose. However, this condition has the same singularity property as the Knowles' free

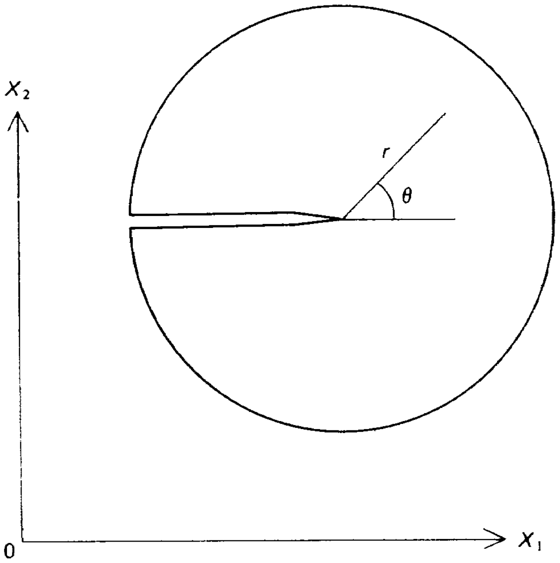

Fig. 3 Cylindrical body with antiplane crack

surface crack problem near crack tip region. This will be shown in the Appendix. This means that the two different conditions are equivalent with respect to the singular boundary value problem.

Now our boundary condition on the crack surface can be written in terms of the component τ_{ij} of $\underset{\sim}{T}$ and u as

$$\left.\begin{aligned} &\tau_{12} = \tau_{22} = 0 \\ &u = 0 \end{aligned}\right\} \qquad (22)$$

For the deformation given by Eq. (21)

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_{,1} & u_{,2} & 1 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 0 & u_{,1} \\ 0 & 1 & u_{,2} \\ u_{,1} & u_{,2} & 1 + |\nabla u|^2 \end{pmatrix}$$

where $|\nabla u|^2 = u_{,\alpha} u_{,\alpha} = u_{,1} u_{,1} + u_{,2} u_{,2}$ and

$$I_1 = 3 + |\nabla u|^2 \qquad (24)$$

where $_{,\alpha}$ indicates $\partial / \partial x_\alpha$

Next we'll show that, when the deformation is of the type described by Eq. (21), equiliblium equation (17) is represented by single equation in x_3 direction, which include only but not p with the aid of boundary constitions (23). Let the component of $\underset{\sim}{\Sigma}$ be σ_{ij}. Then from Eqs. (18), (20) and (23)

$$\left.\begin{aligned} &\sigma_{3\alpha} = 2W'(I_1) u_{,\alpha} = \tau_{3\alpha} \quad \sigma_{\alpha 3} = p u_{,\alpha} \\ &\sigma_{33} = 2W'(I_1) - p \\ &\sigma_{\alpha\beta} = [2W'(I_1) - p] \, \delta_{\alpha\beta} = \tau_{\alpha\beta} \\ &\alpha = 1, 2 \end{aligned}\right\} \qquad (25)$$

from Eq. (25) boundary condition is rewritten in terms of $\sigma_{\alpha\beta}$ as

$$\sigma_{21} = \sigma_{22} = 0 \qquad u = 0 \qquad (26)$$

The condition previously endowed to $\underset{\sim}{f}$ is written as

$$\underset{\sim}{f} = \underset{\sim}{f}(0, 0, f(x_1, x_2)) \qquad (27)$$

Then the equations of motion (17) can be rewritten as

$$\left.\begin{aligned} &[2W'(I_1) - p]_{,\alpha} + p_{,3} u_{,\alpha} = 0 \\ &[2W'(I_1) u_{,\beta}]_{,\beta} - f(x_1, x_2) = p_{,3} \end{aligned}\right\} \qquad (28)$$

In the second equation of (28), left side is independent of x_3, so p should be linear function of x_3.
Moreover $p_{,\alpha}$ ($\alpha = 1,2$) is independent of x_3, because of the first of Eq. (28).
These imply that

$$p(x_1, x_2, x_3) = d_0 x_3 + p_1(x_1, x_2) \qquad (29)$$

Then from (28)

$$p_1 = 2W'(I_1) + d_0 u + d_1 \qquad (30)$$

where d_0 and d_1 is integral constant. Substituting Eqs. (29) and (30) into the forth of Eq. (25), we conclude that d_0 and d_1 should be zero in order that σ_{22} equals zero for any x_3 at the crack surface. At the same time, the first of Eq. (28) is automatically satisfied by $d_0 = d_1 = 0$. After all, the governing equation for our problem is

$$[2W'(I_1) u_{,\beta}]_{,\beta} - f = 0 \qquad (31)$$

or

$$\operatorname{div} \beta \equiv \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} = f \quad \text{in } \Omega \qquad (32.1)$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega \qquad (32.2)$$

