Simple Algorithms of CFD for Compressible Flows

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Abstract

Upwind scheme is one of the most important components of modern numerical scheme for gas dynamics. No scheme, however, is perfect on accuracy, robustness and efficiency. Among a lot of schemes, AUSM developed by Liou et al. satisfies many requirement . Today, there are many practical scheme which can be written in the same form with AUSM. We call these schemes as AUSM type schemes. In order to investigate the key of favorable feature of AUSM type schemes, three new simple schemes are presented and numerically tested. It is shown that a new scheme (ST-AUSM) is simpler than other schemes, however, it exhibits better results.

1.Introduction

Robust and accurate numerical algorithms are required for the viscous supersonic flow analysis of an SST (Super Sonic Transport). Although Mach number of the flow around an SST is not so high, complex flows due to the realistic aircraft shape sometimes blow up computations. In order to stabilize the computation, increasing numerical viscosity is effective, it, however, degrade the accuracy of viscous boundary layer. The boundary layer must be carefully computed, because approximately 50% of total drag is generated by skin friction due to the turbulent boundary layer. A numerical inviscid flux have significant influence on both robustness and accuracy, therefore its selection is vital to the computation.

In order to give the numerical flux in a robust and accurate way, upwind schemes for gas dynamics have been widely used as the basis of high resolution schemes. FDS (Flux Difference Splitting) [1][2] schemes and FVS (Flux Vector Splitting) [3][4][5] schemes were developed as extensions of an upwind scheme for a liner equation and have achieved great successes. It is known that FDS and FVS schemes, however, have some weak points. FDS schemes sometime gives unphysical flux at high Mach number and suffer from the violent carbuncle phenomenon. Numerical diffusion of FVS schemes are too big for the viscous flow problems. HLLE (Harten-Lax-van Leer) scheme and HLLEM (HLLE Modified) scheme were developed [6] to improve FDS, but HLLE's numerical viscosity is as large as FVS and HLLEM is more complex than FDS and suffers from the carbuncle too.

On the other hand, by simplifying FVS, Liou & Steffen [7] invented AUSM (Advection Upstream Splitting Method). AUSM is very simple, robust for strong shock and accurate for boundary layer, however, show overshoot at a shock front. Inspired by AUSM, many schemes have been proposed. Jameson[8][9] showed CUSP (Convective Upstream Split Pressure) which is similar to AUSM but is expressed in combination of central difference and numerical diffusion. Jounouchi et al.[10] showed SFS (Simplified Flux vector Splitting method) in the similar form with AUSM but using mass flux of van Leer's FVS and improved overshoot at a shock. Wada and Liou[11] showed AUSMDV (AUSM with flux Difference splitting and flux Vector splitting) as an improvement from AUSM and showed precise research on their scheme and others. Shima and Jounouchi[12] showed that many schemes can be made which should be called AUSM type schemes in the common form with AUSM and introduced uni-particle upwind schemes in AUSM type schemes and exhibited SHUS (Simple High-resolution Upwind Scheme). Jounouchi et al.[13] pointed out the physical interpretation and theoretical background of AUSM type schemes and then showed that SFS is applicable to the two-phase flow. On the other hand, Nakamori and Nakamura proposed new FVS (NNFVS hereafter) as the improvement of Steger-Warming's FVS in order to improve the accuracy of viscous flow.[14] Liou[15] also presented AUSM$^-$ and AUSM$^+$-W to improve AUSM.

Although some of these schemes have been developed independently, these schemes can be written in the common form. These schemes can be called as AUSM type schemes. Since AUSM type schemes are simple, robust and accurate enough for practical application, they have been used for many applications especially for hypersonic viscous problems already.

High resolution schemes using AUSM type schemes have already been used successfully for many applications as mentioned above, however, there are other high resolution schemes which are completely different from AUSM type schemes. For example, multi-dimensional upwind schemes use multi-dimensional splitting of characteristic waves and show better performance for oblique discontinuity.[16]-[23] CIP(Cubic Interpolated Polynomial) scheme uses non-conservative semi-Laglange scheme and von Neuman's numerical viscosity and achieves great success in computation of multiphase flows containing strong shock waves. [24][25]

It is expected that the way in which AUSM type schemes simplified one-dimensional Riemann solver will be effective for simplifying multidimensional upwind schemes. And also, upwind schemes are interpreted as non-diffusive scheme with numerical diffusion, therefore it may be possible to apply numerical diffusion working in AUSM type scheme to non-conservative scheme.

In this study, first, we show a common form for AUSM type schemes and introduce several AUSM type schemes. Second, a new member of AUSM type scheme, which may be the simplest one, is introduced in order to investigate key feature of AUSM type schemes. In order to check this feature, two more schemes are demonstrated.

