Flutter LCO in Isentropic Flow: Analytical Theory

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Abstract: Using full continuum models, we establish purely theoretically that in two-dimensional isentropic flow, the flutter speed for a slender high-aspect ratio wing is a Hopf bifurcation point of the aeroelastic structure dynamics which can be expressed as a non-linear convolution evolution equation. The flutter speed is determined by the linearized model and the LCO is periodic with period $\left(\frac{2\pi}{\omega}\right)$where ω is the angular flutter frequency in the linear model, and can be expressed as a harmonic series.

Introduction

Using full continuum models, we establish purely theoretically that in two-dimensional isentropic flow, the flutter speed for a slender high-aspect ratio wing is a Hopf bifurcation point of the aeroelastic structure dynamics which can be expressed as a non-linear convolution evolution equation. The flutter speed is determined by the linearized model and the LCO is periodic with period $\left(\frac{2\pi}{\omega}\right)$where ω is the angular flutter frequency in the linear model, and can be expressed as a harmonic series.

We have taken some pains to describe the model in enough detail since continuum models are rare. Because of the page limitation we have had to omit all details of proofs of results.

The structure model, which goes back to Goland [1], is described in section 2. Of course, the main simplification is to neglect camber, but it is not expected that this significantly alters the qualitative nature of the results and certainly not the flutter speed, which is based on the linearized model. In section 2 we also describe the isentropic flow model and the boundary conditions in more detail than has been done in the standard texts on aeroelasticity [2, 3].

Some attention is paid to the linearized model in section 3, in particular to the role of the Possio equation, which is practically ignored in [3]. The importance of the linear model is that the solution to the non-linear problem can be boot-strapped on the linear, as we show in section 4. It is shown that the aeroelastic structure equation can be described as a non-linear convolution-evolution equation, for fixed M, with U as the speed parameter for which the Hopf bifurcation theory applies. The flutter LCO is not sinusoidal but periodic, with period $\left(\frac{2\pi}{\omega}\right)$ where ω is the angular flutter frequency determined from the linear model, and is expressed as a harmonic series.

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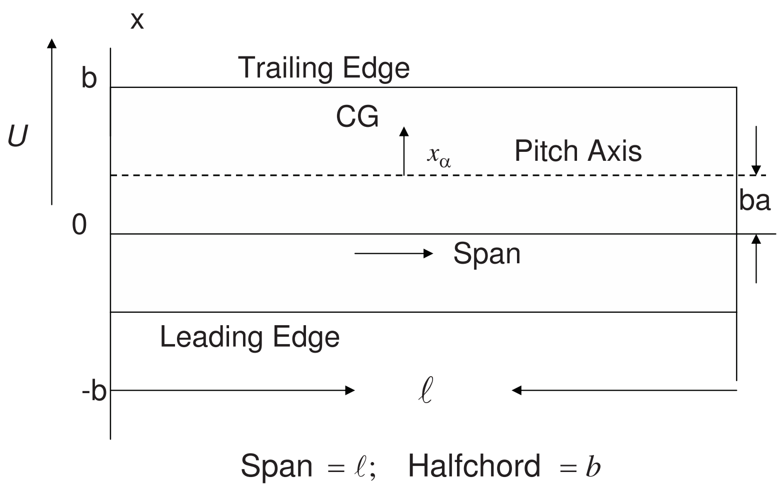

Figure 1: Wing Structure Beam Model

1 The Structure Model

The earliest model of a wing structure stated in terms of a partial differential equation would appear to be that of Goland [1, 2], which utilizes a uniform slender (i.e. zero thickness disregarding camber) rectangular 'beam' model with two degrees of freedom—plunging (beam bending) and pitching (beam torsion)—a cantilever beam attached to the fuselage and free at the other end. With $h(t, y)$ denoting the displacement normal to the structure plane and $\theta(t, y)$ denoting the pitch angle about an axis parallel to the y-axis (see figure 1),

$$-b < x < b;\ 0 < y < \ell < \infty,\ t > 0$$

the structure dynamics can be described as:

$$\left.\begin{array}{r} m\ddot{h} + S\ddot{\theta} + EIh'''' = L(t, y) \\ I_\theta\ddot{\theta} + S\ddot{h} - GJ\theta'' = M(t, y) \end{array}\right\}, \quad 0 < y < \ell \tag{1.1}$$

where prime denotes derivative with respect to the y-variable, with appropriate boundary conditions (cantilever or free-free). The forcing functions on the right-hand side, the lift $L(t, \cdot)$ and the moment $M(t, \cdot)$, are determined by the aerodynamics model described in the next section, and will depend on the structure dynamic variables $h(\cdot)$ and $\theta(\cdot)$.