In the following we will consider the energy density function

$$W(I_1) = \frac{\nu}{2b}\left[\left\{1+\frac{b}{2}(I_1-3)\right\}^2 - 1\right],$$
$$b,\nu > 0 \qquad (33)$$

where b, ν denote material constants.
Then the required stress components for Eq. (32) are

$$\sigma_{32} = \nu\left[1+\frac{b}{2}|\nabla u|^2\right]u_{,\alpha} \quad \alpha = 1,2 \qquad (34)$$

Knowles solved the free surface crack problem for energy density (33) and give the closed form singuler solution near the crack tip. We'll show that the same order singular solution

$$u \sim r^{3/4}\, V(\theta) \quad \text{as } r\to 0 \qquad (35)$$

where $r = \sqrt{x_1^2+x_2^2}$ exists under the boundary condition (32.2).

Appendix I

Let us consider the equation

$$\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} = f \qquad (1)$$

where

$$\sigma_{3\alpha} = \nu\left[1+\frac{b}{2}|\nabla u|^2\right]u_{,\alpha} \quad \alpha = 1,2 \qquad (2)$$

with the boundary condition on the crack surface

$$u = 0 \qquad (3)$$

substituting Eq. (2) into Eq. (1) and we'll get nonlinear equation

$$\left[1+\frac{3}{2}bu_{,1}^2 + \frac{1}{2}bu_{,2}^2\right]u_{,11} + 2bu_{,1}u_{,2}u_{,12} + \left[1+\frac{b}{2}u_{,1}^2+\frac{3}{2}bu_{,2}^2\right]u_{,22} = f \qquad (4)$$

Assume that the asymptotic solution of Eq. (4) near crack tip as

$$u = r^m\, V(\theta) \quad \text{as } r\to 0 \quad -\pi<\theta<\pi \qquad (5)$$

We are interested in whether there is a solution of the form (5) in the range $0 \le m < 1$. (If $m \ge 1$, the singularity caused by the crack tip won't influence severely to the accuracy of gradient u in f.e.m. solution. If $m < 0$, then the solution assumes that $u \to \infty$ near crack tip, which contradicts to our physical instinct.)
Substituting Eq. (5) into Eq. (4) and comparing the lowest order term w.r.t. (with respect to) r, we will get the following asymptotic equation.

$$\dot{\overline{(P\dot V)}} + m(3m-2)PV = 0 \qquad (6)$$

where

$$P = \dot V^2 + m^2V^2 \qquad (\,\cdot\,) = \frac{\partial}{\partial\theta}$$

Boundary condition (3) is now written as

$$V(\pm\pi) = 0 \qquad (7)$$

Let us introduce the following change of variable

$$\left.\begin{aligned} mV &= \xi(\theta)\sin\Psi(\theta) \\ \dot V &= \xi(\theta)\cos\Psi(\theta)\end{aligned}\right\} \qquad (8)$$

Then from compatibility of Eq. (8) yields

$$\dot\xi(\theta)\sin\Psi(\theta) + \dot\Psi\,\xi(\theta)\cos\Psi(\theta) - m\xi(\theta)\cos\Psi(\theta) = 0 \qquad (9)$$

Substituting Eq. (8) into Eq. (6), we will get

$$3\dot\xi\cos\Psi - \dot\Psi\xi\sin\Psi + (3m-2)\xi\sin\Psi = 0 \qquad (10)$$

Deleting $\dot\xi$ from Eqs. (9) and (10) and dividing the resulting equation by ξ

$$\dot\Psi[2+\cos\Psi] - [3m-1+\cos 2\Psi] = 0 \qquad (11)$$

We will show $\dot\Psi > 0$. Multiplying V to Eq. (6),

$$P\ddot V V + \dot P\dot V V + m(3m-2)PV = 0 \qquad (12)$$

Using that $\dot{\overline{P\dot V V}} = \dot P\dot V V + P\ddot V V + P\dot V^2$, Eq. (12) becomes

$$\dot{\overline{P\dot V V}} - P\dot V^2 + m(3m-2)PV^2 = 0 \qquad (13)$$

Integrating Eq. (13) with respect to θ from $-\pi$ to π,

$$P\dot{V}V\Big|_{-\pi}^{\pi} = 0 = m(3m-2)\int_{-\pi}^{\pi} PV^2 d\theta - \int_{-\pi}^{\pi} P\dot{V}^2 d\theta$$

That is

$$m(3m-2) = \frac{\int_{-\pi}^{\pi} P\dot{V}^2 d\theta}{\int_{-\pi}^{\pi} PV^2 d\theta} > 0 \qquad (14)$$