2. Formulation of AUSM type schemes

2.1. The common form of AUSM type schemes

We show the formulation of AUSM type schemes for three dimensional Euler equation. The equation can be written in the integral form as follows.

$$\int \mathbf{Q} dv + \int \tilde{\mathbf{F}} ds = 0 \tag{1}$$

$$\mathbf{Q} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \end{pmatrix} \tag{2}$$

$$\tilde{\mathbf{F}} = m\mathbf{\Phi} + p\mathbf{N},\ \mathbf{\Phi} = \begin{pmatrix} 1 \\ u \\ v \\ w \\ h \end{pmatrix},\ N = \begin{pmatrix} 0 \\ x_n \\ y_n \\ z_n \\ 0 \end{pmatrix} \tag{3}$$

$$m = \rho V_n,\ V_n = x_n u + y_n v + z_n w \tag{4}$$

where ρ, u, v, w, e, p and $h = (e+p)/\rho$ represent density, velocity in x,y,z direction, total energy per unit volume and total enthalpy respectively. (x_n, y_n, z_n) is a unit normal of a surface. The variable m denotes mass flux. This form indicates that Euler flux can be divided into the convective term and the pressure term.

AUSM is based on the fact that the convective term and the pressure term can be upwinded separately. The original AUSM and all other AUSM type schemes can be written in a following form,

$$\tilde{\mathbf{F}} = \frac{m+|m|}{2}\mathbf{\Phi}_+ + \frac{m-|m|}{2}\mathbf{\Phi}_- + \tilde{p}\mathbf{N} \tag{5}$$

where subscript ± denote physical values at the left(+) and right(-) sides of cell boundaries suggesting directions of propagation. And $\tilde{p}$ is mixing of left and right pressures using functions of Mach number of left and right state which is defined by,

$$p = \beta_+ p_+ + \beta_- p_- + p' \tag{6}$$

$$\beta_\pm = \begin{cases} \frac{1}{4}(2 \mp M_\pm)(M_\pm \pm 1)^2, \text{ if } |M_\pm| < 1 \\ \max(0, \min(1, \frac{1 \pm M_\pm}{2})), \text{if} |M_\pm| \geq 1 \end{cases} \tag{7}$$

These functions are the simplest smooth polynomial functions that satisfy consistency and upwind nature, however other selections are possible. The symbol p' is a pressure correction term mentioned in section 2.3.

2.2 Selection of mass flux

In original AUSM, the mass flux is calculated using simple switches of Mach number, as

$$m = \frac{\overline{M} + |\overline{M}|}{2}\rho_+ c_+ + \frac{\overline{M} - |\overline{M}|}{2}\rho_- c_- \tag{8}$$

$$\overline{M} = M_+ + M_- \tag{9}$$

$$M_\pm = \begin{cases} \pm\frac{1}{4}\left(\frac{V_{n\pm}}{c_\pm} \pm 1\right)^2, \text{ if } \left|\frac{V_{n\pm}}{c_\pm}\right| < 1 \\ \dfrac{\frac{V_{n\pm}}{c_\pm} \pm \left|\frac{V_{n\pm}}{c_\pm}\right|}{2}, \text{ if } \frac{V_{n\pm}}{c_\pm} \geq 1 \end{cases} \tag{10}$$

Various schemes can be made replacing mass flux. Formulations of several AUSM type schemes are shown in Appendix. SFS and AUSMDV use variations of van Leer's FVS for their mass flux. Shima and Jounouchi [12] showed that mass fluxes of any Riemann solvers can be applied for that of AUSM type scheme and showed improvements when those mass fluxes are used. They derived SHUS (Simple High Resolution Upwind Scheme) using mass flux of Roe scheme, such as,

$$m = \frac{1}{2}\{(\rho V_n)_+ + (\rho V_n)_- - |\overline{V}_n|\Delta\rho - \frac{|\overline{M}+1| - |\overline{M}-1|}{2}\overline{\rho}\overline{V}_n - \frac{|\overline{M}+1| + |\overline{M}-1| - 2|\overline{M}|}{2|\overline{M}|}\Delta p\} \tag{11}$$

$$\overline{M} = \frac{1}{\theta}\frac{\overline{V}_n}{\overline{c}} \tag{12}$$

$$\theta = \max(1, \frac{1}{2\rho_+}(\frac{\overline{\rho}\Delta V_n}{\overline{c}} - \frac{\Delta p}{\overline{c}^2}), \frac{1}{2\rho_-}(\frac{\overline{\rho}\Delta V_n}{\overline{c}} + \frac{\Delta p}{\overline{c}^2})) \tag{13}$$

$$\Delta q = q_- - q_+ \tag{14}$$

$$\rho = (\rho_+ + \rho_-)/2 \tag{15}$$

$$\overline{V}_n = (V_{n+} + V_{n-})/2 \tag{16}$$

$$c = (c_+ + c_-)/2 \tag{17}$$

Note that there is no need for Roe averaged value for this scheme. A parameter θ is introduced for measure against low density. This scheme looks a little complex than original AUSM, however, the computation is simpler because no conditional branch is necessary.

