Design of 3D Transit Orbits and Application to Mission Design

Wataru Noma, Mai Bando, Shinji Hokamoto（Kyushu University）

Abstract

In recent years, the use of low energy trajectories using invariant manifolds in many-body problems has been studied. However, although many studies on transit orbits in two-dimensional space have been made up to now, there are not many studies on design of transit orbits in three-dimensional space. For this reason, in this paper, we analyze the properties of transit orbits in a three-dimensional space, describe its design method, and show an application example to the actual mission in the earth-moon system three-body problem.

3次元トランジット軌道の設計とミッションへの応用

摘要

近年，多体問題における不変多様体を用いた低エネルギー軌道の利用が研究されている．しかし，これまで２次元空間のトランジット軌道に関する多くの研究が行われてきたが，3 次元空間のトランジット軌道の設計に関する研究はあまり行われていない．このため，本論文では，３次元空間におけるトランジット軌道の特性を分析し，その設計方法を説明する．さらに，地球-月系の三体問題における実際のミッションへの応用について述べる．

1. Intriduction

In the design of low-energy trajectories in the circular restlicted three-body problem(CRTBP), it is known that transport trajectories using trajectories called “transit orbit” play an important role [1] [2] [3]. A transit orbit is defined as a trajectry that passes through the bottleneck region that connects the outside and inside of the zero velocity curve, which is a forbidden region in the CRTBP. Until now, much research has been done on planar transit orbits [4]. However, three-dimensional transit orbits with six-dimensional state variables have not been considered so far. The purpose of this study is to propose a design method of 3D transit orbit. As a result, it is possible to design a transit orbit that can be injected with the smallest velocity change from the position information of the spacecraft and to be useful for space missions.

In this paper, characteristics of transit orbits and design method in 6-dimensional phase space are explained. Specifically, after introducing the conventional design method for transit orbits, we propose a new design method that reduces the amount of calculation by approximating the 2-dimensional surface created by the velocity components of the Poincare section. The new design method reduces the number of variables to be considered by assuming the initial position first, and as a result, the design can be perfomed in a lower calculation cost. By the proposed method, we investigate transit orbits for various values of the Jacobi constant. As a result of examining the relationship between the transit orbit and the Jacobi constant, it was found that the Jacobi constant is very important in designing transit orbits as well as the invariant manifold in the CRTBP. Therefore, it is shown that the transit orbit can be designed more easily by choosing the Jacobi constant.

2. Basic equations

2. 1 Circular restricted three-body problem

The problem that three bodies are moving under universal gravitation is called the three-body problem [5]. Consider the motion of two celestial bodies of mass $m_1$and m_2 and mass point m. When the mass m is sufficiently small compared to m_1 and m_2, the two objects revolve around their common center of gravity. The problem that the mass of one of these three bodies is sufficientlly small compared with the mass of the other two bodies and does not affect the motion of the remaining two bodies is called the restricted three-body problem. In particular, the revolving motion of two celestial bodies is assumed to be a circular motion, which is called the CRTBP.

2. 2 Equation of Motion

As shown in Fig.**1**, the origin is the center of gravity of two celestial bodies (called the first celestial body and the second celestial body respectively) with masses of m_1 and m_2（assuming $m_1 > m_2$）. The direction from the first celestial body to the second celestial body is the x axis, the axis perpendicular to the x axis on the orbital plane of the two celestial bodies is the y axis, and the z-

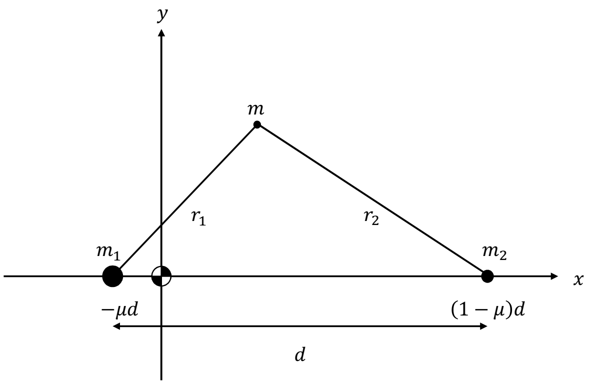

Fig. **1**: the Circular Restricted Three-body Problem

axis complete the right-handed system. Let the distance from the mass point m to the first and second bodies be r_1 and r_2, respectively. And the coordinates of the mass point m, $m_1$and m_2 are $(x,\ y,\ z)$, $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$, respectively. The universal gravitational constant is G, the angular velocity of the celestial body is ω, and the distance between the celestial bodies is d. The equation of motion [7] for the mass m is