So $m > \frac{2}{3}$. On the other hand from Eq. (11)

$$\dot{\Psi} = \frac{d\Psi}{d\theta} = \frac{3m - 1 + \cos 2\Psi}{2 + \cos 2\Psi} \qquad (15)$$

Under the condition $m > \frac{2}{3}$, Eq. (15) implies that $\dot{\Psi} > 0$ in the interval $-\pi < \theta < \pi$. This means that function Ψ is monotone strictly increasing on its interval of definition. This fact makes an important role in the following discussion. From Eq. (15)

$$\frac{d\theta}{d\Psi} = 1 + \frac{3(1-m)}{3m - 1 + \cos 2\Psi} \qquad (16)$$

integrate Eq. (13) w.r.t. Ψ with the condition $\Psi(-\pi) = 0$, which is indicated by the boundary conditions (7) and (8). Then

$$\pi + \theta = \Psi + \frac{3(1-m)}{\sqrt{3m(3m-2)}} \times \left[\tan^{-1}\left(\frac{\sqrt{3m-2}}{\sqrt{3m}}\tan\Psi\right)\right]\Big|_0^{\Psi} \qquad (17)$$

From the boundary condition at $\theta = \pi$ and the fact $\dot{\Psi} > 0$, we will conclude that

$$\Psi(\pi) = i\pi \qquad (i: \text{positive integer})$$

In addition, as the second term in the right side of Eq. (16) is positive, θ must increase when Ψ increases. From these two condition the term in the parentheses of Eq. (14) should be $n\pi$ $(n \geq 1)$ at $\theta = \pi$ Now the compatibility condition for both side of Eq. (17) at $\theta = \pi$ requires $i = 1$, and the minimum value for m is given by

$$\frac{3(1-m)}{\sqrt{3m(3m-2)}} = 1 \qquad (18)$$

this gives

$$m = \frac{3}{4} \qquad (19)$$

Substituting Eq. (19) into Eq. (17)

$$\tan\theta = \tan\left[\Psi + \tan^{-1}\left(\frac{1}{3}\tan\Psi\right)\right] \qquad (20)$$

or

$$\tan\Psi = \frac{-2\cos\theta - \sqrt{4\cos^2\theta + 3\sin^2\theta}}{3\sin\theta} \qquad (21)$$

Care should be paid that only the positive sign can be permitted before the root of Eq. (21) in the interval

From Eqs. (8) and (21)

$$\frac{\dot{V}}{V} = m\frac{2\cos\theta - \sqrt{4\cos^2\theta + 3\sin^2\theta}}{\sin\theta} \qquad (22)$$

After integration

$$V(\theta) = C\left[\cos^2\frac{\theta}{2}\left(\frac{2\sqrt{s^2+s+1}+s+2}{2\sqrt{s^2+s+1}-s+1}\right)\right]^m \qquad (23)$$

where $s = \tan^2\frac{\theta}{2}$ and $m = \frac{3}{4}$

The term in the parenthes of Eq. (20) is well defined in the interval $[-\pi, \pi]$, positive bounded for all interval except both ends, and $V(\pm\pi) = 0$. The eigenvalue $m = \frac{3}{4}$ is the same as Knowles' solution for the free crack surface problem.

§2. The Interpolation Theorem for the function with singularity

In the previous section, the nonlinear boundary value problem

$$\text{div}\,\sigma = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

has the asymptotic solution $u \sim r^{3/4}\, V(\theta)$ near crack tip. This implies that grad u has the singularity of order $r^{-1/4}$. Our main subject is to investigate the convergence rate of the approximate solution of the boundary value problem, which is obtained by Galerkin finite element method. In this section, we will introduce some mathematical concept and Babuska's finite element interpolation theorem[18] for singular function. These are necessary for the error estimate of the next section.

We will consider the bounded set Ω in the two dimentional Euclidian space. The boundary of Ω is denoted by $\partial\Omega$ and $\bar{\Omega} = \Omega \cup \partial\Omega$. We will use multi-index notation $x = (x_1, x_2)$, $\|x\|^2 = x_1^2 + x_2^2$, $dx = dx_1 dx_2$, $i_1, i_2 > 0$ $|i| = i_1 + i_2$. The class of infinitely differentiable function on $\bar{\Omega}$ will be denoted by $C^\infty(\bar{\Omega})$, and the subspace of such functions with compact support in Ω will be denoted by $\overset{\circ}{C}{}^\infty(\Omega)$.
Lebesque space and its norm is defined as

$$L_2(\Omega) = \left\{ u : \|u\|^2_{L_2(\Omega)} = \int_\Omega |u|^2 d\Omega < \infty \right\} \tag{1}$$

Let us introduce the Sobolev spaces W_2^k with $k \geq 0$, an integer, with the norm

$$\|u\|^2_{W_2^k(\Omega)} = \sum_{S=0}^{k} |u|_{W_2^s(\Omega)} \tag{2.1}$$

where

$$|u|^2_{W_2^s(\Omega)} = \sum_{|i|=s} \int_\Omega [D^i u(x)]^2 dx \tag{2.2}$$

and

$$D^i = \frac{\partial^{|i|}}{\partial x_1^i \, \partial x_2^{i2}} \tag{2.3}$$

The space W_2^k is understood to be the completion of $C^\infty(\Omega)$ w.r.t. the norm (2.1). In the same way $\overset{\circ}{W}{}_2^k(\Omega) \subset W_2^k$ will be the completion of $C^\infty(\Omega)$ w.r.t. norm (2.1).

