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Difference Approximations for Hyperbolic Conservation Laws and Consistency with Characteristics*

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ABSTRACT

In the theory of conservation laws, characteristics may be considered as paths along which infinitesimal perturbations propagate. The concept of characteristics can not be directly applied to difference approximations for conservation laws, but it is possible to discuss difference approximations from the viewpoint of the discrete propagation of infinitesimal perturbations.

In this report, we propose a new concept, 'the consistency with characteristics', for difference approximations for scalar conservation laws and give a sufficient condition under which the consistency with characteristics is satisfied. We also discuss difference approximations of conservation form for scalar conservation laws from the viewpoint of the consistency with characteristics. Furthermore, we consider a straightforward extension of the concept to difference approximations for systems of conservation laws.

Keywords: partial differential equation, conservation law, difference scheme, TVD, consistency with characteristics

概　　要

保存則に於いて重要な概念の一つである特性曲線は，無限小の変動が伝達していく経路として理解することができる。保存則の差分近似に於いては特性曲線に対応する概念はないが，無限小の変動の伝達の観点から差分近似を考察することは可能である。

本稿では，スカラー保存則の差分近似に対する新しい適合性条件として特性曲線適合性を提案し，差分近似がその適合性を満足する為の十分条件を与える。また，これまでに提案されている保存型差分近似をこの適合性の観点から解析する。さらに，この特性曲線適合性の概念を保存則の系の差分近似へ拡張して解析を行う。

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1. Introduction

In this paper, we are mainly concerned with difference approximations of the form

$$u_j^{n+1} = H(u_{j-1}^n, u_j^n, u_{j+1}^n)$$

for a scalar conservation law

$$u_t + f(u)_x = 0 \,, \quad -\infty < x < \infty, 0 \leqslant t < \infty \,.$$

Here the flux f is assumed to be a function of C^1-class.

So far many kinds of difference approximations for hyperbolic conservation laws have been proposed and used as important tools for calculating numerical solutions as well as for developing the mathematical theory. Each class of difference approximations has the theoretical foundation of its own. For example, difference approximations of conservation form[10] conserve the quantities corresponding to that of the original conservation laws, monotone difference approximations[1] are consistent with the so-called entropy condition, TVD difference approximations[5] are stable with respect to total variation, and high-resolution difference approximations[5,13,16,23] claim the merit that errors are of high-order and discontinuities are sharply captured. Many tests of numerical calculation for several standard problems have been executed and advantage or disadvantage of each class of difference approximations has been indicated (for example, refs. 11 and 22).

Now, it is widely recognized that the concept of characteristics plays an important role in the study of hyperbolic conservation laws. Nevertheless there seems to be no works on difference approximations from the view point of characteristics. The purpose of this paper is to discuss difference approximations from that point of view. For this purpose we propose a new concept 'consistency with characteristics' for difference approximations for hyperbolic conservation laws. The concept is based on the fact that characteristics are interpreted as paths along which infinitesimal perturbations propagate. We then give sufficient conditions on the form of difference approximations under which the consistency with characteristics is satisfied.

This paper is organized as follows. First, we introduce the concept of 'consistency with characteristics' (Section 3), and discuss a sufficient condition under which the consistency is satisfied (Section 4). Next, we apply the result to difference approximations of conservation form (Section 5 and 6). We further extend the concept of consistency with characteristics and the relating results to difference approximations for systems of conservation laws (Sections 7 and 8).

2. The difference approximation for a scalar conservation law

In this section, we state basic suppositions on difference approximations for a scalar conservation law.

We consider the following 3-point explicit difference approximation

$$u_j^{n+1} = H(u_{j-1}^n, u_j^n, u_{j+1}^n), \quad n \geqslant 0, \ \ j = 0, \pm 1, \pm 2, \cdots \tag{1}$$

for the scalar conservation law

$$u_t + f(u)_x = 0, \quad t > 0, \ -\infty < x < \infty \,, \tag{2}$$

where the flux f is a real valued function of C^1-class. Strictly speaking, the 3-variable function H in (1) also depends on the mesh widths Δx and Δt, but the notation as in (1) is used for simplicity. Each u_j^n is considered to be an approximate value of $u(x, t)$ at the grid point $(x, t) = (j\Delta x, n\Delta t)$. Usually the approximate values u_j^n's have an a priori bound depending only on the initial

data u_j^0's,
i.e.

$$|u_j^n| \leqslant M \text{ for some } M = 0\,, \tag{3}$$

and for the stability the ratio of mesh widths

$$\lambda = \frac{\Delta t}{\Delta x}$$

must satisfy the so-called CFL condition

$$\lambda \max_{|u| \leqslant M} |f'(u)| \leqslant 1. \tag{4}$$

In the following, we fix a positive number M and the ratio λ satisfying the CFL condition (4). Then the variable $\alpha = \lambda f'(u)$ varies on the interval $[-1, 1]$ whenever u varies on the set $[-M, M]$. In this sense we call α the normalized characteristic speed.

We state basic hypotheses on H.

(C1) The 3-variable function H is of C^1-Class.

Under the condition (C1), we write partial derivatives of H as follows.

$$\begin{cases} H_{-1} = \dfrac{\partial H(u_{-1}, u_0, u_1)}{\partial u_{-1}} \\[2ex] H_0 = \dfrac{\partial H(u_{-1}, u_0, u_1)}{\partial u_0} \\[2ex] H_1 = \dfrac{\partial H(u_{-1}, u_0, u_1)}{\partial u_1} \end{cases}$$

(C2) For any real number u,

$$u = H(u, u, u)\,. \tag{5}$$

(C3) For any smooth function $u(x, t)$,

$$\lim_{\Delta t \to 0} \frac{H(u(x - \Delta x, t), u(x, t), u(x + \Delta x, t)) - u(x, t)}{\Delta t} = -f(u)_x$$

(C4) $H_0(u, u, u)$ is a function of $\alpha = \lambda f'(u)$.

Conditions (C2) and (C3) mean that the difference approximation (1) is consistent with the conservation law (2). Note that, if f is convex or concave, condition (C4) is automatically satisfied. Under the above conditions, we obtain the following lemma.

Lemma 1. *For any real number u, the following equalities hold:*

$$H_{-1}(u, u, u) + H_0(u, u, u) + H_1(u, u, u) = 1, \tag{6}$$

$$H_{-1}(u, u, u) - H_1(u, u, u) = \lambda f'(u)\,. \tag{7}$$

proof.

By differentiating both sides of (5) with respect to u, we obtain (6). To show (7), let $u(x, t)$ be a smooth function. Then, it follows from (C3) that

$$\begin{aligned} &\Big\{ H_{-1}(u(x,t), u(x,t) u(x,t)) \\ &\quad - H_1(u(x,t), u(x,t), u(x,t)) \Big\} u_x(x,t) \\ &= \lambda f'(u(x,t)) u_x(x,t), \end{aligned}$$

which implies (7)

(q.e.d.)

Remark 1. *The above lemma and condition (4) mean that both $H_{-1}(u, u, u)$ and $H_1 1(u, u, u)$ and also functions of $\alpha = \lambda f'(u)$, and the following equalities hold:*

$$\begin{cases} H_{-1}(u, u, u) = \dfrac{1 + \lambda f'(u) - H_0(u, u, u)}{2} \\[2ex] H_1(u, u, u) = \dfrac{1 - \lambda f'(u) - H_0(u, u, u)}{2} \end{cases} \tag{8}$$

3. The consistency with characteristics

In this section, we introduce the concept of 'consistency with characteristics' for the difference approximation (1).

It is well known that a solution $u = u(x, t)$ to the conservations law (2) is constant on the curve

$x = x(t)$ defined by

$$\frac{dx}{dt} = f'(u), \tag{9}$$

where u is smooth. The curve $x = x(t)$ is called a characteristic curve. Since u is constant on each characteristic curve, each characteristic curve becomes a half straight line or a definite straight line which starts from a point on the line $t = 0$. This is one of the main properties of solution to scalar conservation laws.

Each characteristic curve can be interpreted as a path along which infinitesimal perturbations propagate.

Though the concept of characteristics can not be directly applied to the difference approximation (1), it is possible to consider a discrete version of propagation of infinitesimal perturbations. This consideration will be important, because it seems to lead to a criterion to estimate difference approximations.

Now, let u be a real number such that $|u| \leqslant M$ and suppose that $u_j^0 = u$ for any j. When an infinitesimal perturbation is given to u_0^0, the influence of the perturbation on u_j^n can be evaluated by the first derivative of u_j^n

$$C_{n,j} \equiv \frac{\partial u_j^n}{\partial u_0^0} \tag{10}$$

at u. By applying the chain rule, we obtain

$$C_{n,j} = \sum_{(a,b,c)\in S_{nj}} \frac{n!}{a!b!c!} \left\{H_{-1}\right\}^a \left\{H_0\right\}^b \left\{H_1\right\}^c , \tag{11}$$

where

$$\begin{cases} H_{-1} = H_{-1}(u,u,u) \\ H_0 = H_0(u,u,u) \\ H_1 = H_1(u,u,u) \end{cases}$$

and

$$S_{nj} = \left\{(a,b,c) \in Z^3 ; a+b+c = n, a-c = j, a,b,c \geqslant 0\right\}$$

Note that each S_{nj} is a finite set and $C_{n,j} = 0$ when $|j| > n$.