The result of shock tube problem (Sod's standard problem) by second order Roe scheme, original AUSM and SHUS are shown in Fig.1-3. Second order accuracy is achieved by MUSCL using van Albada's differentiable limitter. As shown in figures original AUSM exhibit small over shoot. On the other hand SHUS shows no overshoot and almost identical result with Roe scheme. SHUS has been already applied for wide range of practical computations including three-dimensional Navier-Stokes analysis of the super sonic flyable demonstrator of NAL.

2.3 Pressure correction term

Wada and Liou[11] showed that the pressure term without the pressure correction is sufficient for an usual case, therefore, over shoots are found at a strong propagating shock like a supersonic colliding jet without it. Note that they include this correction as a modification of normal component of momentum flux, not in the pressure correction. They pointed out that the use of normal momentum flux of Hänel's FVS, which is equivalent to van Leer's one, improve this problem. They use this momentum flux for their AUSMDV scheme. Nakamori and Nakamura[13] used similar method in their FVS (NNFVS). These modifications can also be written as pressure corrections as follows and this way is more convenient to explain various AUSM type schemes in a unified manner.

Let $\tilde{\mathbf{F}}$ be an uncorrected flux and $\tilde{\mathbf{F}}_{\mathbf{FVS}}$ be a flux whose normal momentum flux is replaced by that of a FVS scheme. This correction can be written in the form of pressure correction as,

$$\tilde{\mathbf{F}}_{\mathbf{corrected}} = \tilde{\mathbf{F}} + p'\mathbf{N}, \tag{18}$$

$$p' = (\tilde{\mathbf{F}}_{\mathbf{FVS}} - \tilde{\mathbf{F}}, \mathbf{N}). \tag{19}$$

If the mass flux of Hänel's FVS , which is same as van Leer's, is used , p' is written as,

$$p' = \frac{1}{2}(m_+ + m_- - |m|)(V_{n+} - V_{n-}), \quad (20)$$

where mass flux of Hänel's FVS is given by

$$m = m_+ + m_- \quad (21)$$

$$m\pm = \begin{cases} \pm(\rho c)_\pm (M_\pm \pm 1)^2, & \text{if } |M_\pm| < 1 \\ \frac{(\rho V)_{n\pm} \pm |(\rho V)_{n\pm}|}{2}, & \text{if } |M_\pm| \geq 1 \end{cases} \quad (22)$$

Because Hänel's FVS is given by,

$$\mathbf{F}_{FVS} = m_+ \mathbf{F}_+ + m_- \mathbf{F}_- + (\beta_+ p_+ + \beta_- p_-)\mathbf{N} \quad (23)$$

See Appendix for actual form of this term for AUSMDV and NNFVS.

Note that, these corrections are only needed for a strong propagating shock and that these are not used in numerical examples of this report.

3.New simple AUSM type schemes

3.1 ST-AUSM (Simplest-AUSM): Use simpler mass flux

It has been shown in some reports that variations of AUSM type scheme works well. Let us think simpler scheme in order to investigate the key of the favorable nature of AUSM type schemes. We use a simple upwind mass flux using just convective velocity, such as,

$$m = \frac{1}{2}((\rho V_n)_+ + (\rho V_n)_- - |\overline{V}_n|(\rho_- - \rho_+)) \quad (24)$$

Several methods can be used for computing average velocity, we use mass averaged one here.

$$\overline{V}_n = \frac{(\rho V_n)_+ + (\rho V_n)_-}{\rho_+ + \rho_-} \quad (25)$$

In this case, mass flux can be also written in following form.

$$m = \frac{\rho_+ + \rho_-}{2}\overline{V}_n + \frac{\rho_+ - \rho_-}{2}|\overline{V}_n| \quad (26)$$

Numerical experiments showed that other averaging of velocity, such as arithmetic average or maximum of two, worked also well for equation (26). Although this scheme is very simple, the solution have much smaller overshoot than original AUSM (Fig.4) and give best result among Roe, AUSM, SHUS and ST-AUSM for supersonic flow before cylinder(Fig.5). In latter case, Roe scheme exhibits violent carbuncle, AUSM does overshoot at shock, SHUS does slight unnatural concave around stagnation point but ST-AUSM has no problem.

ST-AUSM is robust enough for usual subsonic and supersonic flows, because it gives only small overshoot at shock and correct flux (i.e. no flux) for strong, which means vacuum is found in expansion region, symmetric expansion, and which occurs behind body in initial stage of computation. However, computation breaks at strong, asymmetric expansion problem in one-dimensional test case.

