A study of estimating the Angular Position of the moving spacecraft

Tsutomu Ichikawa

Japan Aerospace Exploration Agency, Institute of Space and Astronautical science
3-1-1, Yoshinodai, Sagamihara, Kanagawa 229-8510, JAPAN
E-mail :ben@isas.jaxa.jp

Abstract: The problem of estimating the angular position of a spacecraft moving at a constant velocity using two rotating tracking stations is considered. This reports on an initial phase of analytical studies on the optimal attainable estimation performance and associated receiver design. Perametric dependence of the optimum attainable estimation is also studied.

地上局を用いた探査機の角度位置推定の一考察

宇宙航空研究開発機構　宇宙科学研究本部　市川　勉

概要

2つの地上局を用いて、定速度で巡航している探査機の角度位置推定について本稿で論じる。角度での位置変動に対する周波数での観測データを用いて、カルマンフィルタによる逐次推定を行い解析モデルによる角度位置推定を精度含め解析及び評価を行った。

1. INTRODUCTION

Consider the situation in Fig.1 where at a reference time t=0 there is a spacecraft at point V and two stations at S_1 and S_2 relative to Earth geocenter O. The spacecraft is assumed to be at a distance d_0 from the geocenter O at time t=0. It is also assumed to be moving at a constant velocity v relative to geocenter O for all time in a direction which is at an angle α with respect to OV. The stations are both assumed to be rotating at a constant angular velocity ω about O. At time t=0, station S_1 is at an angle ε_0 and the spacecraft at an angle γ_0 with respect to a star reference. The angle between the stations is denoted by η and the distances of the stations from geocenter by R_1 and R_2. We assumed that the spacecraft is continuously transmitting a signal $s(t)$. Each station receives an additive noise-corrupted version of this transmitted signal. The receives waveforms from both stations over a given time interval are then used to estimate the unknown vehicle angle γ_0. This article considers this estimation problem based on the assumption that d_0, v, α, η, ω, and ε_0 are known. These assumptions are made so that effect of the rotating stations and the moving target on angular position estimation can be studied.

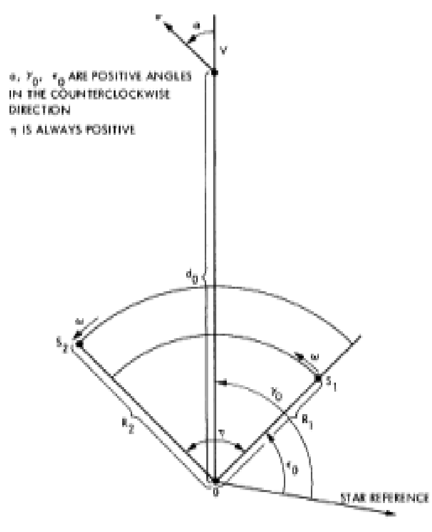

Fig. 1. Spacecraft and station locations

The estimation problem is defiend in Section 2 below along with a discussion of the minimum attainable mean square estimation error performance. Since we could not determine the optimal estimator, a suboptimal estimator is derived in section 3.The performance of this suboptimal estimator is examined relative to the attainable performance on angular position, station rotation, station distance from geocenter and observation time duration. The specific case of a sinusoidal ranging signal is considered and numerical computations of the optimum attainable angular estimation accuracy are performed for several parameter values.

2. THE ESTIMATION

Let us first derive the equations for the received waveforms at each station. The spacecraft is assumed continuously transmit a signal $s(t)$. The received waveform at station S_1

is assumed to

$$y_1(t) = s[t - \phi_1(t,\gamma_0)] + v_1(t) \qquad (1)$$

where $v_1(t)$ is additive white Gaussian noise with power spectral density N_1 and $\phi_1(t,\gamma_0)$ is effectively the signal delay time. Similarly, the received waveform at station S_2 is assumed to be

$$y_2(t) = s[t - \phi_2(t,\gamma_0)] + v_2(t) \qquad (2)$$

where $v_2(t)$ is additive white Gaussian noise with power spectral density N_2 and $\phi_2(t,\gamma_0)$ is the signal delay time to both deterministic functions of the unknown spacecraft angle γ_0, $v_1(t)$ and $v_2(t)$ are mutually stastically independent.
The receiver's function is to estimate γ_0 based on observations $(y_1(t), y_2(t))$, $-T_1 \le t \le T_2$, with a goal of minimizing error estimator is the conditional mean estimator when the prior distribution of γ_0 is known.