We now introduce the following property (C) and the definition of consistency with characteristics for the difference approximation (1).

Property (C):

Each $C_{n,j}$ takes the maximum value at $\lambda f'(u) = \frac{j}{n}$ when u varies on the set $[-M, M]$.

Definition 1. *The difference approximation (1) for the conservation law (2) is said to be consistent with characteristics if (1) satisfies the property (C).*

Remark 2. *Since* $\lambda = \frac{\Delta t}{\Delta x}$, *the value of* u *at which* $C_{n,j}$ *attains the maximum satisfies the relation*

$$\frac{j\,\Delta x}{n\,\Delta t} = f'(u).$$

This relation may be regarded as a discrete version of the definition of characteristics.

4. The main result

In this section, we give a sufficient condition on H under which the difference approximation (1) is consistent with characteristics.

At first, we note that each $C_{n,j} = \frac{\partial u_j^n}{\partial u_0^0}$ is a function of the normalized characteristic speed $\alpha = \lambda f'(u) \in [-1, 1]$, because $H_{-1}(u,u,u)$, $H_0(u,u,u)$, and $H_1(u,u,u)$ are functions of α. We also note that (6) and (7) can be rewritten as follows, respectively.

$$H_{-1}(\alpha) + H_0(\alpha) + H_1(\alpha) = 1 \tag{12}$$

$$H_{-1}(\alpha) - H_1(\alpha) = \alpha \tag{13}$$

For convenience, define a function $Q(\alpha)$ of $\alpha \in [-1, 1]$ by

$$Q(\alpha) = 1 - H_0(\alpha).$$

It is obvious that

$$\begin{cases} H_{-1}(\alpha) = \frac{1}{2}\left\{Q(\alpha) + \alpha\right\} \\ H_0(\alpha) = 1 - Q(\alpha) \\ H_1(\alpha) = \frac{1}{2}\left\{Q(\alpha) - \alpha\right\}. \end{cases} \tag{14}$$

Now, suppose that $Q(\alpha)$ is of C^1-class and hence that each $C_{n,j}(\alpha)$ is of C^1-class. Then the property (C) implies that

$$C'_{n,j}\left(\frac{j}{n}\right) = 0 \quad \text{for } n \geq 1 \text{ and } -n < j < n, \tag{15}$$

where $C'_{n,j}(\alpha)$ denotes the derivative of $C_{n,j}(\alpha)$ with respect to α. We see from (11) that

$$C'_{n,j}(\alpha) = \sum_{(a,b,c)\in S_{nj}} \frac{n!}{a!b!c!}\left\{H_{-1}(\alpha)\right\}^a \left\{H_0(\alpha)\right\}^b \left\{H_1(\alpha)\right\}^c \times\left[a\cdot\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} + b\cdot\frac{H'_0(\alpha)}{H_0(\alpha)} + c\cdot\frac{H'_1(\alpha)}{H_1(\alpha)}\right], \tag{16}$$

for $n \geq 1$, $-n \leq j \leq n$ and $\alpha \in (-1, 1)$. We also see from (12), (13) and the definition of S_{nj} that

$$a = \frac{1}{2}(n + j - b) = \frac{n}{2}\left(1 + \frac{j}{n}\right) - \frac{b}{2} = \frac{n}{2}\left\{2H_{-1}\left(\frac{j}{n}\right) + H_0\left(\frac{j}{n}\right)\right\} - \frac{b}{2}$$

$$c = \frac{1}{2}(n - j - b) = \frac{n}{2}\left(1 - \frac{j}{n}\right) - \frac{b}{2} = \frac{n}{2}\left\{2H_1\left(\frac{j}{n}\right) + H_0\left(\frac{j}{n}\right)\right\} - \frac{b}{2}$$

for $n \geq 1$ and $-n \leq j \leq n$. By these relations, we have

$$\begin{aligned}
&a\cdot\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} + b\cdot\frac{H'_0(\alpha)}{H_0(\alpha)} + \frac{H'_1(\alpha)}{H_1(\alpha)} \\
&= \frac{n}{2}\left\{2\left(H'_{-1}(\alpha) + H'_1(\alpha)\right) + H_0(\alpha)\left(\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} + \frac{H'_1(\alpha)}{H_1(\alpha)}\right)\right\} \\
&\quad - \frac{b}{2}\left\{\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} - 2\cdot\frac{H'_0(\alpha)}{H_0(\alpha)} + \frac{H'_1(\alpha)}{H_1(\alpha)}\right\} \\
&= \frac{n}{2}\left\{-2H'_0(\alpha) + H_0(\alpha)\left(\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} + \frac{H'_1(\alpha)}{H_1(\alpha)}\right)\right\} \\
&\quad - \frac{b}{2}\left\{\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} - 2\cdot\frac{H'_0(\alpha)}{H_0(\alpha)} + \frac{H'_1(\alpha)}{H_1(\alpha)}\right\} \\
&= \frac{1}{2}\left\{nH_0(\alpha) - b\right\} \times\left\{\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} - 2\frac{H'_0(\alpha)}{H_0(\alpha)} + \frac{H'_1(\alpha)}{H_1(\alpha)}\right\}
\end{aligned}$$

for $\alpha = \frac{j}{n}$, $n \geq 1$ and $-n \leq j \leq n$. Therefore (15) is equivalent to

$$\left\{\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} - 2\cdot\frac{H'_0(\alpha)}{H_0(\alpha)} + \frac{H'_1(\alpha)}{H_1(\alpha)}\right\} \times \sum_{(a,b,c)\in S_{nj}} \frac{n!}{a!b!c!}\left\{H_{-1}(\alpha)\right\}^a\left\{H_0(\alpha)\right\}^b \times\left\{H_1(\alpha)\right\}^c\left\{nH_0(\alpha) - b\right\} = 0 \tag{17}$$

for $\alpha = \frac{j}{n}$, $n \geq 1$ and $-n \leq j \leq n$. Employing the relation

$$\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} - 2\cdot\frac{H'_0(\alpha)}{H_0(\alpha)} + \frac{H'_1(\alpha)}{H_1(\alpha)}$$

$$= \left\{\log \frac{H_{-1}(\alpha)H_1(\alpha)}{\{H_0(\alpha)\}^2}\right\}' , \qquad (18)$$

we conclude that if

$$\frac{H_{-1}(\alpha)H_1(\alpha)}{\{H_0(\alpha)\}^2} \equiv \frac{k}{4} \qquad (19)$$

for some positive constant k, the relation (15) is satisfied. By (14), the relation (19) is equivalent to

$$Q(\alpha)^2 - \alpha^2 = k\Big\{1 - Q(\alpha)\Big\}^2 . \qquad (20)$$

It is elementary that for each $k \geqslant 0$, (20) has a solution

$$Q(\alpha) = \begin{cases} \dfrac{-k \pm \sqrt{k + (1-k)\alpha^2}}{1-k} & (\text{for } k \neq 1) \\[2ex] \dfrac{1+\alpha^2}{2} & (\text{for } k = 1) . \end{cases} \qquad (21)$$

Thus, we obtained candidates for $Q(\alpha)$ which makes difference approximations consistent with characteristics.

Before proceeding to our main result, we prepare the following lemma.

Lemma 2. *Let $k > 0$ and $Q(\alpha)$ be a solution to (20). Then we have the following:*

(a) $\dfrac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} = \dfrac{1-\alpha}{Q(\alpha)-\alpha^2}$ *for* $\alpha \in (-1, 1)$

(b) $\dfrac{H'_0(\alpha)}{H_0(\alpha)} = \dfrac{-\alpha}{Q(\alpha)-\alpha^2}$ *for* $\alpha \in (-1, 1)$

(c) $\dfrac{H'_1(\alpha)}{H_1(\alpha)} = \dfrac{-1-\alpha}{Q(\alpha)-\alpha^2}$ *for* $\alpha \in (-1, 1)$

Proof.

Since $Q(\alpha)$ is a solution to (20), we see that

$$Q'(\alpha) = \frac{\alpha}{Q(\alpha) + k\{1 - Q(\alpha)\}}$$

$$= \frac{\alpha\{1 - Q(\alpha)\}}{Q(\alpha) - \alpha^2} \qquad (22)$$

for $\alpha \in (-1, 1)$. In the second equality we used the fact that

$$\begin{aligned} Q(\alpha) - \alpha^2 &= Q(\alpha) - Q(\alpha)^2 + Q(\alpha)^2 - \alpha^2 \\ &= Q(\alpha)\{1 - Q(\alpha)\} + k\{1 - Q(\alpha)\}^2 \\ &= \{1 - Q(\alpha)\}\,[Q(\alpha) + k\{1 - Q(\alpha)\}] \end{aligned}$$

By (14) and (22), we obtain

$$\frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} = \frac{Q'(\alpha)+1}{Q(\alpha)+\alpha}$$

$$= \frac{\{Q(\alpha)+\alpha\}(1-\alpha)}{\{Q(\alpha)+\alpha\}\{Q(\alpha)-\alpha^2\}} = \frac{1-\alpha}{Q(\alpha)-\alpha^2} .$$

This proves (a). In a similar manner, we can check that (b) and (c) hold. (q.e.d.)