In order to deal with singularities which are induced by the crack tips, we will introduce the weighted Sobolev Space.

Let O_i^κ be the circles with the centers in the verticles $P_1, \ldots.. P_\nu$ of polygonal curve and radii $\kappa > 0$, i.e.

$$O_i^\kappa = \{ x : \|x - P_i\| < \kappa \}$$

in addition we shall assume that the circles $O_i^{2\kappa}$ are disjoint.

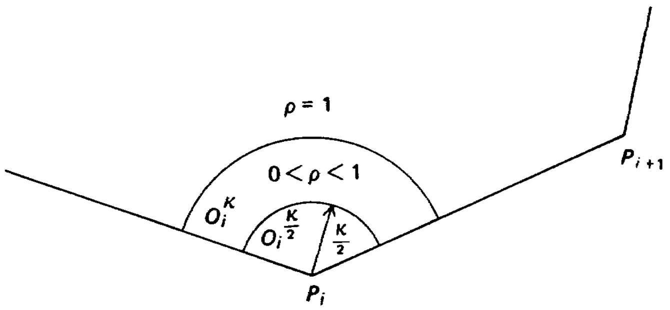

Fig. 4 Domain with corners

Let a vector $\beta \equiv (\beta_1, \beta_2, \ldots.., \beta_\nu)$, β_i real be given.
we shall write $\beta \geq \gamma$ if $\beta_i \geq \gamma_i$ for $i = 1, \ldots.. \nu$.

Let us introduce the function ρ^β with the following property

(1) $\rho = 1$ on $\Omega - \bigcup_{i=1}^{\nu} O_i^\kappa$

(2) $\rho(x) = (r_i(x)\kappa^{-1})$ on $\Omega \cap O_i^{\kappa/2}$
where $r_i(x) = \|x - P_i\|$

(3) $0 < c(\beta) < \rho^\beta(x) \leq 1$ on $\bar{\Omega} - \bigcup_{i=1}^{\nu} P_i$

(4) ρ has all derivatives on $\bar{\Omega} - \bigcup_{i=1}^{\nu} P_i$

With above preparations, we will define the weighted Sobolev space with the norm

$$\|u\|^2_{W^k_{2,\beta(\Omega)}} = \sum_{S=0}^{k} |u|^2_{W^k_{2,\beta(\Omega)}} \tag{3.1}$$

$$|u|^2_{W^k_{2,\beta}} = \sum_{i=S} \int_\Omega \left\{ D^i u(x)^2 \rho\beta dx \right\} \tag{3.2}$$

The space $W^k_{2,\beta}$ will be denoted to be the completion of $C^\infty(\Omega)$ in the norm (3.1). Similarly $\overset{\circ}{W}{}^k_{2,\beta}(\Omega)$ will be the completion of $\overset{\circ}{C}{}^\infty(\Omega)$ under norm (3.1). Sometimes we will write H_β instead of $\overset{\circ}{W}{}^1_{2,\beta}(\Omega)$.

Next we will triangulate the domain Ω by a finite number of arbitrary triangles such that any two triangles are either disjoint or have a common vertex and/or common side. Let us denote by $\Gamma_h(\Omega)$ the space of all continuous function on which are piecewise linear on each triangle τ and vanish on $\partial\Omega$, where h denotes the meshsize parameter, $0 < h \leq 1$. Obviously $\Gamma_h \subset \overset{\circ}{W}{}_2^1$. The following interpolation theorem in weighted Sobolev space is given by Babuska.[18]

Theorem 1 The function $u(x)$ is defined on the domain Ω such that

$$\| u \|_{W^s_{2,\kappa_s}} \leq Q \tag{4.1}$$

where

$$\kappa_s = 2(2 - \frac{\sigma_1}{2} - s, \ldots\ldots, 2 - \frac{\sigma_\nu}{2} - s) \tag{4.2}$$

then for $\gamma_i + \frac{\beta_i}{2} + 1 - \frac{\sigma_i}{2} \geq 0$, there exist a function $v \in \Gamma_h(\Omega)$ such that

$$\| \rho^r u - v \|_{H\beta} \leq CQh^u \tag{5.1}$$

where

$$u = \min_i (1, \gamma_i + \frac{\beta_i}{2} + 1 - \frac{\sigma_i}{2}) > 0 \tag{5.2}$$

$$\beta = (\beta_1, \ldots\ldots, \beta_\nu)$$

and C is a constant independent of h.