3.2 SCHEME2: Consideration on Split pressure Term

It has been shown that various AUSM type schemes including new simple scheme work fairly well. Now we consider the common nature of AUSM type schemes. The success of new scheme (ST-AUSM) shows that even a simple upwind method based on convective velocity works well as a mass flux term. Thus it is considered that selection of mass flux has merely minor influence on the favorable nature of AUSM type schemes. Although various methods are used for mass fluxes, all AUSM type schemes use similar split pressure terms. Terefore it is easily expected that this pressure term will be the key for favorable character of these schemes. The split pressure term looks just a smooth upwind switch, however, the Taylor expansion of this term in subsonic case shows that this term work as numerical diffusion too, as follows.

$$\overline{p} = \beta_+ p_L + \beta_- p_R$$
$$\approx \frac{p_L + p_R}{2}\{1 - \frac{3}{4c}(1 - M^2)\Delta V_n\} - \frac{1}{4}(3M - M^3)\Delta p + O(\Delta^2), |M| < 1 \quad (27)$$

The second term in first blanket works as diffusion term for velocity when Mach number is smaller than unity, because first derivative of velosity in pressure term become a second order derivative in momentum equations. The Δp term works as smooth upwind switch of pressure.

Here, let us think about the origin of the split pressure term. The split pressure term comes from momentum term of van Leer's FVS scheme. van Leer's FVS is essensialy a first order accurate upwind scheme for Euler eauation using smooth swithing functions. An upwind scheme consists of central difference and diffusionterm, thus , it is natual that the split pressure term have also diffusive effect.

In order to see the effectiveness of this term as numerical diffusion, new scheme (SCHEME2) using a slightly modified split pressure derived from equation (27). The mass flux and split pressure is given by,

$$m = \frac{1}{2}\{(\rho V_n)_L + (\rho V_n)_R - |\overline{V}_n|(\rho_R - \rho_L)\} \quad (28)$$

$$\overline{p} = \frac{p_L + p_R}{2}(1 - \frac{3}{4c}\Delta V_n) - \frac{1}{4}(3\overline{M} - \overline{M}^3)\Delta p \quad (29)$$

This scheme works well in one-dimensional test case. (Fig.6) However, a multidimensional code that uses numerical diffusion shown above is not robust enough for general cases. In addition, this scheme only with upwind switch term but without the diffusion term could not proceed computation in stable even in one-dimensional case. Thus we can conclude that the split pressure term in AUSM type scheme works as sophisticated numerical diffusion and that nonlinear form of the split pressure term inherited from van Leer's FVS is superior to numerical diffusion form in robustness.

3.3 SCHEME3: Application to non-conservative semi-Lagrange scheme

As shown in the previous section, the split pressure term plays important part in AUSM type schemes. The split pressure term is similar to von Newman's numerical viscosity in the sense that it works in pressure term.

We try to apply the split pressure term for one-dimensional Euler equation in non-conservative primitive variable form.

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ u \\ p\end{pmatrix} + u\frac{\partial}{\partial x}\begin{pmatrix}\rho \\ u \\ p\end{pmatrix} = \begin{pmatrix}-\rho u_x \\ -p_x/\rho \\ -\gamma p u_x\end{pmatrix} \quad (30)$$

The numerical scheme is explained as follows. See also Fig.7.

step1:Gradient at each point is approximated using van Albada's differentiable slope limitter.

step2:Profile of each variable between point is reconstructed using Third order polynomial using values and slope on both ends.

step3:Left hand side of equation is evaluated by Lagrange step.

step4:Pressure gradient term in RHS is evaluated using split pressure as AUSM.

step4: Velocity gradient terms in RHS are evaluated using central difference.

Since this scheme is not conservative, shock speed is not accurate. However, no significant oscillation is found, (Fig.8) thus the split pressure terms also works well as numerical diffusion for this scheme.

4.Conclusion

Several AUSM type schemes are presented, and three new schemes are introduced in order to indicate the nature of AUSM type schemes. These schemes, ST-AUSM, SCHEME2 and SCHEME 3 are initially developed merely for demonstration, however, ST-AUSM, in which simple upwind mass flux and split pressure are used, have also practical benefit. It shows best solution for supersonic flow around cylinder among several upwind schemes including original AUSM and Roe scheme.

It is shown that the split pressure term that is inherited from van Leer's FVS works as sophisticated numerical diffusion. SCHEME2, which utilize the same mass flux with ST-AUSM and the numerical diffusion term derived from the Taylor expansion of the split pressure, works also well, however, it is not so robust as ST-AUSM which use split pressure itself. In addition, the split pressure term can be used to stabilize non-conservative semi Laglange scheme in SCHEME3.