We are now in a position to prove the main result in this paper. In the statement we use a parameter $q \in [0, 1]$, instead of $k \geqslant 0$, defined by

$$q = \frac{\sqrt{k}}{1+\sqrt{k}} .$$

Theorem 1. *Suppose that the function H in (1) satisfies conditions (C1)–(C4) and let $Q(\alpha)$ be the function defined by (14). If for some $q \in [0,1]$*

$$Q(\alpha) = \begin{cases} \dfrac{-q^2 + (1-q)\sqrt{(1-2q)\alpha^2 + q^2}}{1-2q} , & q \neq \dfrac{1}{2} \\[2ex] \dfrac{1+\alpha^2}{2} , & q = \dfrac{1}{2} , \end{cases} \qquad (23)$$

then the difference approximation (1) for the scalar conservation law (2) is consistent with characteristics.

Proof.

Let $n \geqslant 1$ and $-n \leqslant j \leqslant n$ be fixed. Let $q \in [0, 1]$, and $Q(\alpha)$ be the function corresponding to q. Then it suffices to show that

$$C_{n,j}(\alpha) = C_{n,j}\left(\frac{j}{n}\right)\exp\left(\int_{\frac{j}{n}}^{\alpha} (j - ns)L(s)ds\right) \tag{24}$$

for $\alpha \in [-1, 1]$, where $L(\alpha)$ is a positive function defined by

$$L(\alpha) = \frac{1}{Q(\alpha) - \alpha^2} \quad \text{for } \alpha \in [-1, 1] . \tag{25}$$

In fact, (24) implies that

$$C_{n,j}(\alpha) \leqslant C_{n,j}\left(\frac{j}{n}\right) \quad \text{for all } \alpha \in [-1, 1] ,$$

because $C_{n,j}(\alpha) \geqslant 0$ and $\int_{\frac{j}{n}}^{\alpha} (j - ns)\, L(s)ds \leqslant 0$ for $\alpha \in [-1, 1]$. Here, the term $\exp\left(\int_{\frac{j}{n}}^{\alpha} (j - ns)\ L(s)ds\right)$ is understood to be 0, if $\int_{\frac{j}{n}}^{\alpha} (j - ns)\ L(s)ds = -\infty$.

Now if $0 < q < 1$, then $\dfrac{\{H_0(\alpha)\}^2}{H_{-1}(\alpha)H_1(\alpha)}$ is constant on $[-1, 1]$ and hence $C_{n,j}(\alpha)$ is written as

$$C_{n,j}(\alpha)$$

$$= \sum_{(a,b,c)\in S_{nj}} \frac{n!}{a!b!c!} \left\{H_{-1}(\alpha)\right\}^a \left\{H_0(\alpha)\right\}^b \left\{H_1(\alpha)\right\}^c$$

$$= \left\{H_{-1}(\alpha)\right\}^{\frac{1}{2}(n+j)} \left\{H_1(\alpha)\right\}^{\frac{1}{2}(n-j)}$$

$$\times \sum_{b\in S'_{nj}} \frac{n!}{b!\left(\frac{n+j-b}{2}\right)!\left(\frac{n-j-b}{2}\right)!}$$

$$\times \left[\frac{\{H_0(\alpha)\}^2}{H_{-1}(\alpha)H_1(\alpha)}\right]^{\frac{b}{2}}$$

$$= K \cdot \left\{H_{-1}(\alpha)\right\}^{\frac{n+j}{2}} \left\{H_1(\alpha)\right\}^{\frac{n-j}{2}},$$

where

$$S'_{nj} = \left\{b \in Z ; 0 \leqslant b \leqslant n - |j|,\ b \equiv n - j \pmod 2\right\}$$

and K is a positive constant independent of α. By differentiating $C_{n,j}(\alpha)$ and applying Lemma 2, we have

$$C'_{n,j}(\alpha) = \frac{1}{2} C_{n,j}(\alpha) \left\{(n+j) \cdot \frac{H'_{-1}(\alpha)}{H_{-1}(\alpha)} + (n-j) \cdot \frac{H'_1(\alpha)}{H_1(\alpha)}\right\}$$

$$= (j - n\alpha)L(\alpha)\, C_{n,j}(\alpha) \tag{26}$$

for $\alpha \in (-1, 1)$. Solving this differential equation, we obtain (25)

If $q = 0$, then $Q(\alpha) = |\alpha|$ and so $C_{n,j}(\alpha)$ is written as

$$C_{n,j}(\alpha) = \begin{cases} \dfrac{n!}{j!\,(n-j)!}\,\alpha^j (1-\alpha)^{n-j} & \text{for } j \geqslant 0 \text{ and } \alpha \in [0, 1] \\ \dfrac{n!}{(-j)!\,(n+j)!}\,(-\alpha)^{-j}(1+\alpha)^{n+j} & \text{for } j \leqslant 0 \text{ and } \alpha \in [-1, 0] \\ 0 & \text{otherwise.} \end{cases}$$

By an elementary calculation, we see that $C_{n,j}(\alpha)$ satisfies the differential equation (26) for $\alpha \in (-1, 1)$ except for $\alpha = 0$.

If $q = 1$, then $Q(\alpha) \equiv 1$ and so $C_{n,j}(\alpha)$ is written as

$$C_{n,j}(\alpha) = \begin{cases} \dfrac{n!}{(\frac{n+j}{2})!\,(\frac{n-j}{2})!} \left(\dfrac{1+\alpha}{2}\right)^{\frac{n+j}{2}} \left(\dfrac{1-\alpha}{2}\right)^{\frac{n-j}{2}} & \text{for } n-j \text{ even,} \\ 0 & \text{for } n-j \text{ odd.} \end{cases}$$

From this we easily see that $C_{n,j}(\alpha)$ satisfies the differential equation (26) for $\alpha \in (-1, 1)$. Thus the proof is completed. (q.e.d.)

Remark 3. *We give some remarks on the functions $Q(\alpha)$ in Theorem 1. See also Fig. 1.*

(i) *$Q(\alpha) = |\alpha|$ for $q = 0$ and*
$Q(\alpha) \equiv 1$ for $q = 1$.

(ii) *If $0 < q < \frac{1}{2}$, $Q(\alpha)$ is a part of a hyperbola.*

(iii) *If $q = \frac{1}{2}$, $Q(\alpha)$ is a part of a parabola.*

(iv) *If $\frac{1}{2} < q < 1$, $Q(\alpha)$ is a part of an ellipse.*

(v) *For every $q \in [0, 1]$,*
$Q(0) = q$, $Q(\pm 1) = 1$ and $|\alpha| \leqslant Q(\alpha) \leqslant 1$.

(vi) *For every $q \in [0, 1)$, $Q'(\pm 1) = \pm 1$.*

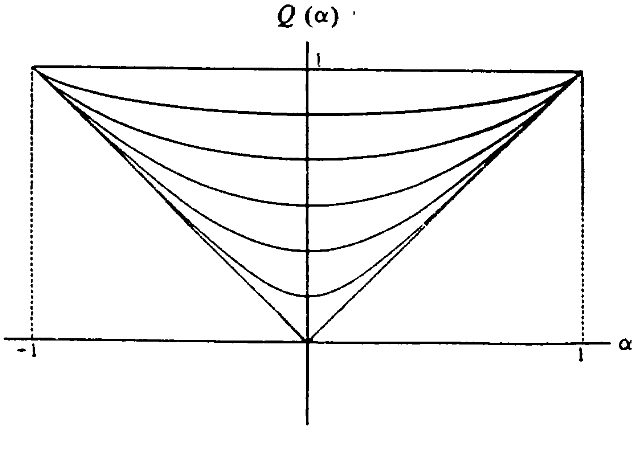


Fig. 1

Remark 4. *(21) provides another family consisting of functions*

$$Q(\alpha) = \frac{-q^2 - (1-q)\sqrt{(1-2q)\alpha^2 + q^2}}{1-2q}$$

$$\textit{for } q \notin [0, 1],$$

where the parameter q is defined by $q = \dfrac{\sqrt{k}}{\sqrt{k}-1}$ for $k \geqslant 0$ with $k \neq 1$. However, it is not hard to check that the consistency with characteristics is not satisfied for this family.

5. Numerical fluxes and consistency with characteristics

In this section, we apply the result obtained in the previous section to difference approximations of conservation form for the conservation law (2).

First we consider the relationship between numerical fluxes and consistency with characteristics and construct conservative difference approximations consistent with characteristics.