We can also add a control term as in [4] but the emphasis in this paper is on the aerodynamics, by far the more complicated part.

2 The Aerodynamic Model

The basic references here are [5, 6, 7]. The airflow is described in terms of Eulerian dynamics where q is the 3×1 flow vector

$$q(t, x, y, z) \quad t > 0 \quad x, y, z \in R^3$$

supplemented by the positive-valued thermodynamical variables:

Pressure p,	$p(t,x,y,z) \geq 0$
Density ρ,	$\rho(t,x,y,z) \geq 0$
Temperature T,	$T(t,x,y,z) \geq 0$

We may also include the entropy $S(t,x,y,z)$. Any two of the variables will determine the other two. A basic assumption is the Perfect Gas Law,

$$p = \rho R T$$
$$R = c_p - c_v, \quad \gamma = \frac{c_p}{c_v} > 1$$

where $c_p,\ c_v$ denote specific heat at constant pressure and volume, respectively. Far field values ($|z|+|y|+|x| \to \infty$) will be denoted $q_\infty,\ p_\infty,\ \rho_\infty,\ T_\infty$ and assumed finite.

2.1 The Field Equations

All relations we need to describe the dynamics are to be deduced from two laws:
1. Law of Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho q) = 0 \tag{2.1}$$

2. Law of Conservation of Momentum:

$$\rho \frac{Dq}{Dt} + \nabla p = \lambda \Delta q + (\lambda + \mu/3)\nabla(\nabla \cdot q) \tag{2.2}$$

where μ, λ are constants (may depend on T) describing the fluid (air). μ is the shear viscosity and λ is the bulk viscosity. To this we must add an "energy equation" [5, p. 33] which we shall need to discuss more below. We note that μ is very small for perfect gases. In air, $\mu = 1.85 \times 10^{-5}$kg/m/s, $\lambda = 0.6\mu$. The question of smallness of μ, λ has to be ultimately referred to the Reynolds number [see 5, 6].

2.2 Aeroelastic Boundary Conditions

In viscous flow, the boundary condition on the wing boundary is characterized by

$$q(t,x,y,0) = q_\infty(t,x,y,0) + k\frac{Dz}{Dt}, \quad |x| < b,\ 0 < y < \ell \tag{2.3}$$

where z is the wing displacement: in the direction normal to the wing,

$$z(t) = h(t,y) - (x-a)\theta(t,y), \quad 0 < y < \ell,\ |x| < b. \tag{2.4}$$

All we are interested to obtain from the flow is the pressure differential over the wing

$$\delta p(t,x,y) = p(t,x,y,0+) - p(t,x,y,0-)$$

from which we calculate what we need in (1.1), the lift and monent:

$$L(t,y) = \int_{-b}^{b} \delta p(t,x,y)dx, \quad 0 < y < \ell \tag{2.5}$$

$$M(t,y) = \int_{-b}^{b} (x-a)\delta p(t,x,y)dx, \quad 0 < y < \ell. \tag{2.6}$$

It is convenient to consider $\frac{Dz(t)}{Dt)}$ as the "input" and $\delta p(t, \cdot)$ as the "output"—relating the Lagrangian dynamics of wing structure to the Eulerian flow dynamics. q_∞ is the air speed, the far-field $(|x|, |y|, |z| \to \infty)$ flow,

$$q_\infty = U(i \cos\alpha + j \cos\beta + k \cos\gamma)$$

in the usual way. α is the "angle of attack".