It can be concluded that the key feature of AUSM type scheme is its split pressure term through research using these schemes. The split pressure term works as sophisticated numerical diffusion term and its nonlinear form does better job for strong shock and expansion than a numerical diffusion form.

When we consider simple multidimensional upwind schemes, it is thought that a key is the design of split pressure term bearing multidimensional nature, since a multi-dimensional upwind convective term can be constructed in a relatively easy way. It is expected that we will get simple multidimensional upwind schemes in the similar way in which we have got simple approximated Riemann fluxes by AUSM type schemes.

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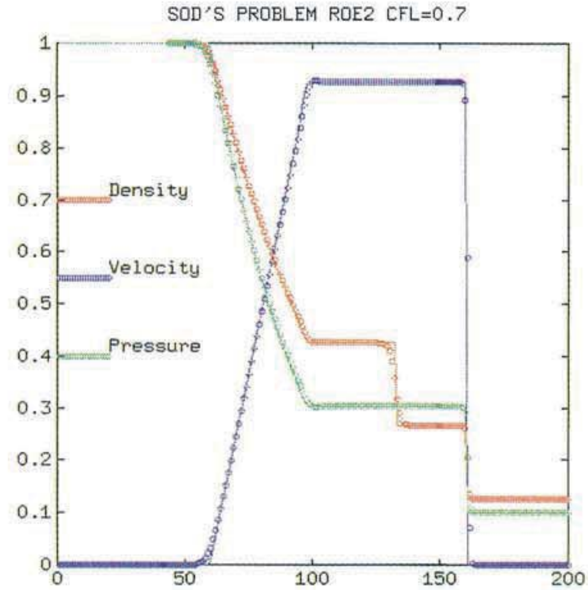