Let $\bar{f}$ be a 2-variable function of C^1-class such that

$$\bar{f}(u, u) = f(u) \tag{27}$$

and suppose that the difference approximation (1) is of conservation form, that is,

$$u_j^{n+1} = H(u_{j-1}^n, u_j^n, u_{j+1}^n) \tag{28}$$

$$= u_j^n - \lambda\left\{\bar{f}(u_j^n, u_{j+1}^n) - \bar{f}(u_{j-1}^n, u_j^n)\right\},$$

where $\lambda = \dfrac{\Delta t}{\Delta x}$. The function $\bar{f}$ is called a numerical flux. By assumption, it is obvious that conditions (C1), (C2) and (C3) on H are satisfied. Also it follows from (28) that

$$\begin{cases} H_{-1}(u, u, u) = \lambda \bar{f}_1(u, u) \\ H_0(u, u, u) = 1 - \lambda\left\{\bar{f}_1(u, u) - \bar{f}_2(u, u)\right\} \\ H_1(u, u, u) = -\lambda \bar{f}_2(u, u) \end{cases} \tag{29}$$

where

$$\begin{cases} \bar{f}_1(u,v) = \dfrac{\partial \bar{f}(u,v)}{\partial u} \\ \bar{f}_2(u,v) = \dfrac{\partial \bar{f}(u,v)}{\partial v} \end{cases}$$

So, condition (C4) on H is satisfied if and only if

$$\begin{cases} \bar{f}_1(u,u) = \dfrac{1}{2} f'(u) + \dfrac{1}{2\lambda} Q(\lambda f'(u)) \\ \bar{f}_2(u,u) = \dfrac{1}{2} f'(u) - \dfrac{1}{2\lambda} Q(\lambda f'(u)) \end{cases} \tag{30}$$

for some continuous function $Q(\alpha)$ of $\alpha \in [-1, 1]$.

Now, let u and v be real numbers. With the aid of (30), we obtain the following expansion of the numerical flux $\bar{f}(u, v)$;

$$\begin{aligned} \bar{f}(u,v) &= \bar{f}(u,u) + \int_u^v \bar{f}_2(u,s)ds \\ &= f(u) + \int_u^v \bar{f}_2(s,s)ds + o(|v-u|) \\ &= f(u) + \frac{1}{2}\int_u^v f'(s)ds - \frac{1}{2\lambda}\int_u^v Q(\lambda f'(s))ds \\ &\quad + o(|v-u|) \\ &= \frac{1}{2}\{f(u)+f(v)\} - \frac{1}{2\lambda}\int_u^v Q(\lambda f'(s))ds \\ &\quad + o(|v-u|). \end{aligned} \tag{31}$$

Similarly, we obtain another expansion

$$\bar{f}(u,v) = \frac{1}{2}\{f(u)+f(v)\} - \frac{1}{2\lambda} Q(\lambda f'(s))(v-u) + o(|v-u|), \tag{32}$$

where s is a number between u and v. Following the above expansion, we construct two kinds of numerical flux

$$\bar{f}(u,v) = \begin{cases} \dfrac{1}{2}\{f(u)+f(v)\} - \dfrac{1}{2\lambda} Q\left(\lambda \dfrac{f(v)-f(u)}{v-u}\right)(v-u) & \text{if } v \neq u \\ f(u) & \text{if } v = u, \end{cases} \tag{33}$$

and

$$\bar{f}(u,v) = \frac{1}{2}\{f(u)+f(v)\} - \frac{1}{2\lambda}\int_u^v Q(\lambda f'(s))ds. \tag{34}$$

It should be noted that (33) and (34) are numerical fluxes corresponding to flux-vector splitting and to flux-difference splitting, respectively. The above discussion gives the following theorem.

Theorem 2. *Let $Q(\alpha)$ be a function as in Theorem 1,*

i.e.

$$Q(\alpha) = \begin{cases} \dfrac{q^2 + (1-q)\sqrt{(1-2q)\alpha^2 + q^2}}{1-2q} & \text{if } q \in [0,1] \text{ and } q \neq \dfrac{1}{2}, \\ \dfrac{1+\alpha^2}{2} & \text{if } q = \dfrac{1}{2} \end{cases}$$

Then the difference approximation (28) with numerical flux $\bar{f}(u,v)$ defined by (33) or (34) is consistent with characteristics.

Remark 5.

(i) *By a result in ref. 5, it turns out that the difference approximation in Theorem 2 is TVD (Total Variation Diminishing), because*

$$|\alpha| \leqslant Q(\alpha) \leqslant 1 \textit{ for } \alpha \in [-1, 1]. \qquad (35)$$

Here a difference approximation is said to be TVD if

$$\sum_j \left| u_{j+1}^{n+1} - u_j^{n+1} \right| \leqslant \sum_j \left| u_{j+1}^n - u_j^n \right|$$

for every $n \geqslant 0$ (See ref. 5). Note that (35) is equivalent to the condition that propagation coefficients $H_{-1}(\alpha)$, $H_0(\alpha)$ and $H_1(\alpha)$ are non-negative.

(ii) *In particular, if $\bar{f}$ is given by (34), then the difference approximation is monotone and hence is consistent with the so-called entropy condition. See refs. 1 and 6.*

Corollary 1. *The Godunov difference approximation is consistent with characteristics.*

proof.

The Godunov difference approximation is written as

$$u_j^{n+1} = u_j^n - \lambda\left\{\bar{f}(u_j^n, u_{j+1}^n) - \bar{f}(u_{j-1}^n, u_j^n)\right\}, \qquad (36)$$

where the numerical flux $\bar{f}$ is defined by

$$\bar{f}(u,v) = \begin{cases} \min\limits_{u \leqslant w \leqslant v} f(w) & \text{if } u \leqslant v \\ \max\limits_{v \leqslant w \leqslant u} f(w) & \text{if } v \leqslant u. \end{cases} \qquad (37)$$

See refs. 4 and 15. By virtue of (37), the partial derivatives $f_1(u,u)$ and $f_2(u,u)$ are given by

$$\bar{f}_1(u,u) = \begin{cases} f'(u) & \text{if } f'(u) \geqslant 0 \\ 0 & \text{if } f'(u) \leqslant 0 \end{cases} \qquad (38)$$

$$\bar{f}_2(u,u) = \begin{cases} f'(u) & \text{if } f'(u) \leqslant 0 \\ 0 & \text{if } f'(u) \geqslant 0. \end{cases}$$

Thus we see that

$$\begin{cases} \bar{f}_1(u,u) = \max\left\{0, f'(u)\right\} = \frac{1}{2}\left\{f'(u) + |f'(u)|\right\} \\ \bar{f}_2(u,u) = \min\left\{0, f'(u)\right\} = \frac{1}{2}\left\{f'(u) - |f'(u)|\right\} \end{cases}$$

This means that $Q(\alpha) = |\alpha|$ and the proof is completed. (q.e.d.)

We end this section by considering the Glimm difference approximation[3] (See also refs. 7 and 21). As is well known, the Glimm difference approximation includes random choice process, is defined on a staggered mesh and is not of conservation form. By these facts, our argument can not be directly applied. However, it can be shown that the Glimm difference approximation is consistent with characteristics in the sense of expectation (or equivalently, in the sense of mean in probability).

Let u_L and u_R be real numbers, and let $w(\frac{x}{t}; u_L, u_R)$ denote the exact solution to the Riemann problem for (2):

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = \begin{cases} u_L & \text{for } x < 0 \\ u_R & \text{for } x > 0. \end{cases} \end{cases} \qquad (39)$$

The Glimm difference approximation is then interpreted as a two-step difference approximation:

$$u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = w\left(\frac{2}{\lambda} P_{j+\frac{1}{2}}^{n+\frac{1}{2}}; u_j^n, u_{j+1}^n\right),$$

$$u_j^{n+1} = w\left(\frac{2}{\lambda} P_j^{n+1}; u_{j-\frac{1}{2}}^{n+\frac{1}{2}}, u_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right), \qquad (40)$$

$$n \geqslant 0, \; j = 0, \pm 1, \cdots.$$

Here P_j^{n+1} and $P_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ denote independent random variables each of which is uniformly distributed in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

As each u_j^n depends on $n(2n+1)$ independent random variables, we denote by $\bar{u}_j^n$ the expectation with respect to the $n(2n+1)$ random variables for $n \geqslant 1$. We show that the derivative $\frac{\partial \bar{u}_j^n}{\partial u_0^0}$ at u is equal to $C_{n,j}(u)$ of the two step Lax-Friedrichs difference approximation defined by the function

$$H(u_{-1}, u_0, u_1)$$
$$= \frac{1}{4}(u_{-1} + 2u_0 + u_1) - \frac{\lambda}{4}\left\{f(u_1) - f(u_{-1})\right\}$$
$$- \frac{\lambda}{2}\left\{f\left(\frac{u_0 + u_1}{2} - \frac{\lambda}{2} f(u_1) + \frac{\lambda}{2} f(u_0)\right)\right.$$
$$\left. - f\left(\frac{u_{-1} + u_0}{2} - \frac{\lambda}{2} f(u_0) + \frac{\lambda}{2} f(u_{-1})\right)\right\}$$