Proof We will need the meanings of $\gamma_i, \beta_i, \sigma_i$ of Eq. (5.2) in the next section. We will introduce the outline of the proof given by Babuska here. The proof is devided in a few steps. We will consider the single corner case only.

(1) Let $\lambda_p(r) \in C^\infty(0, \infty)$ be such that

$$\lambda_p = \begin{cases} 1 & \text{for } 0 \leq r < p \\ 0 & \text{for } r > 2P \end{cases}$$

(2) for h sufficiently small and $c > 0$, let us introduce the function $u_h = \lambda_{Ch} \rho^\gamma u$. Let us estimate the norm $\| u_h \|_{H\beta(\Omega)}$, where

$\beta = (\beta_1, 0 \ldots\ldots, 0)$. We have

$$\| u_h \|^2_{H\beta(\Omega)} =$$

$$\int_{D_1^{2Ch}} [(\frac{\partial u_h}{\partial x_1})^2 + (\frac{\partial u_h}{\partial x_2})^2 + {u_h}^2] \, \rho^\rho dx$$

$$\leq C_1 \Big\{ \int_{O_1^{2Ch}} [(\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2] \, \rho^{\beta+\gamma} dx$$

$$+ \int_{O_1^{2Ch}} u^2 \rho^{\bar{\beta}+2\gamma} dx \Big\} \tag{6}$$

where $\bar{\beta} = [\beta_1 - 2, 0, \ldots\ldots, 0]$

then

$$\| u_h \|^2_{H\beta} \leq C_1 h^u \Big\{ \| u \|^2_{W'_{2,-\kappa},(\Omega)}$$

$$+ \| u \|^2_{W^0_{2,-\kappa_0}(\Omega)} \Big\} \tag{7.1}$$

where

$$\left. \begin{aligned} \kappa_1 &= (2 - \sigma_1, 0, \ldots\ldots, 0) \\ \kappa_2 &= (4 - \sigma_1, 0, \ldots\ldots, 0) \end{aligned} \right\} \tag{7.2}$$

and

$$u = \frac{\beta_1}{2} + \gamma_1 + 1 - \frac{\sigma_1}{2} \tag{8}$$

(3) Taking C large enough and denoting $\bar{u}_h = u(1 - \lambda_{Ch})\rho\gamma$, we may construct a piecewise linear function $\nu \in \Gamma_h$ which coinsides with $\bar{u}_h$ in the vertices of τ's. Obviously $\bar{u}_h - \nu = 0$ in a neighbourhood of the vertex P_i. So we need not worry about the neighbourhood of crack tip in estimating $\| \bar{u}_h - \nu \|_{H\beta(\Omega)}$ and we will get the estimate

$$\| \bar{u}_h - \nu \|_{H\beta} \leq C_2 Q h^\mu \tag{9}$$

where

$$\mu = \min [1, \gamma_1 + \frac{\beta_1}{2} + 1 - \frac{\sigma_1}{2}] \tag{10}$$

(4) Adding estimates (7) and (9) with the aid of triangular inequality, we will get estimate (5), i.e.

$$\begin{aligned} &\| \rho^\gamma u - \nu \| \\ &= \| \lambda_{Ch} \rho^\gamma u + u\rho^\gamma - \lambda_{Ch} \rho^\gamma u - \nu \|_{H\beta} \\ &\leq \| u_h \|_{H\beta} + \| \bar{u}_h - \nu \|_{H\beta} \\ &= Ch^\mu \end{aligned}$$

where $\mu = \min (1, \frac{\beta_1}{2} + \gamma_1 + 1 - \frac{\sigma_1}{2})$

Now we will consider the value of μ which gives the maximum convergence rate under the condition $\gamma = 0$. When $\mu < 1$, the minimum possible value of σ_1 gives maximum μ. In our problem $u \sim r^{3/4}$, so taking into account of Eq. (7.2) minimum σ_1 which gives meaning to estimate (7) is

$$\sigma_1 = \frac{1}{2} + \epsilon \qquad \epsilon > 0 \tag{11}$$

§3. Error estimate for the Finite Element Galerkin Approximation.

