Dynamical Systems Approach to Optimization Problems — Optimization through Bifurcation —

Kazuo Tsuchiya
Graduate School of Engineering, Kyoto University,
Yoshida-Honmachi, Sakyo-ku, Kyoto city, Kyoto 606-8501, Japan

Abstract

This presentation introduces a heuristic method of optimization problem. The method which utilizes the successive bifurcations of replicator equations is explained. The performance of the method is demonstrated by applying to a combinatorial optimization problem, a quadratic assignment problem (Q.A.P.).

1 Introduction

One of the objectives of engineering science is to develop the methodology of system design. System design is to determine the structure of the system which realizes the given function. Here, the structure means the connections between the elements in the system. The computational theory of system design is the theory of optimization. The optimization problem is formulated as follows;

$$\text{Minimize } L(x_i) \quad (i = 1, \cdots, N) \tag{1}$$

$$\text{subject to } g_k(x_i) \leq 0 \quad (k = 1, \cdots, M)$$

where function $L(x_i)$ is called the objective function and variables x_i are called the decision variables. The functions g_k express the constraints on the problem. The methods of optimization are classified into two groups. One is the exact method which guarantees the optimality of the solution. The other is the heuristic method which may find a good solution with a feasible computational cost. Now, we must deal with a complex optimization problem. Complex means that the size of the problem is large and the imposed constraints are complicated. Design of supersonic transport is one of the examples. At that time, from the practical view point, it becomes well suited to obtain an approximate solution with a feasible computational cost [1]. The heuristic methods are classified into two groups; one is the stochastic method and the other is the deterministic method. For the algorithms of the stochastic method, there are the simulated annealing algorithm and the genetic algorithm, etc.. The deterministic method was first formulated from the stochastic method by the use of the mean field approximation. The basic idea of the heuristic method is as follows; first, an energy function is composed of the objective function and the constraints,

$$E(x_i) = L(x_i) + \sum_{k=1}^{M} \lambda_k g_k(x_i) \tag{2}$$

where λ_k are the constants. A dynamic system of the variables x_i is constructed as a gradient vector field of the energy function $E(x_i)$

$$\dot{x}_i = -\frac{\partial}{\partial x_i} E(x_i). \tag{3}$$

The approximate solution of the problem is obtained as an equilibrium point of Eq. (3). To improve the performance of the solution, the annealing algorithm (deterministic annealing) is applied. The deterministic annealing is to vary the parameter in the system slowly. Usually, the rate of variation (Annealing schedule) is determined empirically. We have proposed another model of the deterministic method of optimization problem. As the dynamic system, we adopt a replicator system instead of a gradient system and the annealing schedule is determined based on the bifurcation characteristics of the replicator equation.

In the following, the basic characteristics of the replicator equation is explained briefly in Sec. 2 and the proposed method of optimization proposed is explained with numerical examples in Sec. 3.

2 The Replicator Equation [2]

The replicator equation is the equation where the derivative of the variables is proportional to the state of the variable. Here, we use the following simple model,

$$U^T = (u_1, \cdots, u_N) \tag{4}$$

$$\dot{u}_i = f_i(u_j; \lambda_i, \alpha) u_i$$

$$f_i(u_j; \lambda_i, \alpha) = (\lambda_i - u_i^2) - \alpha \sum_{j \neq i}^{N} u_j^2$$

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N > 0, \ \alpha > 0$$

The proportional coefficient (growth rate) f_i is composed of two terms; the first one expresses the self activatory and inhibitory influences, and the second one expresses the mutual inhibitory interactions between elements. There are three types of equilibrium solution. The first one is the uniform solution U_{uni} where all the elements take non zero values, the second one is the transient solution $U^{(k)}_{\mathrm{trans}}$ where the values of the last k elements are zero, and the third one is the feasible solution $U^{(i)}_{\mathrm{feas}}$ where the only ith element has non zero value; that is,

$$U_{\mathrm{uni}} = (*, *, \cdots, *) \tag{5}$$

$$U^{(k)}_{\mathrm{trans}} = (*, \cdots, *, \overbrace{0, \cdots, 0}^{k})$$

$$U^{(i)}_{\mathrm{feas}} = (0, \cdots, 0, \overset{\overset{i}{\vee}}{*}, 0, \cdots, 0)$$

where $*$ means a non zero value.