To this end, let Δu be a sufficiently small number. Set $u_0^0 = u + \Delta u$ and $u_j^0 = u$ for $j \neq 0$. Then

$$\left| u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u \right| \leqslant \left| \Delta u \right|$$

and

$$\left| u_j^n - u \right| \leqslant \left| \Delta_u \right|$$

for every $n \geqslant 0$ and j. Therefore, we see from (40) that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u_j^{n+1} dP_j^{n+1}$$
$$= \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w\left(\frac{2x}{\Delta t} ; u_{j-\frac{1}{2}}^{n+\frac{1}{2}}, u_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) dx$$
$$= \frac{1}{2}\left(u_{j-\frac{1}{2}}^{n+\frac{1}{2}} + u_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right)$$
$$-\frac{\lambda}{2}\left\{f(u_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - f(u_{j-\frac{1}{2}}^{n+\frac{1}{2}})\right\}$$
$$= \frac{1}{2}\left(u_{j-\frac{1}{2}}^{n+\frac{1}{2}} + u_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right)$$
$$- \frac{\lambda}{2} f'(u)\left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) + o(|\Delta u|),$$

and similarly

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u_{j+\frac{1}{2}}^{n+\frac{1}{2}} dP_{j+\frac{1}{2}}^{n+\frac{1}{2}}$$
$$= \frac{1}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{\lambda}{2} f'(u) \ (u_{j+1}^n - u_j^n)$$
$$+ o(|\Delta u|).$$

Since $|\bar{u}_j^n - u| \leqslant |\Delta u|$, these relations imply that

$$\bar{u}_j^{n+1} = H(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n) + o(|\Delta u|), \tag{42}$$

where H is the function in (41). In view of (41) and (42), it is not hard to check that $\frac{\partial \bar{u}_j^n}{\partial u_0^0}$ is equal to $C_{n,j}(u)$ defined by (11), with the functions

$$H_{-1}(\alpha) = \frac{(1+\alpha)^2}{4} = \frac{1}{2}\left(\frac{1+\alpha^2}{2} + \alpha\right),$$

$$H_0(\alpha) = \frac{1-\alpha^2}{2} = 1 - \frac{1+\alpha^2}{2},$$

$$H_1(\alpha) = \frac{(1-\alpha)^2}{4} = \frac{1}{2}\left(\frac{1+\alpha^2}{2} - \alpha\right).$$

This corresponds to the case where

$$Q(\alpha) = \frac{1+\alpha^2}{2}.$$

Thus we may conclude that the Glimm difference approximation is consistent with characteristics in the sense of expectation.

6. Examples of difference approximations

In this section we give some examples of difference approximations and brief comments on them.

We consider the following five cases:

(case 1) $Q(\alpha) = |\alpha|$.

(case 2) $Q(\alpha) = \frac{1}{2}(1+\alpha^2)$.

(case 3) $Q(\alpha) \equiv 1$.

(case 4)

$$Q(\alpha) = \begin{cases} \dfrac{\epsilon}{2} + \dfrac{\alpha^2}{2\epsilon} & \text{for } |\alpha| \leqslant \epsilon \\ |\alpha| \text{ for } |\alpha| \geqslant \epsilon, & 0 < \epsilon < 1 . \end{cases} \tag{43}$$

(case 5) $Q(\alpha) = \alpha^2$.

The graphs of $C_{n,j}(\alpha)$ with $n = 10$ and $j = 0, 1, 2, \cdots, 10$ are displayed in Fig. 2. Each dotted line in Fig. 2 denotes the value of α which attains the maximum of $C_{n,j}(\alpha)$.

(case 1) The difference approximation defined by the flux (33) with $Q(\alpha) = |\alpha|$ is known as the

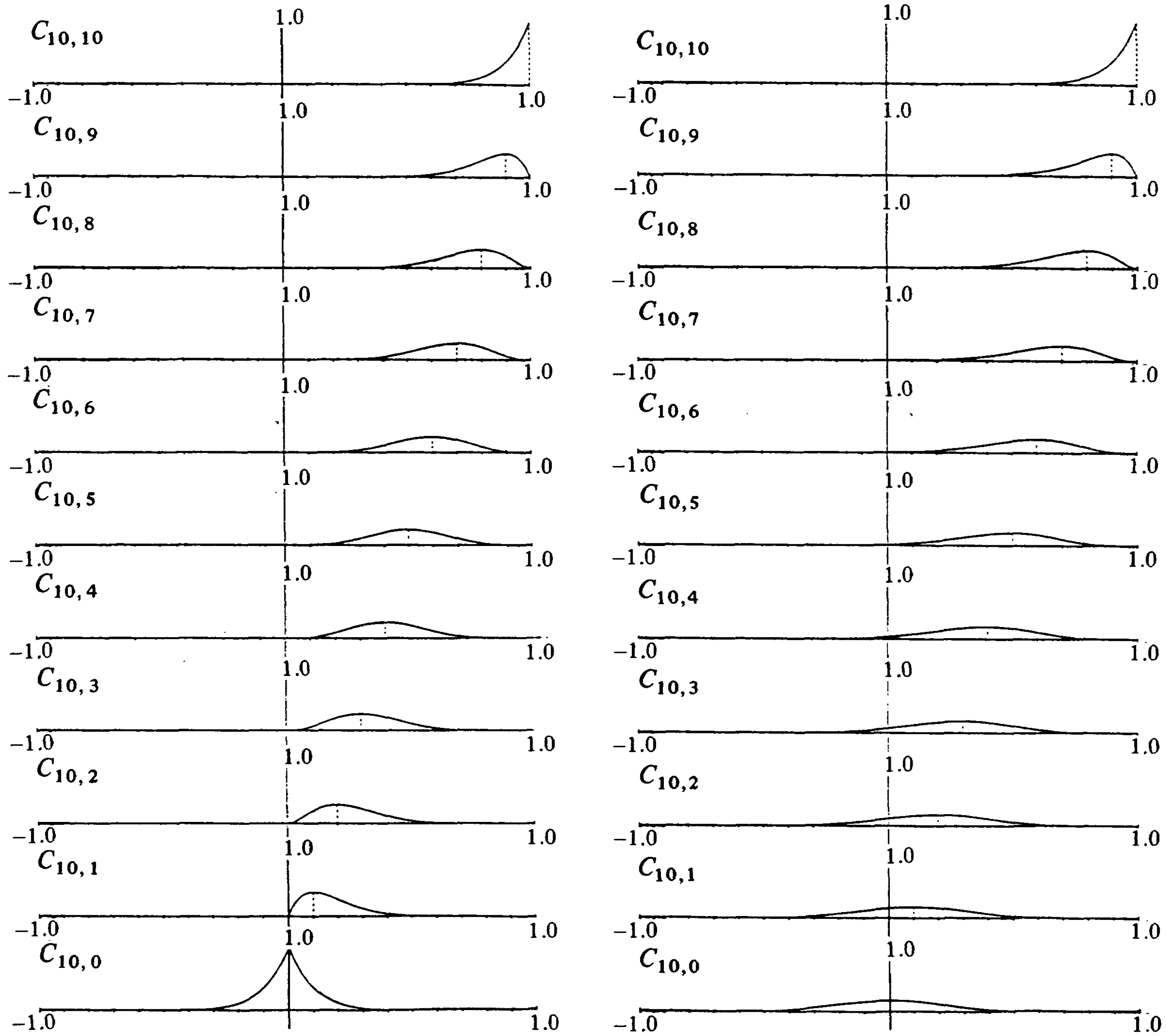


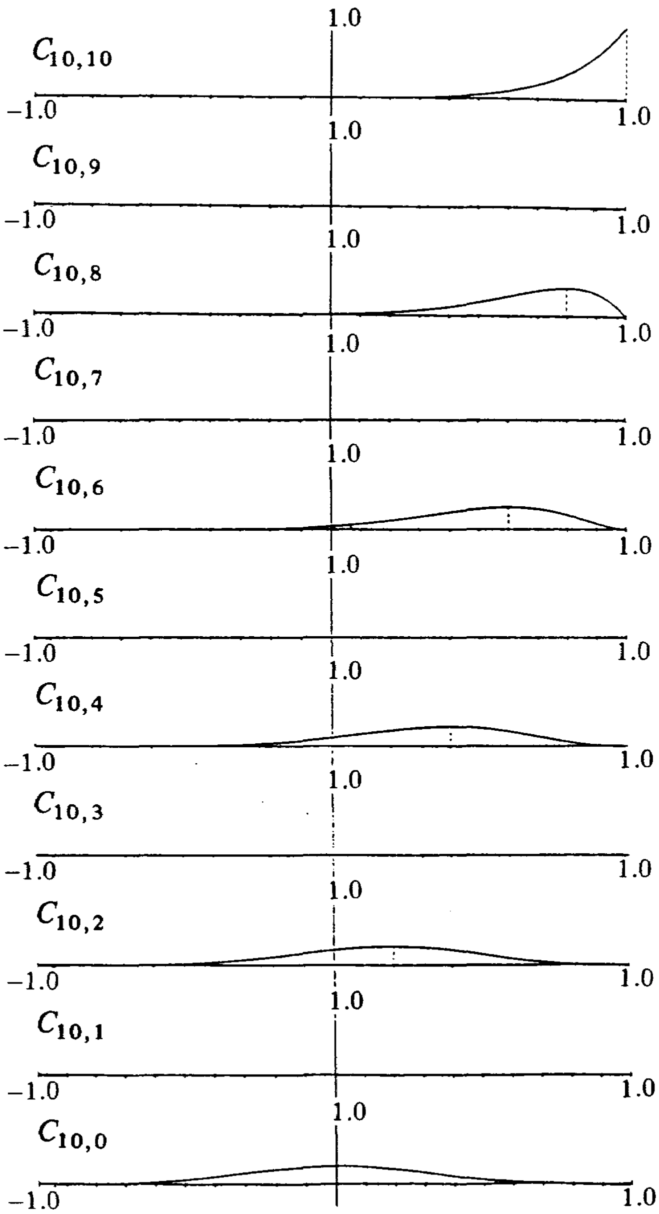
Fig. 2 Case 1 Case 2

upstream (or upwind) difference approximation (sometimes called Murmann's, Roe's or generalized Courant-Isaacson-Ree's). On the other hand the difference approximation defined by the numerical flux (34) with $Q(\alpha) = |\alpha|$ is the Engquist-Osher difference approximation (See refs. 2 and 15).