In this case, the conditional mean estimator is nonlinear. Moreover, it appears that the problem of determining explicit estimator equations is not tractable. An approach to overcome this problem is to derive suboptimum receivers that can be implemented instead.
ϕ is the following expressions:

$$\begin{aligned}\phi_1(t,\gamma_0) = &\left[1-(v/c)^2\right]^{-1}[-(v/c)^2 t - (vd_0\cos\alpha)/c^2 \\ &+ vR_1\cos(\gamma_0+\alpha-\varepsilon_0-\omega t)/c^2] \\ &+\Big\{\left[1-(v/c)^2\right]^{-1}\Big[t+(vd_0\cos\alpha)/c^2 \\ &- vR_1\cos\left(\gamma_0+\alpha-\varepsilon_0-\omega t\right)/c^2\Big]^2 \\ &+\left[1-(v/c)^2\right]^{-1}\Big[\Big(d_0^2+R_1^2-2R_1d_0\cos\big(\gamma_0 \\ &-\varepsilon_0-\omega t\big)\Big)/c^2-t^2\Big]\Big\}^{1/2}\end{aligned} \qquad (3)$$

$$\begin{aligned}\phi_2(t,\gamma_0) = &\left[1-(v/c)^2\right]^{-1}[-(v/c)^2 t - (vd_0\cos\alpha)/c^2 \\ &+ VR_2\cos(\gamma_0+\alpha-\varepsilon_0-\omega t)/c^2] \\ &+\Big\{\left[1-(v/c)^2\right]^{-1}\Big[t+(vd_0\cos\alpha)/c^2 \\ &- vR_2\cos\left(\gamma_0+\alpha-\varepsilon_0-\omega t\right)/c^2\Big]^2 \\ &+\left[1-(v/c)^2\right]^{-1}\Big[\Big(d_0^2+R_2^2-2R_2d_0\cos\big(\gamma_0 \\ &-\varepsilon_0-\omega t\big)\Big)/c^2-t^2\Big]\Big\}^{1/2}\end{aligned} \qquad (4)$$

In the next section of this paper we consider one method of obtaining a suboptimal receiver by using extended Kalman filetr estimation approach. This approach results in a relatively simple receiver structure.

It is also of interest to determine the optimum mean square estimation error so that the performance of suboptimal receivers can be evaluated. Unfortunately, it appears in this case that the problem of determining this optimum performance value is also not tractable. However, it is possible to obtain lower bounds on the minimum mean square estimation error. The Cramer-Rao lower bound appears to be most tractable to use. In the case when γ_0 is an unknown but nonrandom parameter, the Cramer-Rao lower bound on the mean square estimation error of any unbiased estimator $\hat{\gamma}_0$ is given by

$$E\left[(\hat{\gamma}_0-\gamma_0)^2\right] \ge \left[\sum_{i=1}^{2}(2/V_1)\int_{-T_1}^{T_2}\dot{s}^2(t-\phi_i(t,\gamma_0))\left(\frac{\partial\phi_i(t,\gamma_0)}{\partial\gamma_0}\right)^2 dt\right]^{-1} \qquad (5)$$

where $\dot{s}(t) = ds(t)/dt$. In the case when γ_0 is a random parameter with known density $p(\gamma_0)$, the Cramer-Rao lower bound on the mean square estimation error of any estimator $\hat{\gamma}_0$ is given by

$$E\left[(\hat{\gamma}_0-\gamma_0)^2\right] \ge \left\{E\left[\sum_{i=1}^{2}(2/V_1)\int_{-T_1}^{T_2}\dot{s}^2(t-\phi_i(t,\gamma_0))\left(\frac{\partial\phi_i(t,\gamma_0)}{\partial\gamma_0}\right)^2 dt - \frac{\partial^2 np(\gamma_0)}{\partial\gamma_0}\right]\right\}^{-1} \qquad (6)$$

where the expectation in the right hand side of Eq.(6) is with respect to the prior distribution of γ_0. For a normal γ_0 with variance σ_γ^2, Eq.(6) reduced to:

$$E\left[(\hat{\gamma}_0-\gamma_0)^2\right] \ge \left\{E\left[\sum_{i=1}^{2}(2/V_1)\int_{-T_1}^{T_2}\dot{s}^2(t-\phi_i(t,\gamma_0))\left(\frac{\partial\phi_i(t,\gamma_0)}{\partial\gamma_0}\right)^2 dt + 1/\sigma_\gamma^2\right]\right\}^{-1} \qquad (7)$$

These lower bounds will be used in the remainder of this article to estimate the performance of the suboptimal estimator as well as the optimum theoretically attainable performance.

.

3. ESTIMATOR

Consider the problem of estimating γ_0 in the following equivalent state variable formulation. Let $\gamma(t)$ be a variable state satisfying

$$\dot{\gamma}(t) = 0 \quad , \quad \gamma(-T_1) = \gamma_0 \qquad (8)$$

Then $\gamma(t) = \gamma_0$, the parameter to be estimated, for all t.
Rewrite Eqs.(1) and (2) as

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} s(t-\phi_1(t,\gamma_0)) + v_1(t) \\ s(t-\phi_2(t,\gamma_0)) + v_2(t) \end{bmatrix} \qquad (9)$$

So the equivarent problem is to estimate $\gamma(T_2)$ based on observation $y(t)$ in the interval $[-T_1, T_2]$.