Fig.1 Sod's problem by second order Roe scheme

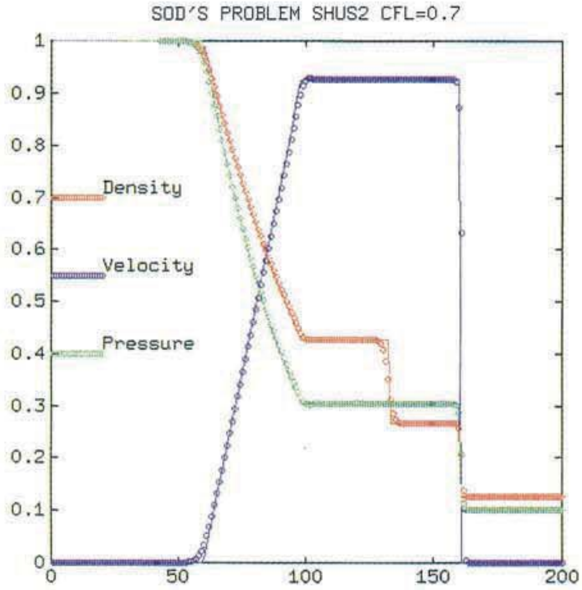

Fig.3 Sod's problem by second order SHUS

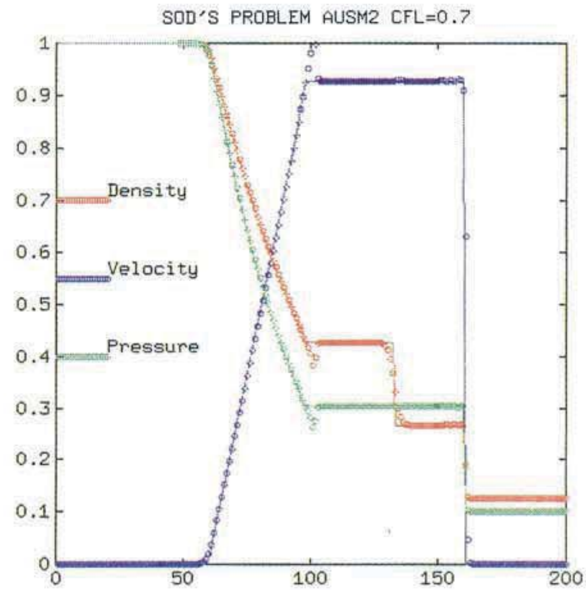

Fig.2 Sod's problem by second order AUSM

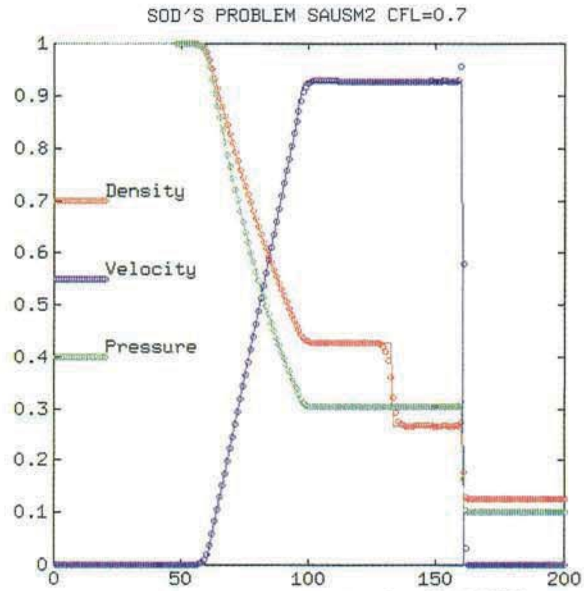

Fig.4 Sod's problem by second order ST-AUSM

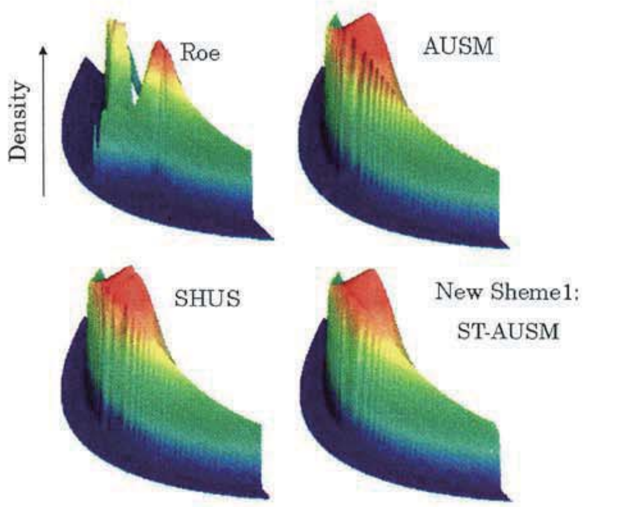

Fig.5 3D plot of density profile of supersonic flow at Mach 6 before cylinder. Violent carbuncle is found in result of Roe scheme, over shoot is found in AUSM's result and slight unnatural concave is found in SHUS's result but no problem is found in ST-AUSM's result.

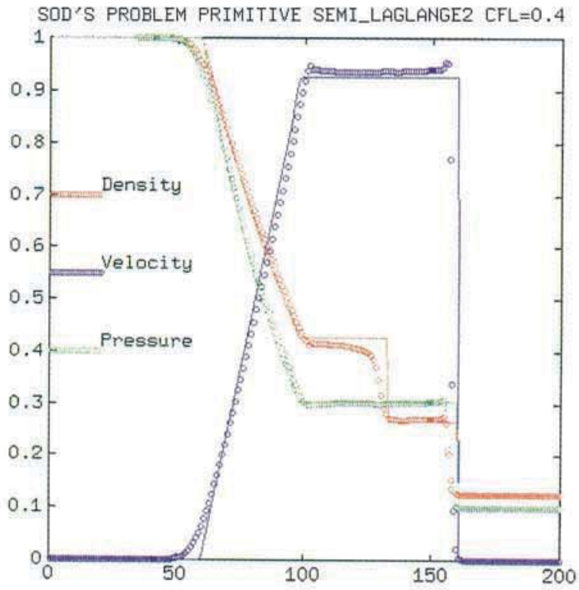

Fig.8 Sod's problem by second order, non-conservative, semi-Laglange scheme.

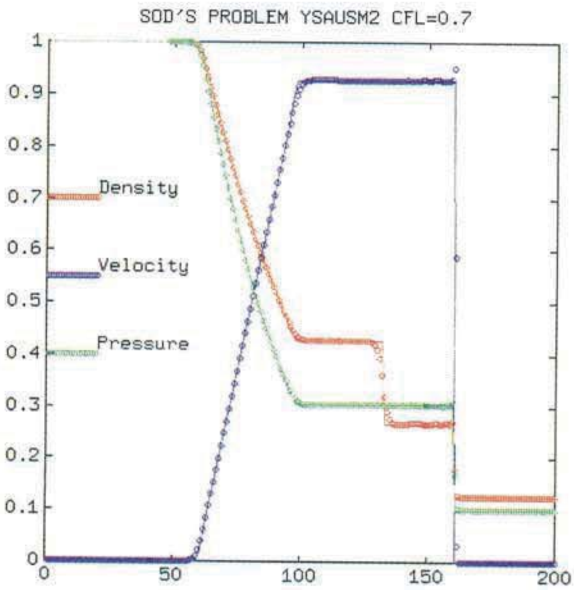

Fig.6 Sod's problem by second order scheme using upwind flux SCHEME2 .