2.3 Isentropic Flow

Our first simplification is to consider the non-viscous case

$$\mu = 0 = \lambda \tag{2.7}$$

where the flutter phemonema are not lost. However, we need to invoke a thermodynamic assumption, that the entropy $S(t, x, y, z)$ is constant, and thus the flow is "isentropic". This is a remarkably simplifying assumption that makes the flow irrotational. For this however we need to invoke the Gibbs relation,

$$T\nabla S + \frac{\nabla p}{\rho} = \nabla(c_p T) \tag{2.8}$$

By the Perfect Gas Law,

$$\nabla(c_p T) = \frac{c_p}{R}\nabla(p/\rho)$$

and hence

$$\frac{\nabla p}{\rho} = \frac{c_p}{R}\left[\frac{\nabla p}{\rho} - \frac{p}{\rho^2}\nabla\rho\right],$$

relating the pressure to density. With

$$\gamma = \frac{c_p}{c_v} > 1,$$

this yields

$$\nabla \log\left(\frac{p}{\rho^\gamma}\right) = 0$$

or

$$p = A\rho^\gamma, \tag{2.9}$$

where A is a constant. Now by definition

$$\frac{dp}{d\rho} = a_\infty^2$$

where a_∞ is the speed of sound, and we have that

$$\gamma\frac{p_\infty}{\rho_\infty} = a_\infty^2$$

where p_∞, ρ_∞ are the undisturbed or far-field values of pressure and density assumed constant.

Getting back now to the inviscid version of the momentum conservation law, we have

$$\frac{\partial q}{\partial t} + (q \cdot \nabla)q + \frac{\nabla p}{\rho} = 0. \tag{2.10}$$

Hence with

$$\Omega = \nabla \times q$$

we have

$$\frac{\partial \Omega}{\partial t} + \nabla \times (q \cdot \nabla)q = 0.$$

Using the identity

$$(q \cdot \nabla)q = \frac{1}{2}\nabla||q||^2 - q \times \Omega$$

we obtain

$$\frac{\partial \Omega}{\partial t} + \nabla \times (q \times \Omega) = 0, \quad t \geq 0.$$

For given q, we may consider this as a linear equation

$$\dot{\Omega} = L(t)\Omega$$

where

$$\Omega(0) = 0.$$

Hence it follows that

$$\Omega(t) = 0$$

or

$$\nabla \times q = 0.$$

Hence

$$q = \nabla\phi \tag{2.11}$$

where ϕ is the velocity potential, and we have "potential flow". The point to be noted here is that we do not invoke Crocco's Theorem as in [8]. This idea is borrowed from [6, p. 71]. Note that we can have isentropic flow which is rotational depending on the initial flow (see [5, p. 24], for more).

Hence

$$\frac{\partial q}{\partial t} + \frac{1}{2}\nabla||q||^2 + \frac{\Delta p}{\rho} = 0$$

or

$$\nabla\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho}\right] = 0.$$

Hence

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} = \text{Far Field Values} = \frac{1}{2}U^2 + \frac{p_\infty}{\rho_\infty}$$

$$U = ||q_\infty||$$

and

$$\frac{p}{\rho} = A\rho^{\gamma-1} = \frac{a_\infty^2}{\gamma - 1}\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}$$

or

$$\rho^{\gamma-1} = \rho_\infty^{\gamma-1}\frac{(\gamma-1)}{a_\infty^2}\left[\frac{1}{2}U^2 + \frac{a_\infty^2}{\gamma-1} - \frac{\partial\phi}{\partial t} - \frac{1}{2}|\nabla\phi|^2\right].$$

We can now invoke the law of Conservation of Mass and obtain

$$\begin{aligned}\frac{\partial}{\partial t}\rho^{\gamma-1} &= (\gamma-1)\rho^{\gamma-2}\frac{\partial\rho}{\partial t}\\ &= (\gamma-1)\rho^{\gamma-2}[\nabla\cdot\rho\nabla\phi].\end{aligned}$$

After a little analysis this leads to the Euler Full Potential Equation for the velocity potential ϕ

$$\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}|\nabla\phi|^2 = a_\infty^2\left(1+\frac{\gamma-1}{a_\infty^2}\Big(\frac{U^2}{2} - \frac{\partial\phi}{\partial t} - \frac{|\nabla\phi|^2}{2}\Big)\right)\nabla^2\phi - \nabla\phi\cdot\nabla\frac{|\nabla\phi|^2}{2}. \quad (2.12)$$

The main thing to note in this equation in contrast to the Navier-Stokes is that there are no (spatial) second derivatives of the flow velocity $\nabla\phi$. Because of this the boundary condition (2.3) is now simplified to "no slip" on the boundary flow, or

$$k\cdot\nabla\phi = \nabla\phi_\infty\cdot k + \frac{Dz}{Dt} \text{ on } z=0,\ |x|<b,\ 0<y<\ell.$$