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - x\omega^2 &= \frac{Gm_1}{r_1^3}(x_1 - x) + \frac{Gm_2}{r_2^3}(x_2 - x) \\ \ddot{y} + 2\omega\dot{x} - y\omega^2 &= \frac{Gm_1}{r_1^3}(y_1 - y) + \frac{Gm_2}{r_2^3}(y_2 - y) \\ \ddot{z} &= \frac{Gm_1}{r_1^3}(z_1 - z) + \frac{Gm_2}{r_2^3}(z_2 - z) \end{aligned} \qquad (1)$$

Where,

$$r_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$
$$r_2 = \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2}$$

Introduce the nomalized time where the orbital period is set to 2π, then $\omega = 1$, the mass ratio $\mu = m_2/(m_1 + m_2) \leq 0.5$ where the two masses are $1-\mu, \mu$, respectively, and nomalized the distance where the distance of two main bodies is set to $d = 1$. Then the equation of motion of the non-dimensional mass point m is given by

$$\begin{aligned} \ddot{x} - 2\dot{y} - x &= \frac{1-\mu}{r_1^3}(x_1 - x) + \frac{\mu}{r_2^3}(x_2 - x) \\ \ddot{y} + 2\dot{x} - y &= \frac{1-\mu}{r_1^3}(y_1 - y) + \frac{\mu}{r_2^3}(y_2 - y) \\ \ddot{z} &= \frac{1-\mu}{r_1^3}(z_1 - z) + \frac{\mu}{r_2^3}(z_2 - z) \end{aligned} \qquad (2)$$

The position function U is defined as

$$U = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu) \qquad (3)$$

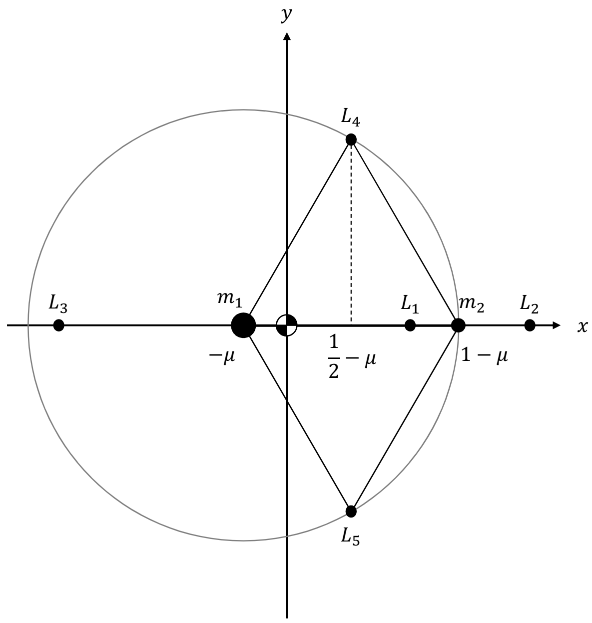

Fig. **2**: Lagrangian points

, then Eq.(2) becomes

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \qquad (4)$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} \qquad (5)$$

$$\ddot{z} = \frac{\partial U}{\partial z} \qquad (6)$$

Equilibrium points are obtained from the Eqs.(4), (5), and (6), and these equiliburium points are called Lagragian points. Lagragian points are classified as the colinear solutions ($L_1,\ L_2,\ L_3$) where gravity and centrifugal force are balanced on a line connecting two celestial bodies ($y = 0$) and equilateral triangle solutions ($L_4,\ L_5$) where the gravity and centrifugal force are balanced at $y \neq 0$. The location of lagrangian points are shown in the Fig.**2**.

2. 3 Jacobi constant

From Eqs.(3)-(6), the Jacobi constant C is defined as

$$C(x, y, \dot{x}, \dot{y}) = -(\dot{x}^2 + \dot{y}^2) + 2U(x, y) \qquad (7)$$

Differentiating Eq.（7） with respect to time,

$$\dot{C} = 0 \qquad (8)$$

can be obtained. Thus, the Jacobi constant is a constant of integration in the CRTBP. It shoule be noted that it is different from the mechanical energy. This will be explained in detail in Chapter 3.