Our problem is

$$A(u) = 0 \qquad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{2}$$

where $A(u) = -\operatorname{div} \sigma(u) + f$

In Galerkin form, instead of directly solving the system (1) and (2), we are given the variational boundary value problem of finding $u \in H$ such that

$$\langle A(u), v \rangle = 0 \qquad \forall\, v \in H \tag{3}$$

where $H = \overset{\circ}{W}{}^1_2$ and $\langle \cdot , \cdot \rangle$ denotes bilinear form. Eq. (3) is rewritten as

$$\langle \sigma(u), \operatorname{grad} v \rangle + \langle f, v \rangle = 0 \qquad \forall\, v \in H \tag{4}$$

Let us first show that the nonlinear operator A: $H \to H^*$ in Eq. (1) is hemicontinuous and strongly monotone. Then we will investigate the convergence rate of the Galerkin approximation solution with the aid of these quantities.

Definition 1 A mapping $A : H \to H^*$ is said to be hemicontinuous at a point $x_0 \in D(A)$ if for any vector x such that

$x_0 + tx \in D(A)$ and for $0 < t \leq \alpha$ $(\alpha > 0)$

$$\langle A(x_0 + tx) - A(x_0), y \rangle_H \to 0 \qquad \text{as } t \to 0 \quad \forall\, y \in H \tag{5}$$

Definition 2 A mapping $A : H \to H^*$ is said to be strongly monotone if

$$\langle A(x+h) - A(x), h \rangle \geq \| h \| \, \gamma(\| h \|) \tag{6}$$

for any x, $x + h \in D(A)$ where $\gamma(t)$ is a real valued nonnegative function defined for $t \geq 0$ and that $\gamma(t) \to \infty$ as $t \to \infty$ *and* $\gamma(t) = 0$ implies $t = 0$. Example of such a function is $\gamma(t) = ct, c > 0$.

If the mapping A is hemicontinuous and strongly monotone at every point of the domain, then it can be shown that the system (1), (2) or system (3) has a unique solution. (See Vainberg [19]. §18.6). Now we will show that our operator A satisfies both conditions.

[Hemicontinuity]

Let $v = u_0 + tu$ and

$$\langle A(u_0 + tu) - A(u_0), w \rangle$$

$$= \langle (1 + \frac{b}{2} |\nabla v|^2) v_{,\alpha} - (1 + \frac{b}{2} |\nabla u_0|^2) u_{0,\alpha}, w_{,\alpha} \rangle$$

$$= \langle v_{,\alpha} - u_{0,\alpha}, w_{,\alpha} \rangle + \frac{b}{2} \langle |\nabla v|^2 v_{,\alpha} - |\nabla u_0|^2 u_{0,\alpha}, w_{,\alpha} \rangle$$

linear term is

$$\langle v_{,\alpha} - u_{0,\alpha}, w_{,\alpha} \rangle = \int_\Omega (t u_{,\alpha} w_{,\alpha}) dx$$

$$= t \int u_{,\alpha} w_{,\alpha} dx \to 0 \qquad \forall\, w \in H$$

For nonlinear term, we will replace

$$\begin{cases} u_{,\alpha} = \dfrac{u_{,\alpha} - v_{,\alpha}}{2} + \dfrac{u_{,\alpha} + v_{,\alpha}}{2} \equiv Z_{,\alpha} + X_{,\alpha} \\ v_{,\alpha} = \dfrac{u_{,\alpha} - v_{,\alpha}}{2} + \dfrac{u_{,\alpha} + v_{,\alpha}}{2} \equiv -Z_{,\alpha} + X_{,\alpha} \end{cases}$$

Then

$$\langle |\nabla v|^2 V_{,\alpha} - |\nabla u_0|^2 u_{0,\alpha}, w_{,\alpha} \rangle$$

$$= \langle (v_{,1}^2 + v_{,2}^2)(-Z_{,1} + X_{,1}) - (u_{0,1}^2 + u_{0,2}^2)(Z_{,1} + X_{,1}), w_{,1} \rangle$$

$$+ \langle (v_{,1}^2 + v_{,2}^2)(-Z_{,2} + X_{,2}) - (u_{0,1}^2 + u_{0,2}^2)(Z_{,2} + X_{,2}), w_{,2} \rangle$$

$$= \langle -(v_{,1}^2 + v_{,2}^2 + u_{0,1}^2 + u_{0,2}^2) X_{,1}, w_{,1} \rangle$$

$$-4 < Z_{,1}X_{,1}^2 + Z_{,2}X_{,1}X_{,2}\,,\ w_{,1} >$$
$$+ < -(V_{,1}^2 + V_{,2}^2 + u_{0,1}^2 + u_{0,2}^2)Z_{,2}\,,\ w_{,2} >$$
$$-4 < Z_{,2}X_{,2}^2 + Z_{,1}X_{,1}X_{,2}\,,\ w_{,2} >$$

Now in our problem $u_{,\alpha} \sim r^{-1/4}$. so using.

$$\int dx_1 dx_2 = \int r dr d\theta$$

even in the nera crack region

$$\lim_{t\to 0} (Z_{,\alpha} r^{1/4}) = \lim_{t\to 0} (-\frac{t}{2} u_{,\alpha} r^{1/4}) = 0$$

Remaining term is, after multiplied by $r^{3/4}$, finite. So

$$\lim_{t\to 0} < A(u_0 + tu) - A(u_0),\ w > = 0$$

$$\forall\ w \in H \qquad (7)$$

This means that A is hemicontinuous.