Unfortunately this is not enough for uniqueness of solution. For that, we have to add

$$\delta p = 0,\ z=0,\ |x|>b,\ y>\ell,\ y<0$$

and the Kutta condition

$$\delta p = 0, \quad z=0,\ x\to b-.$$

We still need to show how to calculate δp from the flow equation. Let ψ denote the acceleration potential

$$\psi = \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2.$$

Then

$$p = \frac{\rho_\infty a_\infty^2}{\gamma}\left(1+\frac{\gamma-1}{a_\infty^2}\Big(\frac{1}{2}U^2-\psi\Big)\right)^{\frac{\gamma}{\gamma-1}},$$

which is usually simplified to

$$p = \frac{\rho_\infty a_\infty^2}{\gamma}\left(1+\frac{\gamma}{a_\infty^2}\Big(\frac{1}{2}U^2-\psi\Big)\right)$$

so that at $z=0$

$$\delta p = -\rho_\infty\delta\psi.$$

Again we may think of $\frac{Dz}{Dt}$ as the input and δp as the output. We shall show how this connection is provided by the Possio Integral Equation [9]. We only consider the subsonic case:

$$\frac{U}{a_\infty} < 1.$$

3 Linear Aeroelasticity

We specialize from now on to 2D, or Typical Section (Airfoil) Theory, where we drop the dependence on the y-coordinate but only in the aerodynamic flow equation. In particular,

$$\phi_\infty = U(x\cos\alpha + z\sin\alpha)$$

where α is the angle of attack.

Our focus is on the question of stability about the 'equilibrium'—steady or time-invariant—state. We can readily verify that

$$\phi = \phi_\infty$$

$$\theta = 0; \quad h = 0$$

is a time-invariant solution of the aeroelastic equations, where U, α are totally arbitrary. There are other time-invariant solutions but only for a discrete sequence of values of U (see [10]), which we shall not consider here.

3.1 Linearization

It is natural to begin with the aeroelastic system linearized about the equilibrium state because stability is completely determined by the linearized system. For this purpose we exploit the unique feature of the problem in the boundary conditions

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi_\infty}{\partial z} + \frac{Dz(t)}{Dt}$$

where

$$\begin{aligned}\frac{Dz(t)}{Dt} &= -\dot{h}(t,y) - (x-a)\dot{\theta}(t,y) + \theta(t,y)\frac{\partial\phi}{\partial x}, \quad z=0,\ |x|<b \\ &= w_a(t,x),\ \text{the downwash for fixed } y\end{aligned}$$

and, as far as the flow is concerned, the structure state variables are just scalar parameters for fixed y. Hence for each $t > 0$ we may start by assuming that the solution is analytic in them, in some neighborhood of the zero structure state. Thus let $\phi(\lambda,t,x,z)$ denote the solution corresponding to $\lambda\theta(t,y)$, $\lambda h(t,y)$, for scalar λ, with

$$\phi(0,t,x,z) = \phi_\infty(x,z)$$

and

$$\frac{\partial\phi(\lambda,t,x,0)}{\partial z} = U\sin\alpha - \lambda\Big(\dot{h}(t,y) + (x-a)\dot{\theta}(t,y)\Big) - \lambda\theta(t,y)\frac{\partial\phi}{\partial x}(\lambda,t,x,0), \quad |x|<b,$$

and $\phi(\lambda,t,x,z)$ satisfies the Full Potential Equation in 2D. We assume the power series expansion

$$\phi(\lambda,t,x,z) = \sum_1^\infty \frac{\lambda^k}{k!}\phi_k(t,x,z) + \phi_\infty(x,z), \quad -\infty < x < \infty,\ z \neq 0 \tag{3.1}$$

where

$$\phi_k(t,x,z) = \left.\frac{\partial^k \phi(\lambda,t,x,z)}{\partial \lambda^k}\right|_{\lambda=0} \tag{3.2}$$

for each $t > 0,\ -\infty < x, z < \infty$ excepting $z = 0,\ |x| > b$, for $|\lambda| < R,\ 0 < R$.