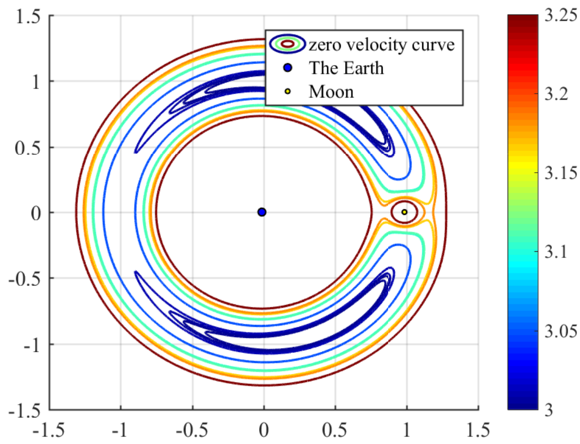

Fig. **3**: the zero velocity curve

2. 4 Zero velocity curve

The Jacobi constant described in the previous section plays a important role like a total mechanical energy in conservaitve mechanical system to place bounds on the motion of the spacecraft. From the relationship

$$v^2 = 2U(x, y, z) - C \geq 0 \tag{9}$$

there is a prohibited region where spacecraft motion is not possible depending on the value of the Jacobi constant. Fig.**3** shows the forbidden regions (areas inside the zero velocity curve) corresponding to the respective C values.

3. Design method of transit orbit

In general, invariant manifolds of periodic orbits are used when designing transit orbits. By cutting an invariant manifold with an appropriate Poincaré section and setting the initial value inside the closed curve representing the intersection of the invariant manifold of the Poincaré section. However, there is a degree of freedom in choosing periodic orbits, and it is possible to design transit orbits even if the initial state is taken approximately. In this section, the method in which the transit orbit was numerically analyzed by previous research is first described, and then the new method is proposed.

3. 1 Previous research with numerical analysis of Transit Orbit

In previous studies [6], transit orbits are comprehensively obtained by numerical search. Specifically, the dimension of the initial six-dimensional variable ($[x, y, z.\dot{x}, \dot{y}, \dot{z}]$) is reduced to five-dimension by fixing the Jacobi constant. Then, by fixing the initial x coordinate, the dimension is further reduced by one dimension. Then, the initial state for transit orbits are found by exhoustive seach for the remaining four-dimensional variables. In this method, the range of the four-variable grid is determined by invariant manifolds of the horizontal and vertical Lyapunov orbits. This is because the invariant manifolds of the horizontal Lyapunov orbit occupy the maximum region on the x-y plane, and the invariant manifolds of the vertical Lyapunov orbit occupy the maximum region on the z axis direction. By using this method, almost all transit orbits that exist in a particular Jacobi constant can be obtained.

It is thought that all transit orbits can be examined by the above method, but the problem is that the calculation cost is too expensive. It was also shown that designing transit orbits becomes more easy by using the relationship with invariant manifolds.

3. 2 Design method of transit orbit in this paper

In the previous subsection, a method to generate the transit orbits for a fixed Jacobi constant is introduced. In this subsection, a new method where the initial position (x, y, z) is fixed first, and a method to find all corresponding transit orbits is considered. Here, the Jacobi constant is not fixed, which is different from the method in the previous section [6].

First, the origin is set to the center of gravity in the CRTBP of the Earth-Moon system, and the axis taken 45 degrees clockwise from the x axis is the $x(ref)$ axis (Fig.**4**), and take the point $[x(ref), z] = [1.79, 0]$ on the $x(ref)$ axis as the initial position. Next, a halo orbit with a Jacobi constant of $C = 3.095866$ is obtained, and an unstable manifold emanating from the halo orbit is computed. At this time, the intersection of the $x(ref)$ axis plane and the invariant manifold is used as a design criterion (Fig.**5**). In the proposed method, the initial position is fixed, so the variables to be considered are the three velocity components $\dot{x}, \dot{y}, and \dot{z}$. Consider a Poincaré section in a three-dimensional phase space. Figure.**6** shows the closed curve consisting of $(\dot{x}, \dot{y}, \dot{z})$ of invariant manifolds in the Poincaré section. A closer look at the Poincaré section shows that it is a closed curve in the $\dot{x} - \dot{y}$ plane but is twisted in the $\dot{z}$ direction. In general, considering that the transit orbit passes through an invariant manifold in the Poincaré section, it is necessary to take the state inside the area where this curve is projected on each coordinate plane. However, if