(Monotonicity)

$$< A(u) - A(v);\ u - v >$$
$$= < \sigma(u) - \sigma(v),\ \text{grad}(u - v) >$$
$$= \int_\Omega \Big\{ (1 + \frac{b}{2} |\nabla u|^2) u_{,\alpha} u_{,\alpha}$$
$$- [2 + \frac{b}{2}(|\nabla u|^2 + |\nabla u|^2)]\, u_{,\alpha} v_{,\alpha}$$
$$+ (1 + \frac{b}{2} |\nabla v|^2\, v_{,\alpha} v_{,\alpha}) \Big\} dx$$

linear term $\doteq \int_\Omega (u_{,\alpha} - v_{,\alpha})^2 dx = |u - v|^2_{W_2^1(\Omega)}$

nonlinear term

$$= \frac{b}{2} \Big\{ (u_{,1}^2 + u_{,2}^2)^2 - (u_{,1}^2 + u_{,2}^2 + v_{,1}^2 + v_{,2}^2)$$
$$(u_{,1} v_{,1} + u_{,2} v_{,2}) + (v_{,1}^2 + v_{,2}^2)^2$$
$$= \frac{b}{2} [\big\{ u_{,1}^2 + u_{,2}^2) - (v_{,1}^2 + v_{,2}^2) \big\}^2$$
$$+ 2(u_{,1}^2 + u_{,2}^2)(v_{,1}^2 + v_{,2}^2)$$
$$- (u_{,1}^2 + u_{,2}^2 + v_{,1}^2 + v_{,2}^2)(u_{,1} v_{,1} + u_{,2} v_{,2})$$
$$= \frac{b}{4} [\big\{ (u_{,1}^2 + u_{,2}^2) - (v_{,1}^2 + v_{,2}^2) \big\}^2$$
$$+ (u_{,1}^2 + u_{,2}^2 + v_{,1}^2 + v_{,2}^2)$$
$$\big\{ (u_{,1} - v_{,1})^2 + (u_{,2} - v_{,2})^2 \big\}]$$

So nonlinear term is positive over the whole domain for $b > 0$. As a consequence

$$< \sigma(u) - \sigma(v), \text{grad}(u - v) > \geq k\, |u - v|^2_{W_2^1} \qquad (8)$$

In case of Dirichlet problem where the boundary condition $u = 0$ on $\partial\Omega$, the following Friedrich's inequality [24] holds.

$$\int_\Omega u^2 dx \leq C \int_\Omega (\text{grad}\, u)^2 dx \qquad (9)$$

So we can conclude that

$$< \sigma(u) - \sigma(v), \text{grad}(u - v) > \geq K \| u - v \|^2_{W_2^1} \qquad (10)$$

From the hemicontinuity and strong monotonicity of our operator A, we have shown that the unique solution exists for our problem. In addition, we can show that the solution u^* is bounded in W_2^1 sense as following.

From (10), for $\| u \|_{W_2^1} > r = K \| f \|_{W_2^1}$

$$< \sigma(u), \text{grad}\, u > + < f, u >$$
$$\geq K \| u \|^2_{W_2^1} + < f, u >$$
$$\geq K \| u \|_{W_2^1} - \| f \|_{W_2^1} \| u \|_{W_2^1} > 0$$

but if u^* is the solution of Eq. (4)

$$< \sigma(u^*), \text{grad}\, v > + < f, v > = 0$$

Compairing the above two conditions, we conclude that

$$\| u^* \|_{W_2^1} \leq r = \frac{1}{K} \| f \|_{W_2^1} \qquad (11)$$

This implies that the magnitude of the solution u depends on the data f.

From above discussion, if we take $\| \bar{u} \|_{W_2^1} = \frac{2}{K} \| f \|_{W_2^1}$ obviously

$$< A(\bar{u}), \bar{u} > \ > 0 \qquad (12)$$

Taking into account that the operator A is hemicontinuous and strongly monotone, Inequality (12) implies that our Galerkin system (= finite element discrete equation) has also a unique solution. [see Vainberg [19] Lemma 23.1, Remark 23.1]

Next, we will devide our region Ω into triangular finite elements denoted by mesh parameter h. Let us consider the continuous function which is linear in each triangular region. Such a family of function Γ_h belongs to the finite dimensional subspace of the space H.