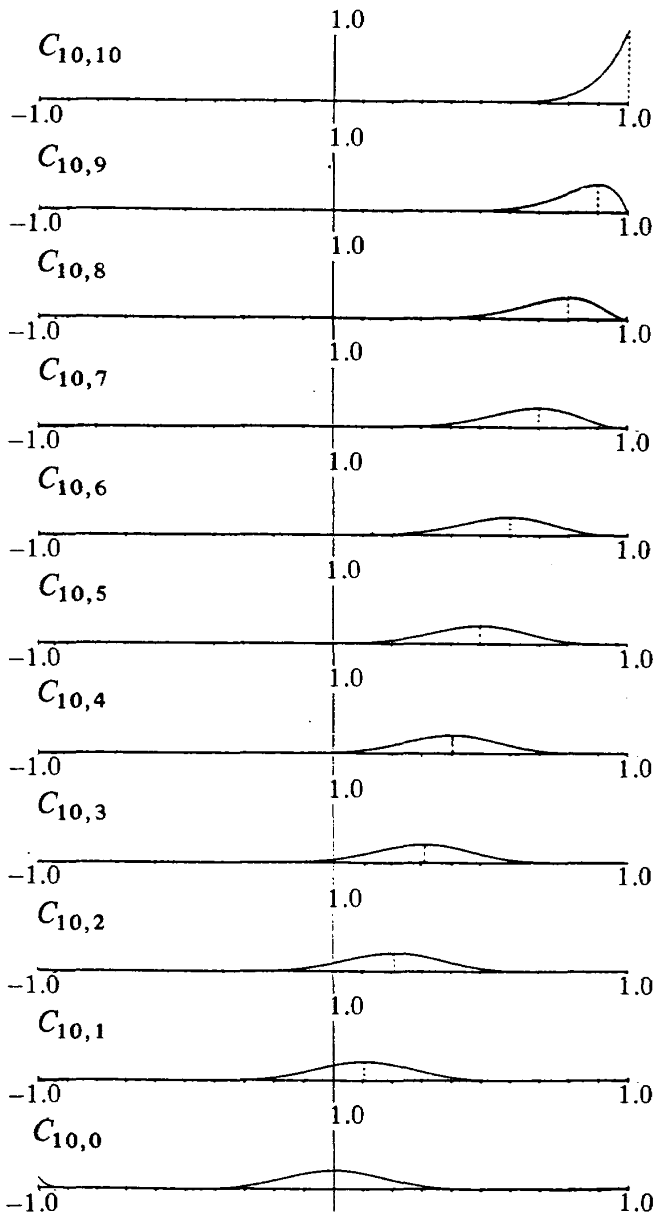
A distinguishing feature of the case of $Q(\alpha) = |\alpha|$ is that $C_{n,j}(\alpha) = 0$ for $-n \leqslant j \leqslant n$ and $\alpha \in [-1, 1]$ with $j\alpha < 0$. This means that perturbations do not propagate to the direction with $j\alpha < 0$ and corresponds to that upstream differencing is to switch the direction of differencing according to the value of $f'(u)$. For upstream differencing, see ref. 8. Thus Theorem 1 gives another interpretation of upstream differencing.

(case 2) As was already shown, the function $Q(\alpha) = \frac{1}{2}(1 + \alpha^2)$ corresponds to the two-step Lax-Friedrichs difference approximation and the two-step Glimm difference approximation in the sense of expectation.

Comparing (case 2) with (case 1), one may observe that the slope of each curve in case 2 is somewhat gentle. This property is caused by the fact that $C_{n,j}(\alpha) \neq 0$ for any $\alpha \in (-1, 1)$, which



Case 3

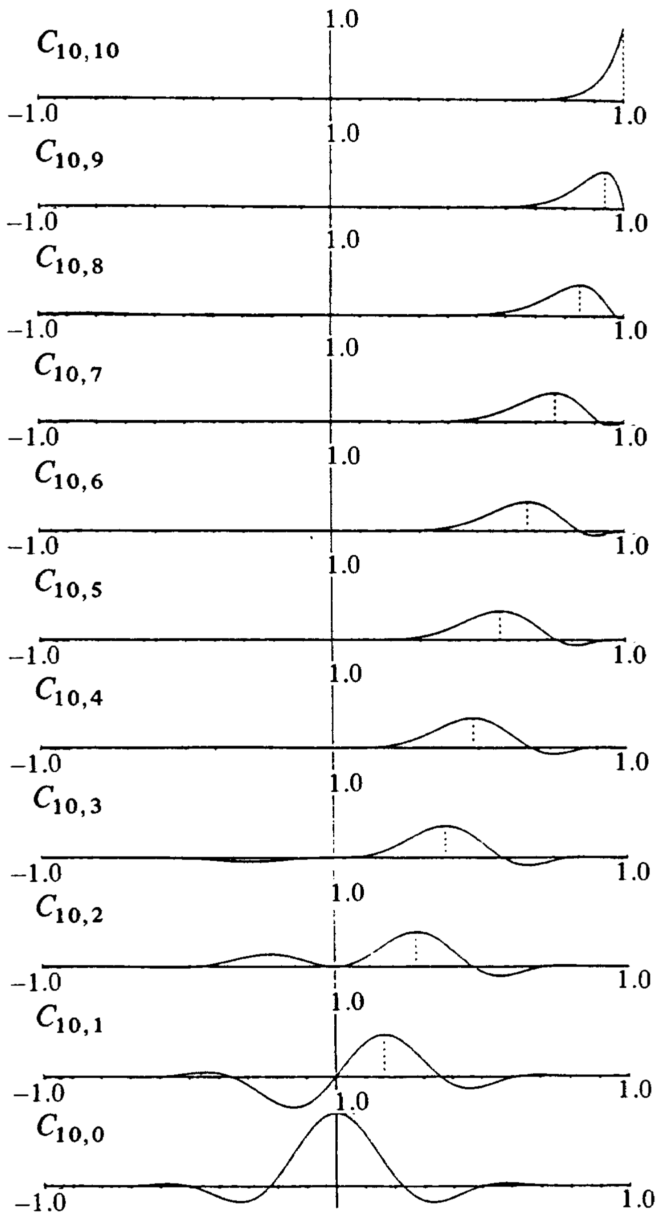


Case 4

means that perturbations propagate to every direction. In general, we can say that as $Q(0) = q$ increases, numerical viscosity of the corresponding difference approximation becomes larger.

(case 3) The function $Q(\alpha) \equiv 1$ provides us with the well-known Lax-Friedrichs difference approximation which has played an important role in the finite difference method for scalar conservation laws.

A feature of Fig. 2 (case 3) is that $C_{n,j}(\alpha) \equiv 0$ if $n-j$ is odd. This corresponds to the fact that Lax-Friedrichs difference approximation is substantially a two-point difference approximation.



Case 5

All difference approximations stated above are consistent with characteristics. We next give two examples of difference approximation which are not consistent with characteristics.

(case 4) It is well known that the upstream difference approximation produces stable inverse (non-physical) shock waves and consequently is not consistent with the entropy condition. For this reason, Harten proposed in ref. 5 the use of $Q(\alpha)$ defined by (43). It is evident that the function $Q(\alpha)$ differs from those in Theorem 1. Accordingly, the resulting difference approximation is not consistent with characteristics, although it seems to be nearly consistent with characteristics. See Fig. 2 (case 4).

(case 5) The function $Q(\alpha) = \alpha^2$ corresponds to the Lax-Wendroff difference approximation with the numerical flux

$$\bar{f}(u,v) = \frac{1}{2}\left\{f(u) + f(v)\right\} - \frac{\lambda}{2} f'\left(\frac{u+v}{2}\right)\left\{f(v) - f(v)\right\}.$$

It should be noted that $Q(\alpha) = \alpha^2$ is the only unique function with which the three-point difference approximation (1) becomes to be second order accurate. It is also interesting to note that the function $C_{n,j}(\alpha)$ in (24) involves a function of the form $Q(\alpha) - \alpha^2$. The graphs of $C_{n,j}(\alpha)$ are given in Fig. 2 (case 5). At a glance, one may recognize that the choice of this $Q(\alpha)$ is not favorable. The bad behavior of $C_{n,j}(\alpha)$ seems to explain also the numerical instability of the Lax-Wendroff difference approximation. Thus it is seen that our notion of consistency with characteristics is rather reasonable and suggests the possibility of a new approach to the study of finite difference approximations.