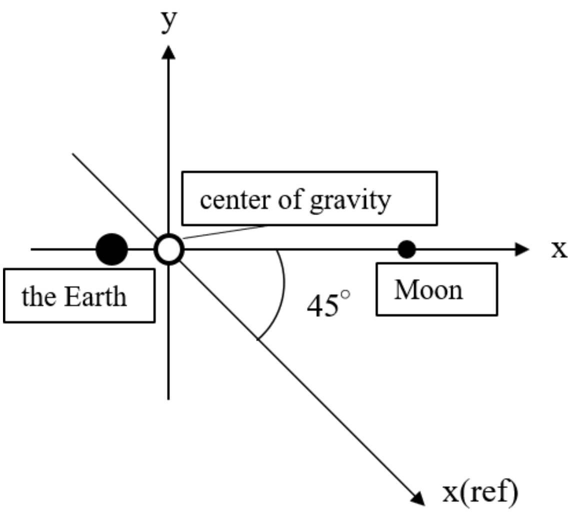

Fig. **4**: Assuming $x(ref)$

full search using a 3D grid requires, a considerable computational cost. Therefore, the closed surface enclosed by this closed curve is approximated by $\dot{z} = f(\dot{x}, \dot{y})$, and the transit orbits are obtained by taking the state on the surface. By doing this, it is possible to get a point inside each projection plane, and to calculate $\dot{z}$ from $\dot{x}$ and $\dot{y}$. Figure.**7** is an approximation of the surface created by the velocity component curve. By using the approximate function of this curved surface, all velocity components can be calculated immediately. Taking the point on this curved surface as the velocity component of the initial state, the trajectories propagating from the points are shown in Fig.**8**. The colors of the orbits correspond to the Jacobi constant of the color bar. In Fig.**8**, there are some trajectories that do not pass through the neck region (for example, blue trajectories). If the Jacobi constant is too large, the zero velocity curve is closed in

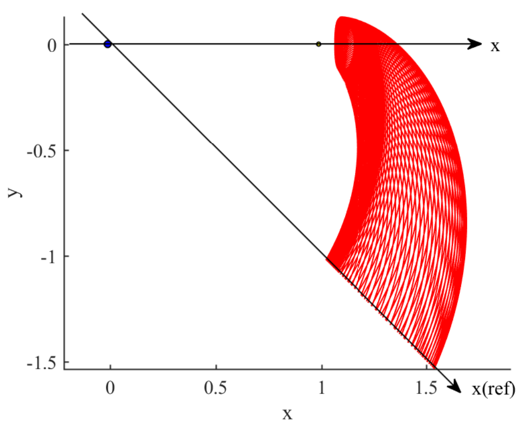

Fig. **5**: Cutting invariant manifold at $x(ref)$

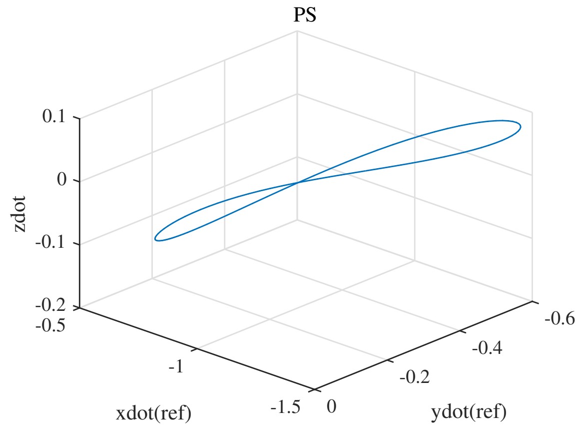

Fig. **6**: Poincaré section

the neck region. Then, trajectories cannot be transit orbits by being excluded by zero velocity curves. On the other hand, when the Jacobi constant is small, the zero velocity curve opens greatly and many transit orbits exist. However, it also indicates that trajectries are not approach the Earth. Furthermore, if the Jacobi constant is too small, the velocity component becomes larger, so the trajectries cannot be said to be a low energy orbits. This result shows that the Jacobi constant determines whether the trajectory can be a transit orbit even when propagating from the state inside the invariant manifold in the Poincaré section. Moreover, it is possible to determine the transit orbit that can approach the Earth by choosing an appropriate Jacobi constant. In the section, by choosing the Jacobi constant, design trajectories that can transition from outside the zero velocity curve to near the Earth are shown.