We will seek $u_h \in \Gamma_h$ which satisfies next equation.

$$< \sigma(u_h), \text{grad}(v_h) > + < f, v_h > = 0 \quad (12)$$

This equation is a finite element Galerkin equation. Our main problem is to estimate the error between $u = u^*$ of the solution of Eq. (4) and $u_h = u_h^*$ of the solution of Eq. 1 (12) under some suitable norm.

Let us introduce arbitrary $w_h \in \Gamma_h$, then $w_h - u_h \in \Gamma_h$ and if u, u_h is the solution of Eqs. (4) and (12), respectively we can derive the following

$$< \sigma(u) - \sigma(u_h), \text{grad}(w_h - u_h) > = 0 \qquad \forall w_h \in \Gamma_h \quad (13)$$

From Eq. (13) and the monotonicity condition (10),

$$K \| u - u_h \|^2_{W_2^1}$$
$$\leq < \sigma(u) - \sigma(u_h), \ \text{grad}(u - u_h) >$$
$$= < \sigma(u) - \sigma(u_h), \ \text{grad}(u - w_h) >$$
$$\leq \frac{b}{2}\int_\Omega \Big\{ \Big| \Big(\frac{2}{b} + u_{,1}^2 + u_{,2}^2 + uh_{,1}^2 + uh_{,2}^2\Big) Z_{,1}\, w_{,1} \Big|$$
$$+ 4 \,|X_{,1}^2 Z_{,1}\, w_{,1}|$$
$$+ \Big|\Big(\frac{2}{b} + u_{,2}^2 + u_{,2}^2 + u_{h,1}^2 + u_{h,2}^2\Big) Z_{,2}\, w_{,2}\Big|$$
$$+ 4\,|X_{,2}^2 Z_{,2}\, w_{,2}|\Big\}\, dx$$

Where $X = \dfrac{u + u_h}{2}$, $Z = \dfrac{u - u_h}{2}$, $w = u - w_h$

Now we will apply Holder's inequality and get following results:

$$K \| u - u_h \|^2_{W_2^1}$$
$$\leq \frac{b}{2}\Big[\int_\Omega \Big\{\Big(\frac{2}{b} + u_{,1}^2 + u_{,2}^2 + u_{h,1}^2 + u_{h,2}^2\Big)^2 \times (Z_{,1}^2 + Z_{,2}^2)$$
$$+ 16\,(X_1^4 Z_{,1}^2 + X_2^4 Z_2^2)\Big\}\rho^\beta dx\Big]^{1/2}$$
$$\cdot \Big[\int_\Omega |\nabla w|^2 \rho^{-\beta} dx\Big]^{1/2}$$
$$\leq g_\beta(u, u_h)\, \| u - u_h \|_{W_2^1} \, \| u - w_h \|_{H_{-\beta}}$$
$$\forall w_h \in \Gamma_h \quad (14)$$

Where

$$g_\beta(u, u_h)$$
$$= \sup \Big\{ \Big| \Big(\frac{2}{b} + u_{,1}^2 + u_{,2}^2 + u_{h,1}^2 + u_{h,2}^2\Big)^2$$
$$+ 16\,(X_{,1}^4 + X_{,2}^4) \,|\rho^\beta \Big\}^{1/2} \quad (15)$$

$$\therefore \| u - u_h \|_{W_2^1} \leq C g_\beta(u, u_h)$$
$$\text{int}\, \| u - w_h \|_{H_{-\beta}} \quad (16)$$

Substituting the previous result, estimate (5) of §2,

$$\| u - u_h \|_{W_2^1} \leq C g_\beta(u, u_h) h^\mu \quad (17)$$

Where

$$\mu = \min\Big(1, \frac{-\beta}{2} + 1 - \frac{\sigma}{2}\Big) \quad (18)$$

Estimate (17) gives the convergence rate of the Galerkin approximation solution of our problem, under the condition that g_β in Eq. (15) is bounded and μ in Eq. (18) is positive. In Eq. (15) $\beta \geq 1$ is necessary for the boundedness of g_β. In §2 we know that $\sigma = \frac{1}{2} + \epsilon$, $\epsilon > 0$. So the maximum admissible μ in Eq. (18) is

$$\mu = \frac{1}{4} - \epsilon \ , \quad \epsilon > 0 \quad (19)$$

The term $-\frac{\sigma}{2}$ in Eq. (18) is the penalty in the interpolation error estimate, as the problem has the singularity grad $u \sim r^{-1/4}$. On the other hand, the term $-\frac{\sigma}{2}$ is the penalty in Galerkin approximation which is due to the nonlinearity and singularity coupled effect in the problem. Note that in linear crack problem the convergence rate of interpolation error is the same as Galerkin approximate error. But in nonlinear problem the convergence rate becomes worse in Galerkin approximation than in interpolation. For this point, careful investigation will be necessary by numerical experiment.

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