7. The extension of consistency with characteristics

In this section, we extend the notion of consistency with characteristics to difference approximations for the hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \quad x \in R^m, \ t \geqslant 0, \qquad (44)$$

where $u(x,t)$ is an m-vector of unknowns and $f: R^m \to R^m$ is a vector-valued function of C^1-class.

Remember that the system (44) is said to be hyperbolic if the Jacobian matrix $A = A(u)$ of $f(u)$ has m distinct real eigenvalues. We denote by $\gamma_i = \gamma_i(u)$, $1 \leqslant i \leqslant m$, the eigenvalues of A, and assume that

$$\gamma_1 < \gamma_2 < \cdots < \gamma_m .$$

Also, we denote by $l_i = l_i(u)$ and $r_i = r_i(u)$ the left and the right eigenvectors, respectively. Eigenvectors are normalized so that

$$\Delta \gamma_i(u) \cdot r_i(u) \geqslant 0$$

and

$$l_i(u) \cdot r_i(u) = 1 ,$$

where the dot $\cdot$ denotes the inner product.

Now, let

$$R(u) = [r_1(u), r_2(u), \cdots, r_m(u)] \qquad (45)$$

be a matrix, with the columns $r_i(u)$'s being the right engenvectors of A. Then

$$R^{-1}AR = \Lambda , \qquad (46)$$

where Λ is the diagonal matrix with $\Lambda_{ij} = \gamma_i \delta_{ij}$ and δ_{ij} is the Kronecker's delta. Define w by

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = R^{-1}u . \qquad (47)$$

Each component of w is called a characteristic variable. Under transformation (47), the system (44) in the matrix form

$$u_t + Au_x = 0 \qquad (48)$$

decouples into m-scalar equations

$$(w_i)_t + \gamma_i (w_i)_x = 0, \ 1 \leqslant i \leqslant m . \qquad (49)$$

By virtue of (48), we can extend the argument in the previous sections to the following 3-point explicit difference approximation for (44)

$$u_j^{n+1} = H(u_{j-1}^n, u_j^n, u_{j+1}^n),$$
$$n \geqslant 0, \ j = \pm 1, \pm 2, \cdots . \qquad (50)$$

As before, we write $\lambda = \dfrac{\Delta t}{\Delta x}$. For the function H in (50), we assume that the following conditions should be satisfied (cf. Section 2):

(C1)*s*
H is a function of C^1-class, and all of $R^{-1}H_{-1}R$, $R^{-1}H_0R$ and $R^{-1}H_1R$ are diagonal matrices, where H_{-1}, H_0 and H_1 are the Jacobians

$$\begin{cases} H_{-1} = \dfrac{\partial H(u_{-1}, u_0, u_1)}{\partial u_{-1}} \\ H_0 = \dfrac{\partial H(u_{-1}, u_0, u_1)}{\partial u_0} \\ H_1 = \dfrac{\partial H(u_{-1}, u_0, u_1)}{\partial u_1} . \end{cases}$$

(C2)*s*
For any $u \in R^m$, $u = H(u,u,u)$.

(C3)*s*
For any smooth function $u(x,t)$,

$$\lim_{\Delta t \to 0} \frac{H(u(x-\Delta x, t), u(x,t), u(x+\Delta x, t)) - u(x,t)}{\Delta t}$$

$$= -f(u)_x$$

(C4)s

Each i-th diagonal element of the diagonal matrix

$$R^{-1}(u)H_0(u,u,u)R(u)$$

is a continuous function of $\alpha_1 = \lambda\gamma_i(u) \in [-1, 1]$.

The next result is proved in a similar manner to Lemma 1.

Lemma 3. *For any $u \in R^m$,*

$$H_{-1}(u,u,u) + H_0(u,u,u) + H_1(u,u,u) = E, \quad (51)$$

$$H_{-1}(u,u,u) - H_1(u,u,u) = \lambda A(u) \quad (52)$$

and

$$R^{-1}(u)H_{-1}(u,u,u)R(u) - R^{-1}(u)H_1(u,u,u)R(u) = \lambda\Lambda, \quad (53)$$

where E is the unit matrix and Λ is the diagonal matrix in (46).

Let $C_{n,j} = C_{n,j}(u)$ be the Jacobian matrix $\frac{\partial u_j^n}{\partial u_0^0}$ of u. For each i, $1 \leqslant i \leqslant m$, let $c_{n,j}^i = c_{n,j}^i(u)$ be the i-th diagonal element of the diagonal matrix $R^{-1}(u)C_{n,j}(u)R(u)$. Then we define property $(C)s$ and consistency with characteristics as follows.

Property $(C)s$

Each $C_{n,j}^i$ takes the maximum value when $\lambda\gamma_i(u) = \frac{j}{n}$.

Definition 2. *The difference approximation (50) for the system of conservation laws (44) is said to be consistent with characteristics if (50) possesses the property $(C)s$.*

By condition $(C)s$, it is obvious that the matrices $R^{-1}(u)H_{-1}(u,u,u)R(u)$, $R^{-1}(u)H_0(u,u,u)R(u)$ and $R^{-1}(u)H_1(u,u,u)R(u)$ are commutative with each other in matrix multiplication. This implies that $H_{-1}(u,u,u)$, $H_0(u,u,u)$ and $H_1(u,u,u)$ are also commutative. So, by applying the chain rule, we obtain

$$C_{n,j} = \sum_{(a,b,c)\in S_{nj}} \frac{n!}{a!b!c!}\left\{H_{-1}\right\}^a\left\{H_0\right\}^b\left\{H_1\right\}^c, \quad (54)$$

where

$$S_{nj} = \left\{(a,b,c) \in Z^3; a+b+c=n, a-c=j, a,b,c \geqslant 0\right\}.$$

Let $h_{-1}^i(\alpha_i)$, $h_0^i(\alpha_i)$ and $h_1^i(\alpha_i)$ be the i-th diagonal element of $R^{-1}(u)H_{-1}(u,u,u)R(u)$, $R^{-1}(u)H_0(u,u,u)R(u)$ and $R^{-1}(u)H_1(u,u,u)R(u)$, respectively. Also define

$$Q_i(\alpha_i) = 1 - h_0^i(\alpha_i) \text{ for } \alpha_i \in [-1, 1] \quad (55)$$

Then it follows from (55) and Lemma 3 that

$$\begin{cases} h_{-1}^i(\alpha_i) = \dfrac{Q_i(\alpha_i) + \alpha_i}{2} \\ h_0^i(\alpha_i) = 1 - Q_i(\alpha_i) \\ h_1^i(\alpha_i) = \dfrac{Q_i(\alpha_i) - \alpha_i}{2} \end{cases} \quad (56)$$

This together with (54) implies that each $c_{n,j}^i$ is a function of α_i on $[-1, 1]$ and written as

$$c_{n,j}^i(\alpha_i) = \sum_{(a,b,c)\in S_{nj}} \frac{n!}{a!b!b!}\left\{h_{-1}^i(\alpha_i)\right\}^a \times\left\{h_0^i(\alpha_i)\right\}^b\left\{h_1^i(\alpha_i)\right\}^c. \quad (57)$$

By applying Theorem 1 to (57), we obtain the following theorem.

Theorem 3. *For each i, $1 \leqslant i \leqslant m$, let $Q_i(\alpha)$*

be the function in (56). If for some $q_i \in [0, 1]$

$$Q_i(\alpha) = \begin{cases} \dfrac{q_i^2 + (1-q_i)\sqrt{(1-2q_i)\alpha^2 + q_i^2}}{1-2q_i}, \\ \qquad\qquad q_i \neq \dfrac{1}{2}, \\ \dfrac{1+\alpha^2}{2}, \qquad q_i = \dfrac{1}{2} \end{cases} \tag{58}$$

then the difference approximation (50) for (44) is consistent with characteristics.

8. Numerical fluxes in the case of systems

In this section we consider difference approximations of conservation form

$$u_j^{n+1} = u_j^n - \lambda\left\{\bar{f}(u_j^n, u_{j+1}^n) - \bar{f}(u_{j-1}^n, u_j^n)\right\} \tag{59}$$

for systems of conservation laws (44), where the numerical flux $\bar{f}$ is a vector-valued function satisfying

$$\bar{f}(u,u) = f(u).$$

Throughout this section, let $Q_i(\alpha)$ be functions as in Theorem 3. First, we discuss difference approximations with numerical fluxes of Roe type. Let $\bar{A}(u,v)$ be a matrix with the following property which is called property U (See ref. 19):

(i) $f(v) - f(u) = \bar{A}(u,v)(v-u)$.

(ii) $\bar{A}(u,u) = A(u)$.