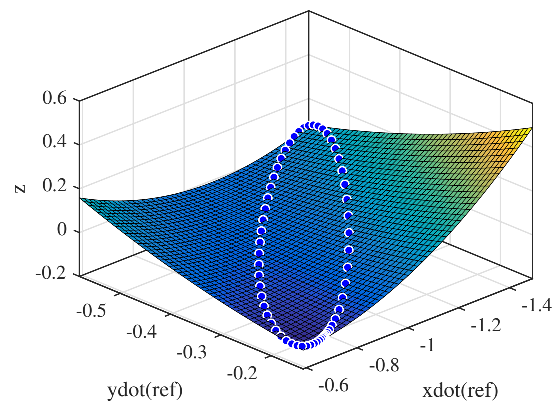

Fig. **7**: Surface approximation in the velocity component

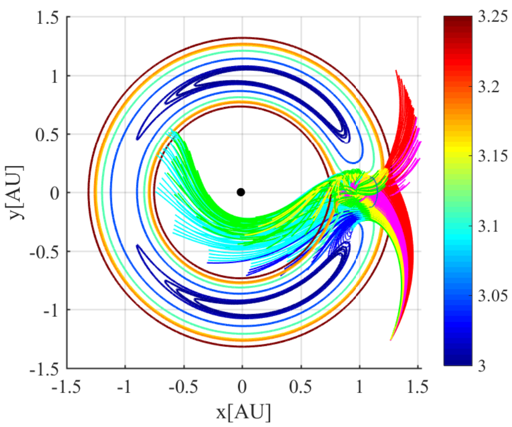

Fig. **8**: Determining the initial state

4. Design of Transit Orbit approaching the Earth

From Fig.**8**, the Jacobi constant of the trajectories approaching the Earth is considered to be in the range of 3.05 to 3.15. In addition, trajectories that are greatly affected by the Moon and cannot finally enter the inside of the zero velocity curve are excluded because they are insufficient as transport trajectories from the outside of the Moon to the vicinity of the Earth. The result obtained above is shown in Fig.**9**. It can be seen that those trajectories pass through the neck of the zero velocity curve and that can transition to the vicinity of the Earth. From this result, it was found that the transit orbits passing through the initial position set by selecting the velocity component appropriately using the proposed method can be obtained. Furthermore, by using them as candidates, it is considered that a sufficient number of transit orbits can be designed to design the optimal trajectory according to the mission purpose. From the above, it was found that the transit orbit can be designed by using

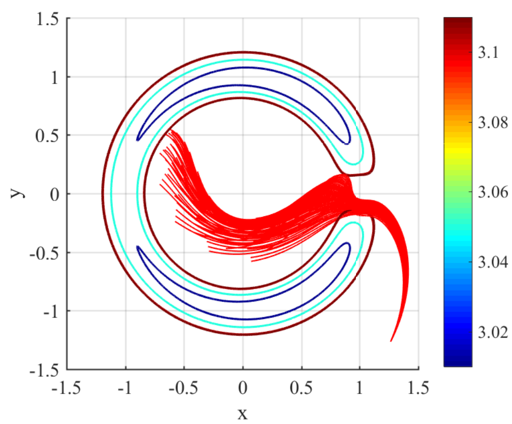

Fig. **9**: Transit Orbits

the velocity curved surface. On the other hand, it became clear that there are innumerable initial states that can become transit orbits.

5. Conclusion

In this paper, a method to reduce the dimension by approximating the curved surface of the velocity component as a simpler method for calculating the transit orbit is obtained. It was found that the transit orbit can be designed by the proposed method. In addition, it was found that the Jacobi constant is greatly related to the characteristics of the transit orbit, and the trajectory can be selected by adjusting the Jacobi constant.

From these results, it is considered that the future prospects can be applied to deep space exploration missions using low-energy trajectories. On the other hand, some problems became clear. First, the invariant manifold used to obtain the curved surface is associated with a certain halo orbit, but the invariant manifold of the halo orbit shows only a part of the whole transit orbits, so that better transit orbit might exist. For this reason, it is necessary to examine the distribution of velocity components that become transit orbits using invariant manifolds corresponding to whole halo orbit family. Also it is necessary to relax the assumption of fixed initial position inside the unstable manifold to consider a method that can design transit orbits with minimum thrust regardless of states of the spacecraft.

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