Note that the no-slip boundary condition can be stated

$$\frac{\varphi(\lambda,t,x,0)}{\partial z} = -\lambda[\dot{h}(t,y) + (x-a)\dot{\theta}(t,y)] - \lambda U\theta(t,y)\cos\alpha - \lambda\theta(t,y)\frac{\partial\varphi}{\partial x},$$

$$z = 0,\ |x| < b \tag{3.3}$$

and the 2D potential field equations can be expressed:

$$\frac{\partial^2\phi(\lambda,\cdot)}{\partial t^2} + \frac{\partial}{\partial t}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + (\gamma-1)\frac{\partial\phi}{\partial t}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$= a_\infty^2\left[1 + \frac{\gamma-1}{2a_\infty^2}\left(U^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2\right)\right]\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$-\frac{1}{2}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] - \frac{1}{2}\frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] \tag{3.4}$$

To obtain the $\phi_k(t,\cdot,\cdot)$, we differentiate (3.4) with respect to λ and set $\lambda = 0$.

3.2 The Linear Problem

For $k = 1$ we obtain the linearized field equation

$$\frac{\partial^2\varphi_1}{\partial t^2} + 2U\cos\alpha\frac{\partial^2\varphi_1}{\partial t\partial x} + 2U\sin\alpha\frac{\partial^2\varphi_1}{\partial t\partial z}$$

$$= a_\infty^2\left[(1 - M^2\cos^2\alpha)\frac{\partial^2\varphi_1}{\partial x^2} + (1 - M^2\sin^2\alpha)\frac{\partial^2\varphi_1}{\partial z^2}\right.$$

$$\left.-2M^2\sin\alpha\cos\alpha\frac{\partial^2\varphi_1}{\partial x\partial z}\right] \tag{3.6}$$

omitting the airfoil $|x| < b,\ z = 0$.

From (3.4), we obtain that the no-slip boundary condition becomes

$$\frac{\partial\varphi_1(t,x,0)}{\partial z} = -\Big(\dot{h}(t,y) + (x-a)\dot{\theta}(t,y)\Big) - \theta(t,y)U\cos\alpha \tag{3.6}$$

with $\psi(\lambda,t,x,z)$ defined by

$$\psi(\lambda,t,x,z) = \frac{\partial\varphi(\lambda,t,x,z)}{\partial t} + \frac{1}{2}|\nabla\phi(\lambda,t,x,z)|^2$$

$$\delta p(\lambda,t,x) = -\rho_\infty\delta\psi(\lambda,t,x), \quad |x| < b.$$

Defining

$$\delta p_1(t,x) = \frac{\partial}{\partial \lambda}\delta p(\lambda,t,x)\bigg|_{\lambda=0} \tag{3.7}$$

we have

$$\delta p_1(t,x) = -\rho_\infty \delta\psi_1(t,x)$$

where

$$\begin{aligned}\psi_1(t,x,z) &= \frac{\partial}{\partial\lambda}\psi_1(0,t,x,z)\\ &= \frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_1}{\partial x}U\cos\alpha + \frac{\partial\varphi_1}{\partial z}U\sin\alpha\end{aligned}$$

and

$$\begin{aligned}\delta\psi_1(t,x) &= 0, \quad x \to b-\\ &= 0, \quad |x| > b.\end{aligned}$$

Hence the linear problem is given by (3.5) with boundary conditions, with the structural equation (1.1) with

$$\delta p_1(t,x) = -\rho_\infty \delta\psi_1(t,x)$$

$$L(t,y) = \int_{-b}^{b} \delta p_1(t,x)dx \tag{3.8}$$

$$M(t,y) = \int_{-b}^{b} (x-a)\delta p_1(t,x)dx \tag{3.9}$$

Let

$$\begin{aligned}A_1(t,x) &= -\frac{\delta\psi_1}{U}(t,x), \quad |x| < b\\ &= \frac{\delta p_1(t,x)}{U\rho_\infty}\end{aligned}$$

which is the Kussner pressure doublet function [2]. Then the Possio equation relates the 'input' $w_a(t,\cdot)$ to the 'output' $\delta p_1(t,\cdot)$, linking the Lagrangian dynamics to the Eulerian. Extant treatises on aeroelasticity [e.g. 3] end at approximately this point.