The stability of Sols. U_{uni} and $U^{(i)}_{\mathrm{feas}}$ are as follows; in the region where α is small, Sol. U_{uni} is the only stable equilibrium solution whereas in the region where α is large, Sols. U_{feas} are the only stable equilibrium solutions. And, the bifurcation characteristics of the branch connected to Sol. U_{uni} are as follows; when parameter α increases, Sol. U_{uni} connects with Sol. $U^{(1)}_{\mathrm{trans}}$ through the pitchfork bifurcation and then, finally, connects with Sol. $U^{(1)}_{\mathrm{feas}}$, where Sol. $U^{(1)}_{\mathrm{feas}}$ is the solution in which the only element with the largest value of λ has a non zero value (Fig. 1), where α_U is the point at which Sol. U_{uni} becomes unstable through the bifurcation and $\alpha_F^{(i)}$ is the point at which Sol. $U^{(i)}_{\mathrm{feas}}$ becomes stable through the bifurcation.

It should be noted that, *from the view point of optimization, this process can be thought to be a process which searches the element with the largest value of* λ.

The optimization method which utilizes the bifurcation characteristics of the replicator equation has been proposed. It will be explained in Sec. 3.

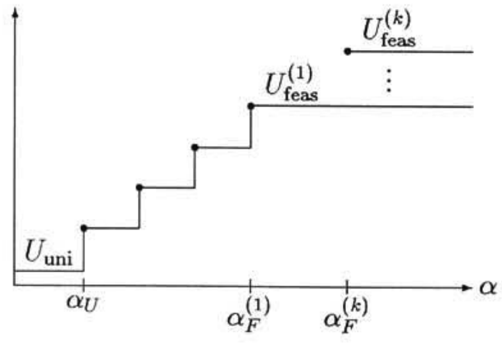

Fig. 1. Schematic diagram of bifurcation of Eq. (4)

3 Optimization Method [3, 4]

The optimization problem is classified into two groups, the nonlinear optimization problem and the combinatorial optimization problem. The former is the problem where the decision variables are continuous whereas the latter is the problem where the decision variables are discrete. Here, the optimization method which utilizes the bifurcation characteristics of a replicator equation is explained for the combinatorial optimization problem. It should be mentioned that this method can be also successfully applied to a nonlinear optimization problem. One of the examples of combinatorial optimization problem is the quadratic assignment problem (Q.A.P.). Q.A.P. is formulated as follows;

$$\text{Minimize } L(X) = \text{trace}(A^T X^T B X) \quad (6)$$

where X is a $N \times N$ permutation matrix, A, B are given $N \times N$ matrices. For the problem, we set the following replicator equation

$$U = [u_{ij}] \quad (i, j = 1, \cdots, N) \qquad (7)$$

$$\dot{u}_{ij} = f_{ij}(u_{i'j'}; \alpha_0, \alpha_1) u_{ij}$$

The growth rate f_{ij} is determined so that matrix U asymptotically converges to matrix X with a small value of the objective function $L(X)$. The growth rate is designed as follows;

$$\begin{aligned} f_{ij} = (1 - u_{ij}^2) &- \frac{\alpha_0}{2}\left(\sum_{i' \neq i} u_{i'j}^2 + \sum_{j' \neq j} u_{ij'}^2\right) \\ &- \frac{\alpha_1}{2}\sum_{i',j'}(a_{jj'}b_{ii'} + a_{j'j}b_{i'i})u_{i'j'}^2 \qquad (8) \\ &(i, j = 1, \cdots, N), \end{aligned}$$

The growth rate is composed of three terms; the first one expresses a self activatory and inhibitory influence and the second one expresses mutual inhibitory interactions between the elements in the same row and column. The third one expresses the inhibitory influence due to the objective function. There are three types

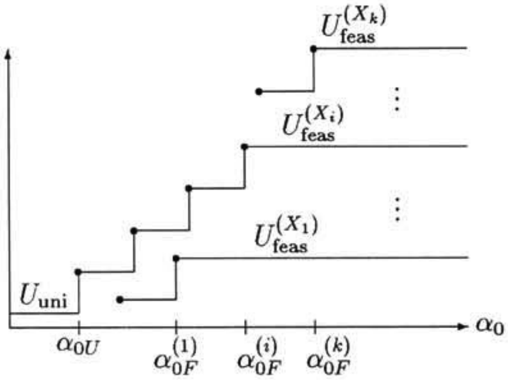

Fig. 2. Schematic diagram of bifurcation of Eq. (7)