(iii) $\bar{A}(u,v)$ has real eigenvalues $\bar{\gamma}_i(u,v)$, $1 \leqslant i \leqslant m$, and complete set $\left\{\bar{r}_i(u,v)\right\}_{1 \leqslant i \leqslant m}$ of eigenvectors.

Also, let $Q(\lambda\bar{A}(u,v))$ be a matrix defined by

$$Q(\lambda\bar{A}(u,v)) = \bar{R}(u,v)\,Q(\lambda\bar{\Lambda}(u,v))\bar{R}(u,v)^{-1}, \tag{60}$$

where

$$\bar{R}(u,v) = [\bar{r}_1(u,v), \cdots, \bar{r}_m(u,v)],$$
$$\bar{\Lambda}(u,v) = \bar{R}(u,v)^{-1}\bar{A}(u,v)\bar{R}(u,v),$$

and

$$Q(\lambda\bar{\Lambda}(u,v))_{ij} = Q_i(\lambda\bar{\gamma}_i(u,v))\delta_{ij}.$$

We then define a numerical flux $\bar{f}(u,v)$ by

$$\bar{f}(u,v) = \frac{1}{2}\left\{f(u) + f(v)\right\} - \frac{1}{2\lambda}Q(\lambda\bar{A}(u,v))\cdot(v-u). \tag{61}$$

As is easily seen, the difference approximation (59) with the numerical flux (61) is consistent with characteristics (See also the proof of Theorem 4 below). If $Q_i(\alpha) = |\alpha|$ for every i, then the resulting difference approximation is known as Roe's. It should be mentioned that Roe[19] constructs a matrix with property U for the Euler equation in gas dynamics (for general case, see ref. 8).

Next, we discuss difference approximations of Osher type. We define a numerical flux $\bar{f}(u,v)$ by

$$\bar{f}(u,v) = \frac{1}{2}\left\{f(u) + f(v)\right\} - \frac{1}{2\lambda}\int_{\Gamma_{uv}} Q(\lambda A(w))dw, \tag{62}$$

where $Q(\lambda A(w))$ is defined in the same way as (60). The integral in (62) is defined as follows.[15] The path $\Gamma_{uv} \in R^m$ of integral has the following property:

(P1) Γ_{uv} consists of m subpaths $\Gamma_1, \Gamma_2, \cdots, \Gamma_m$ and each Γ_i connects $u_{i-1} = \Gamma_i(0)$ and $u_i = \Gamma_i(s_i)$, where $u_0 = u$ and $u_m = v$.

(P2) For a permutation σ of $1, 2, \cdots, m$,

$$\Gamma_i'(s) = r_{\sigma(i)}(\Gamma_i(s)),$$
$$\min\left\{0, s_i\right\} < s < \max\left\{0, s_i\right\}.$$

With this choice of path, the integral in (62) means that

$$\int_{\Gamma_{uv}} Q(\lambda A(w))dw = \sum_{i=1}^{m} \int_0^{s_i} Q_i(\lambda\gamma_{\sigma(i)}(\Gamma_i(s)))\,\Gamma_i'(s)ds$$

Note that the existence of an integration path Γ_{uv} is guaranteed if $||u - v||$ is sufficiently small. If we set $Q_i(\alpha) = |\alpha|$ and $\sigma(i) = m + 1 - i$, we obtain the difference approximation proposed by Osher.[15,17)]

Remark 6. *The exact solution $u(x, t) = w(\frac{x}{t}; u, v)$ to the Riemann problem*

$$u_t + f(u)_x = 0,\ u(x,0) = \begin{cases} u & \text{if } x < 0 \\ v & \text{if } x > 0 \end{cases}$$

consists of $n + 1$ states $u_0 = u, u_1, \cdots, u_n = v$ and n waves. Here i-th wave $(1 \leqslant i \leqslant n)$ connects two neighboring states u_{i-1} and u_i. If i-th eigenvalue is not-degenerate (i.e. $\Delta\gamma_i \cdot r_i > 0$), the i-th wave is a rarefaction wave or a shock wave. In the case of the rarefaction wave, there exists a path Γ_i such that

$$\begin{cases} \Gamma_i(0) = u_{i-1} \\ \Gamma_i(s_i) = u_i \quad (s_i > 0) \\ \Gamma_i'(s) = r_i(\Gamma_i(s)) \quad (0 < s < s_i). \end{cases} \tag{63}$$

The positivity of s_i comes from the physical relevance. In the case of the shock wave, there exists a real number S_i (called the shock speed) such that

$$\begin{cases} f(u_i) - f(u_{i-1}) = S_i(u_i - u_{i-1}) \\ \gamma_i(u_i) < S_i < \gamma_i(u_{i-1}). \end{cases}$$

The above inequality on S_i comes from the physical relevance.

In the construction of the path Γ_{uv}, each subpath Γ_i is defined in the same way as (63), except the restriction $s_i > 0$ and with the permutation σ added. So, we can say that, for every i, two states u_{i-1} and u_i are connected by a rarefaction wave corresponding to $\sigma(i)$-th eigenvalue, although the rarefaction wave is not physically relevant if $s_i < 0$. See ref. 21.

Theorem 4. *The difference approximation (59) with the numerical flux (62) is consistent with characteristics.*

proof.

Without loss of generality, we assume that $\sigma(i) = i$. Define matrices $\bar{f}_1(u, u)$ and $\bar{f}_2(u, u)$ by

$$\bar{f}_1(u,u) = \left.\frac{\partial \bar{f}(u,v)}{\partial u}\right|_{v=u}$$

and

$$\bar{f}_2(u,u) = \left.\frac{\partial \bar{f}(u,v)}{\partial v}\right|_{v=u}$$

respectively. Now, we calculate $\bar{f}_2(u,u)$. Let $u, v \in R^m$ and suppose that $||u - v||$ is sufficnently small. Here $||\cdot||$ denotes the usual l^2-norm. Set

$$\Delta u = v - u.$$

By completeness of $r_i(u)$ $_{1<i<m}$, Δu is written as

$$\Delta u = {}^t(\Delta u_1, \Delta u_2, \cdots, \Delta u_m) = \sum_{i=1}^{m} \Delta a_i r_i(u) \tag{64}$$

for some real numbers Δa_i. On the other hand, it follows from the definition of Γ_{uv} that

$$\Delta u = \sum_{i=1}^{m} \left\{\Gamma_i(s_i) - \Gamma_i(0)\right\} = \sum_{i=1}^{m} \int_0^{s_i} r_i(\Gamma_i(s))ds$$

$$= \sum_{i=1}^{m} \left\{ s_i \; r_i(u) + o(\|\Delta u\|) \right\}$$

$$= \sum_{i=1}^{m} s_i r_i(u) + o(\|\Delta u\|), \qquad (65)$$

because $s_i \to 0$ as $\|\Delta u\| \to 0$. Comparing (65) with (64), we see that

$$s_i = \Delta a_i + o(\|\Delta u\|), \; 1 \leqslant i \leqslant m.$$

Therefore, we have the relation

$$\int_{\Gamma_{uv}} Q(\lambda A(w))dw$$

$$= \sum_{i=1}^{m} \int_0^{s_i} Q_i(\lambda \gamma_i(\Gamma_i(s))) \cdot r_i(\Gamma_i(s))ds$$

$$= \sum_{i=1}^{m} s_i \left\{ Q_i(\lambda \gamma_i(\Gamma_i(s))) + O(\|\Delta u\|) \right\} \cdot \left\{ r_i(u) + O(\|\Delta u\|) \right\}$$

$$= \sum_{i=1}^{m} s_i Q_i(\lambda \gamma_i(\Gamma_i(s))) \cdot r_i(u) + o(\|\Delta u\|)$$

$$= \sum_{i=1}^{m} \Delta a_i Q_i(\lambda \gamma_i(u)) \cdot r_i(u) + o(\|\Delta u\|)$$

$$= [r_1(u), \cdots, r_m(u)] \cdot Q(\lambda \Lambda(u)) \cdot \begin{bmatrix} \Delta a_1 \\ \vdots \\ \Delta a_m \end{bmatrix} + o(\|\Delta u\|)$$

$$= R(u) Q(\lambda \Lambda(u)) R^{-1}(u) \cdot \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_m \end{bmatrix} + o(\|\Delta u\|),$$

which implies

$$\frac{\partial}{\partial v} \int_{\Gamma_{uv}} Q(\lambda A(w))dw \bigg|_{v=u}$$

$$= R(u) Q(\lambda \Lambda(u)) R^{-1}(u).$$

Consequently, we obtain

$$\bar{f}_2(u,u) = \frac{1}{2}\left[A(u) - \frac{1}{\lambda} R(u) Q(\lambda \Lambda(u)) R^{-1}(u) \right].$$

Similarly, we obtain

$$\bar{f}_1(u,u) = \frac{1}{2}\left[A(u) + \frac{1}{\lambda} R(u) Q(\lambda \Lambda(u)) R^{-1}(u) \right].$$

By applying Theorem 3, we conclude that the deference approximation (59) with (62) is consistent with characteristics. (q.e.d.)

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