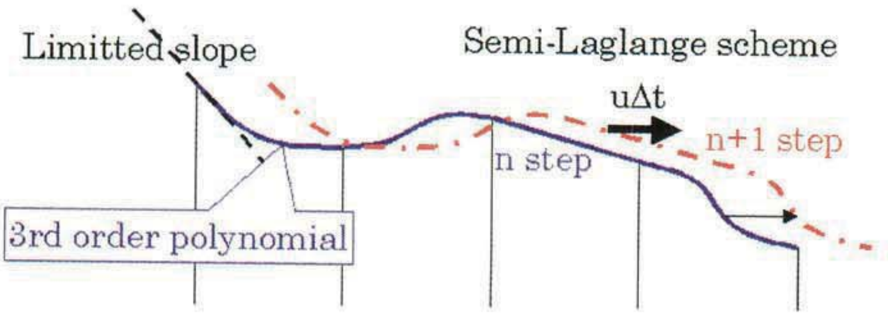

Fig.7 Illustration of Semi-Lagrange step in SCHEME3

Appendix: Formulation of AUSM type schemes

List of mass fluxes of several AUSM type scheme is presented. On the split pressure term, which is another essential component of AUSM type scheme, see Eq.(6) (7) in this paper. This term is same for all AUSM type schemes except for AUSM+.

AUSM

$$m=\frac{\overline{M}+|\overline{M}|}{2}\rho_+c_+ +\frac{\overline{M}-|\overline{M}|}{2}\rho_-c_-$$

$$\overline{M}=M_+ + M_-$$

$$M_\pm=\begin{cases}\pm\frac{1}{4}\left(\frac{V_{n\pm}}{c_\pm}\pm 1\right)^2, & \text{if}\left|\frac{V_{n\pm}}{c_\pm}\right|<1\\ \frac{M_\pm\pm|M_\pm|}{2}, & \text{if}\left|\frac{V_{n\pm}}{c_\pm}\right|\ge 1\end{cases}$$

AUSM+

AUSM+ is very similar to AUSM, but smooth splitting function for average Mach number and pressure are different.

$$M_\pm=\begin{cases}\pm\frac{1}{4}\left(\frac{V_{n\pm}}{c_\pm}\pm 1\right)^2\pm\frac{1}{8}\left(\left(\frac{V_{n\pm}}{c_\pm}\right)^2-1\right)^2, & \text{if}\left|\frac{V_{n\pm}}{c_\pm}\right|<1\\ \frac{M_\pm\pm|M_\pm|}{2} & ,\text{if}\left|\frac{V_{n\pm}}{c_\pm}\right|\ge 1\end{cases}$$

Pressure splitting coefficients are also changed as,

$$\beta_\pm=\begin{cases}\frac{1}{4}\left(2\mp\frac{V_{n\pm}}{c_\pm}\right)\left(\frac{V_{n\pm}}{c_\pm}\pm 1\right)^2\pm\frac{3}{16}\left(\left(\frac{V_{n\pm}}{c_\pm}\right)^2-1\right)^2, & \text{if}\left|\frac{V_{n\pm}}{c_\pm}\right|<1\\ \max\left(0,1,\left(\frac{1\pm\frac{V_{n\pm}}{c_\pm}}{2}\right)\right) & ,\text{if}|M_\pm|\ge 1\end{cases}$$

SFS

Mass flux of SFS is that of FVS using modified Mach number in order to satisfy zero mass flux at contact surface.

$$m=m_+ + m_-$$

$$m\pm=\begin{cases}\pm\frac{\rho_\pm c_\pm^{\ 2}}{4\overline{c}}(M_\pm\pm 1)^2, & \text{if}|M_\pm|<1\\ \frac{(\rho V_n)_\pm\pm|(\rho V_n)_\pm|}{2}, & \text{if}|M_\pm|\ge 1\end{cases}$$

$$M\pm=\frac{V_{n\pm}}{\overline{c}_\pm},c\pm=\frac{c_\pm^{\ 2}}{\overline{c}},c_\pm=\sqrt{\frac{\gamma p_\pm}{\rho_\pm}},\overline{c}=\frac{c_+ + c_-}{2}$$

The common sound speed $\overline{c}$ is also used for split pressure term.

AUSMDV

$$m=(\rho\tilde{V}_n)_+ +(\rho\tilde{V}_n)_-$$

$$\tilde{V}_{n\pm}=\begin{cases}\alpha\pm\left(\pm\overline{c}\frac{(M_\pm\pm 1)^2}{4}-\frac{V_{n\pm}\pm|V_{n\pm}|}{2}\right)+\frac{V_{n\pm}\pm|V_{n\pm}|}{2}, & \text{if}|M_\pm|<1\\ \frac{V_{n\pm}\pm|V_{n\pm}|}{2} & ,\text{if}|M_\pm|\ge 1\end{cases}$$

$$\alpha_\pm=\frac{2(p/\rho)_\pm}{(p/\rho)_+ +(p/\rho)_-},M_\pm=\frac{V_{n\pm}}{\overline{c}},\overline{c}=\max(c_+,c_-)$$