3.3 The Possio Equation: Zero Angle of Attack

To reduce complexity we shall only consider the case $\alpha = 0$, referring to [8, 11] for non-zero angle of attack. We shall also need to state it for more general 'down-wash' functions than (3.6), subject to the condition

$$w_a(t,\cdot) \in L_p[-b,b], \quad 1 \le p < 2$$

and $w_a(t,\cdot)$ is absolutely continuous in $t \ge 0$. The Possio equation is usually stated in terms of the Laplace transform (actually the Fourier transform; see [12] for the time domain version). The Possio equation is

$$\hat{w}_a(\lambda,x) = \int_{-b}^{b} \hat{P}(\lambda,x-\xi)\hat{A}(\lambda,\xi)d\xi, \quad |x| < b,\ \mathrm{Re}\,\lambda > 0 \tag{3.10}$$

where the kernel is given in terms of its spatial Fourier transform:

$$\begin{aligned}\hat{P}(\lambda, i\omega) &= \int_{-\infty}^{\infty} e^{-i\omega x}\hat{P}(\lambda, x)dx, \quad -\infty < \omega < \infty \\ &= \frac{1}{2}\frac{1}{\kappa + i\omega}\sqrt{\kappa^2 M^2 + 2\kappa M^2 i\omega + (1-M^2)\omega^2}, \quad \kappa = \frac{\lambda b}{U} \qquad (3.11)\\ &= \int_0^{\infty} e^{-\lambda t}P(t, i\omega)dt, \quad \mathrm{Re}\lambda > 0\end{aligned}$$

where

$$A(t,x) \to 0 \text{ as } x \to b-$$

$$A(t,\cdot) \in L_p[-b,b], \quad 1 \le p < 2,$$

absolutely continuous in $[0,\infty]$ with $\dot{A}(t,\cdot) \in L_p[-b,b]$ also.

$$\hat{A}(\lambda,\cdot) = \int_0^{\infty} e^{-\lambda t}A(t,\cdot)dt$$

$$\hat{w}_{a,1}(\lambda, x) = \int_{-b}^{b} \hat{P}(\lambda, x-\xi)\hat{A}_1(\lambda,\xi)d\xi, \quad \mathrm{Re}\,\lambda > 0,\ |x| < b$$

where

$$w_{a,1}(t,x) = -\dot{h}(t,y) - (x-a)\dot{\theta}(t,y) - \theta(t,y)U\cos\alpha.$$

Because of the similar property required of the structure state variables we have that

$$\int_0^{\infty} e^{-\sigma t}\Big(||A_1(t,\cdot)||_p + ||\dot{A}_1(t,\cdot)||_p\Big)dt < \infty \qquad (3.12)$$

Moreover, with

$$\hat{\varphi}_1(\lambda, x, z) = \int_0^{\infty} e^{-\lambda t}\varphi_1(t,x,z)dt, \quad \mathrm{Re}\,\lambda > 0$$

with the $L_p - L_q$ transform

$$\hat{\hat{\varphi}}_1(\lambda, i\omega, z) = \int_{-\infty}^{\infty} \hat{\varphi}_1(\lambda, x, z)e^{-i\omega x}dx, \quad -\infty < \omega < \infty$$

we have

$$\hat{\hat{\varphi}}_1(\lambda, i\omega, z) = \frac{-1}{2}\cdot\frac{1}{\kappa + i\omega}\hat{\hat{A}}_1(\lambda, i\omega)e^{-\sqrt{M^2\kappa^2 + 2M^2\kappa i\omega + (1-M^2\omega^2}|z|}, \quad -\infty < \omega < \infty. \qquad (3.13)$$

To obtain the time-domain version, let

$$\begin{aligned}\gamma_1(t,x) &= \int_0^t A_1(t-\sigma, x-U\sigma)dt, \quad -b < x < b+Ut \qquad (3.14)\\ &= 0 \text{ otherwise.}\end{aligned}$$

Then (see [13]), for $z \neq 0$,

$$\varphi_1(t,x,z) = \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial z} G(t-\sigma, x-\xi, z)\dot{\gamma}_1(\sigma,\xi)d\xi \; d\sigma, \tag{3.15}$$

where

$$G(t,x,z) = \frac{1}{2\pi\sqrt{1-M^2}} \int_1^{\frac{1}{r}(t+\frac{Ux}{{c_1}^2})} \frac{d\sigma}{\sqrt{\sigma^2-1}}$$

$$r^2 = \frac{1}{(1-M^2)}\left(\frac{x^2}{c_1^2} + \frac{z^2}{c_2^2}\right)$$

$$c_1^2 = a_\infty^2(1-M^2), \quad c_2^2 = a_\infty^2$$

which is the potential flow solution to the linear case.