of equilibrium solution. The first one is the uniform solution U_{uni} where all the elements take non zero values and the second one is the transient solution $U_{\text{trans}}^{(k)}$ where k elements take zero values. The third one is the feasible solution $U_{\text{feas}}^{(X)}$ where only the one element in a row and a column takes a non zero value. A feasible solution $U_{\text{feas}}^{(X)}$ corresponds to a permutation matrix X. The stabilities of Sol. U_{uni} and $U_{\text{feas}}^{(X)}$ are as follows; in the region where α_0 is small, Sol. U_{uni} is the only stable equilibrium solution whereas, in the region where α_0 is large, Sol. $U_{\text{feas}}^{(X)}$ are only the stable equilibrium solutions. And, the bifurcation characteristics of the branch connected to Sol. U_{uni} are as follows; when parameter α_0 increases, Sol. U_{uni} connects with Sol. $U_{\text{trans}}^{(1)}$ through the pitchfork bifurcation and then, finally connects with Sol. $U_{\text{feas}}^{(X_i)}$ where Sol. $U_{\text{feas}}^{(X_i)}$ corresponds to permutation matrix X_i which gives the ith smallest value of the objective function $L(X)$. Indeed, Sol. $U_{\text{feas}}^{(X_i)}$ is not the optimum solution but, in many cases, is a good approximate solution (Fig. 2), where α_{0U} is the point at which Sol. U_{uni} becomes unstable through the bifurcation and $\alpha_{0F}^{(i)}$ is the point at which Sol. $U_{\text{feas}}^{(i)}$ becomes stable through the bifurcation.

Based on the analysis, the optimization algorithm has been proposed;

1. Trace the branch of Sol. U_{uni} by increasing

α_0 slowly.

2. Accept Sol. $U_{\text{feas}}^{(X_i)}$ obtained as an approximate solution of the problem.

In order to improve the performance of the solution, the parameter α_0 must be increased very slowly near the bifurcation points. The deterministic annealing algorithm is designed as follows; Function S, the entropy over the set of the solutions, is introduced such that

$$S = -\frac{1}{N \ln N} \sum_{i,j} p_{ij} \ln p_{ij} \qquad (9)$$

$$p_{ij} = \frac{u_{ij}^2}{\sum_{j'} u_{ij'}^2}$$

The annealing schedule is given as

$$\Delta\alpha_0 = \left(\frac{\Delta S^{\mathrm{d}}}{\Delta S^{\mathrm{old}}}\right) \Delta\alpha_0^{\mathrm{old}} \qquad (10)$$

where $\Delta\alpha_0$ is the amount of increment of parameter α_0 for the interval of time, $\Delta\alpha_0^{\mathrm{old}}$ and ΔS^{old} are the amounts of increment of parameters α_0 and S for the previous interval of time. Since entropy S is sensitive to the variation of parameter α_0 near the bifurcation points, by the use of the annealing schedule (10), the value of parameter α_0 is increased slowly at the points where the bifurcations occur and as a result, a solution with a high performance may be obtained.

The proposed optimization algorithm is applied to the problems in QAPLIB. Some of the results are shown in Table 1. It can be seen that for many problems, the difference between the value obtained and the optimum value is less than 1%. The CPU time for the problem, $N = 20$, is about 2 min by DEC Alpha Station 500/333.

Table 1. Performance of the proposed method

name	N	L_{opt}	L	L/L_{opt}
Had20	20	6922	6970	1.00693
Nug24	24	3488	3490	1.00057
Tho30	30	(149936)	151256	1.0088
Tho40	40	(240516)	241192	1.00281
Tai50a	50	(4941410)	5051386	1.02226
wil50	50	(48816)	48892	1.00156
Sko56	56	(34458)	34502	1.00128
Tai80a	80	(13557864)	13733524	1.01296
Tai80b	80	(818415043)	821025553	1.00319
Sko100a	100	(152002)	152502	1.00329
wil100	100	(273038)	273294	1.00094
Tho150	150	(8133484)	8158137	1.00303

References

[1] C. R. Reeves, editor. *Modern Heuristic Techniques for Combinatorial Problems.* Blackwell Scientific Publications, 1993.

[2] A. S. Mikhailov. *Foundations of Synergetics I*, chapter 7. Springer-Verlag, 1994.

[3] K. Tsuchiya, T. Nishiyama, and K. Tsujita. An algorithm for a combinatorial optimization problem based on bifurcation. submitted to *Neural Networks.*

[4] K. Tsuchiya, T. Nishiyama, and K. Tsujita. A deterministic annealing algorithm for a combinatorial optimization problem using replicator equations. submitted to *Physica D.*