The common sound speed $\overline{c}$ is also used for split pressure term. The pressure correction term is introduced as,

$$p'=\frac{1}{2}\left(\frac{1}{2}+s\right)\left((\rho V_n)_+ -(\rho V_n)_- -|m|\right)(V_{n+}-V_{n-})$$

$$s=\frac{1}{2}\min\left(1,10\frac{|p_+ - p_-|}{\min(p_+,p_-)}\right)$$

SHUS

$$m=\frac{1}{2}\{(\rho V_n)_+ +(\rho V_n)_- -|\overline{V}_n|\Delta\rho-\frac{|\overline{M}+1|-|\overline{M}-1|}{2}\overline{\rho}\overline{V}_n-\frac{|\overline{M}+1|+|\overline{M}-1|-2|\overline{M}|}{2|\overline{M}|}\Delta p\}$$

$$\overline{M}=\frac{1}{\theta}\frac{\overline{V}_n}{\overline{c}}$$

$$\theta=\max\left(1,\frac{1}{2\rho_+}\left(\frac{\overline{\rho}\Delta V_n}{\overline{c}}-\frac{\Delta p}{\overline{c}^2}\right),\frac{1}{2\rho_-}\left(\frac{\overline{\rho}\Delta V_n}{\overline{c}}+\frac{\Delta p}{\overline{c}^2}\right)\right)$$

$$\Delta q=q_- - q_+$$

$$\overline{\rho}=(\rho_+ +\rho_-)/2$$

$$\overline{V}_n=(V_{n+}+V_{n-})/2$$

$$\overline{c}=(c_+ + c_-)/2$$

The common sound speed $\overline{c}$ is also used for split pressure term.

NNFVS

$$m=m_+ + m_-$$

$$m_\pm=\rho_\pm\lambda_{1\pm}+\frac{p_\pm}{\overline{c}^2}(-2\lambda_{1\pm}+\lambda_{2\pm}+\lambda_{3\pm})$$

with average sound velocity, modified Mach number and smoothed eigen values such as,

$$\overline{c}=\frac{c_+ + c_-}{2},\overline{M}_\pm=\frac{V_{n\pm}}{\overline{c}}$$

$$\lambda_{1\pm}=\begin{cases}\frac{\overline{c}(\overline{M}_\pm\pm|\overline{M}_\pm|)}{2}, & \text{if}|\overline{M}_\pm|>\varepsilon\\ \frac{\overline{c}(\overline{M}_\pm\pm\varepsilon)^2}{2}, & \text{if}|\overline{M}|_\pm\le\varepsilon\end{cases}$$

$$\varepsilon=0.2\frac{\min(p_+,p_-)}{\max(p_+,p_-)}$$

$$\lambda_{2+}=\lambda_{2+}(\overline{M}_+)=\begin{cases}\frac{\overline{c}}{8}(\overline{M}_+ +1)^2(\overline{M}_+^{\ 2}-4\overline{M}_+ +7), & \text{if}|\overline{M}_+|<1\\ \frac{\overline{c}}{2}(\overline{M}_+ +1+|\overline{M}_+ +1|), & \text{if}|\overline{M}_+|\ge 1\end{cases}$$

$$\lambda_{3+}=\lambda_{3+}(\overline{M}_+)=\begin{cases}\frac{\overline{c}}{8}(\overline{M}_+ -1)(\overline{M}_+ +1)^3 & ,\text{if}|\overline{M}_+|<1\\ \frac{\overline{c}}{2}(\overline{M}_+ -1+|\overline{M}_+ -1|), & \text{if}|\overline{M}_+|\ge 1\end{cases}$$

$$\lambda_{2-}=\lambda_{2-}(\overline{M}_-)=\lambda_{3+}(-\overline{M}_-)$$

$$\lambda_{3-}=\lambda_{3-}(\overline{M}_-)=\lambda_{2+}(-\overline{M}_-)$$

The common sound speed $\overline{c}$ is also used for split pressure term. The use of one side upwind density is recommended if $|\overline{M}|<\varepsilon$. The pressure correction term is given by,

$$p'=\frac{1}{2}(m_+ - m_- -|m|)(V_{n+}-V_{n-})$$

It is expected that pressure correction term is not necessary in usual condition judging from experiences with other AUSM type schemes, although it has not been tested.

ST-AUSM (New Scheme1)

$$m=\frac{1}{2}((\rho V_n)_+ +(\rho V_n)_- -|\overline{V}_n|(\rho_- - \rho_+))$$

$$\overline{V}_n=\frac{(\rho V_n)_+ +(\rho V_n)_-}{\rho_+ + \rho_-}$$