Stability of the linearized aeroelastic system is then determined by:

$$m\ddot{h} + S\ddot{\theta} + EIh'''' = \int_{-b}^{b} U\rho_\infty A(t,\xi)d\xi, \quad t>0,\ 0<y<\ell$$

$$I_\theta\ddot{\theta} + S\ddot{h} + GJ\theta'' = \int_{-b}^{b} U\rho_\infty (x-a)A(t,x)dx, \quad t>0,\ 0<y<\ell$$

See [14] for a solution and the detailed study of flutter instability speeds as a function of M.

For the non-linear problem, we need the time-domain solution of (3.10). This is best expressed in operator form

$$A(t,\cdot) = P\mathcal{T}w_a(\cdot,\cdot)$$

where $\mathcal{T}$ is the Tricomi operator

$$\mathcal{T}f = g \quad g(x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}} \int_{-b}^{b} \sqrt{\frac{b+\xi}{b-\xi}} \frac{f(\xi)}{\xi - x} d\xi, \quad |x| < b$$

and P is a Volterra operator of the form

$$PA = g; \quad g(t,\cdot) = \int_0^t P(t-\sigma)A(\sigma,\cdot)d\sigma.$$

The kernel is known explicitly only for $M=0$ and contains delta function derivatives [9], but only δ-functions for $M \neq 0$ (see [14].

4 Flutter as an LCO

Here we begin with the key result [18], the solution of the non-linear Possio equation. For each $y,\ 0<y<z$:

$$\delta p = \rho_\infty U\Big(I - P\mathcal{T}L(\theta)\Big)^{-1} P\mathcal{T}w_{a,1}(\cdot,\cdot) \tag{4.1}$$

where $w_{a,1}$ is the linearized downwash and $L(\theta)$ is the operator defined by

$$L(\theta)A = g$$

$$g(t,\cdot) = \theta(t,y)\Big(- A(t,\cdot) + \dot{\gamma}_1(t,\cdot)\Big) \tag{4.2}$$

We can now state the non-linear aeroelastic system equations. Let $\ell_1,\ \ell_2$ denote the functionals corresponding to lift and moment.

$$\ell_1(A) = \int_{-b}^{b} A(x)dt$$

$$\ell_2(A) = \int_{-b}^{b} (x - ab)A(x)dx.$$

Then we have

$$m\ddot{h}(t,y) + S\ddot{\theta}(t,y) - EIh''''(t,y) = \ell_1(\delta p) \tag{4.3}$$

$$I_\theta\ddot{\theta}(t,y) + S\ddot{h}(t,y) + GJ\theta''(t,y) = \ell_2(\delta p) \tag{4.4}$$

where $\delta(p)$ is given by (4.2). This can be expressed as a non-linear convolution-evolution equation in a Hilbert space for each M with the speed U as a parameter to which Hopf-bifurcation theory applies. We naturally omit the details. The flutter speed is determined by the linearized model as in [14]. Let ω denote the corresponding angular frequency in the linear model, with structure response

$$x_1(t,y) = \sin\omega t \begin{vmatrix} h(0,y) \\ \theta(0,y) \end{vmatrix} \quad 0 < y < \ell$$

being the solution to (4.3), (4.4) with

$$\begin{aligned} \delta p &= \delta p_1 \\ &= \rho_\infty U P \mathcal{T} w_{a,1}(\cdot). \end{aligned}$$

More generally we define

$$\delta p_k = \rho_\infty U \Big(P\mathcal{T}L(\theta_1(\cdot))\Big)^k P\mathcal{T} w_{a,1}(\cdot)$$

and $x_k(\cdot,\cdot)$ the solution to (4.3), (4.4), with

$$\delta p = \delta p_k.$$

Then the LCO can be expressed

$$\sum_1^\infty x_k(t,y)$$

which is no longer sinusoidal, but is a harmonic series with period $\left(\frac{2\pi}{\omega}\right)$. We omit details.

As for the corresponding flow solution, it is shown in [15] that it can be decomposed into the sum of two parts, one part which has no shocks and is solely responsible for the lift, and the second part which may contain shocks but produces no lift and cannot be linearized.

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