As we noted previously, the problem of determining the estimator is not tractable. An alternative is to derive a suboptimal estimator that approximates the minimum mean square error estimator. Another alternative is to abandon the minimum mean square error criterion and to seek estimators based on the maximum likelihood (ML) or maximum a posteriori (MAP) criterion. However, it can be shown that the optimum ML or MAP estimators are also not practically implementable. Hence developing estimators using the ML or MAP criterion will also require consideration of suboptimal estimators.

There are numerous ways of determining such suboptimal estimators. An approach will be to adopt one version of the extended Kalman filter algorithm. This version is the Kalman filter operating on a linearization of the observation equations (9) about the state estimate. The reason for adopting this approach over others is its relative simplicity. In the nonlinear estimation folklore, the extended Kalman filter is regarded as being capable of performing as well as other suboptimal schemes in most problems. So there is a priori no reason to believe that constraining our approach to the extended Kalman filter is overly restrictive.

Let $\hat{\gamma}(t)$ denote the extended Kalman filter estimate of $\hat{\gamma}(t)$.

Then a straightforward application of the equations of Ref.1, shows that $\hat{\gamma}(t)$ satisfies.

$$\frac{d\hat{\gamma}(t)}{dt} = -P(t)\sum_{i=1}^{2}(1/N_i)[y_i(t) - s(t-\phi_i(t,\hat{\gamma}(t)))] \bullet \left[s(t-\phi_i(t,\hat{\gamma}(t)) \frac{\partial\phi_i(t,\hat{\gamma}(t))}{\partial\hat{\gamma}} \right] \tag{10}$$

$$\frac{dP(t)}{dt} = -P^2(t)\sum_{i=1}^{2}(1/N_i\left[s(t-\phi_i(t,\hat{\gamma}(t)) \frac{\partial\phi_i(t,\hat{\gamma}(t))}{\partial\hat{\gamma}} \right]^2 \tag{11}$$

with initial conditions

$$\hat{\gamma}(-T_1) = \bar{\gamma}_0 \tag{12}$$

$$P(-T_1) = \sigma^2_{\gamma_0} \tag{13}$$

where $\bar{\gamma}_0$ and $\sigma^2_{\gamma_0}$ are the prior mean and variance respectively of γ_0. We shall denote

$$\left.\frac{\partial\phi_i(t,\gamma_0)}{\partial\gamma_0}\right|_{\gamma_0-\hat{\gamma}(t)} \quad \text{by} \quad \frac{\partial\phi_i[t,\hat{\gamma}(t)]}{\partial\hat{\gamma}}$$

for simplicity. Also, in Eqs. (10), and (11), $P(t)$ represents an approximation of the conditional variance of $\hat{\gamma}(t)$. The solution of Eq.(11) can easily be shown to be

$$P(t) = \left\{ 1/\sigma^2_{\gamma_0} + \int_{-T_1}^{t} \sum_{i=1}^{2}(1/N_i)\left[\dot{s}^2(t-\phi_i(t,\gamma_0)) \frac{\partial\phi_i(\tau,\gamma_0)}{\partial\gamma_0}\right]^2 d\tau \right\}^{-1} \tag{14}$$

Rewriting Eq.(10) as an integral equation gives

$$\hat{\gamma}(t) = \bar{\gamma}_0 - \int_{-T_1}^{t} P(\tau)\sum_{i=1}^{2}(1/N_i[y_i(\tau) - s(\tau-\phi_i(\tau,\hat{\gamma}(\tau))) \bullet \left[\dot{s}(\tau-\phi_i(t,\hat{\gamma}(\tau))) \frac{\partial\phi_i(\tau,\hat{\gamma}(\tau))}{\partial\hat{\gamma}} \right] d\tau \tag{15}$$

Thus, Eqs. (14) and (15) give the estimator structure with $\hat{\gamma}(T_2)$, the desired estimate of γ_0. The only prior stastical knowledge of γ_0 required is its mean and variance.

The expression for $\partial\phi_i[t,\hat{\gamma}(t)]/\partial\hat{\gamma}$ is given in Eqs. (2) and (3). These waveforms are implemented in the receiver by adjusting the $\hat{\gamma}(t)$ phase contributions in the sinusoidal terms. The structure of the estimator is somewhat similar to the MAP estimator with normal prior distribution for γ_0. In Eqs. (14) and (15), $\phi_i(t,\gamma)$ is given by Eq. (3) and $\partial\phi_i(t,\gamma)/\partial\gamma$. Further simplification of Eqs. (14) and (15) result from using the simpler approximations in Eqs. (24) to (27) given in section 4 for ϕ_i and $\partial\phi_i/\partial\gamma$. Simplification of the basic estimator structure apparently cannot be done without specific assumptions on the signal structure.

The performance of this algorithm unfortunately cannot be determined analytically. In evaluating extended Kalman filters. $P(t)$ is only an approximation to the conditional variance of $\gamma(t)$ (Ref. 1,2). Moreover, $P(t)$ depends on the observations and so cannot be determined other than from simulation runs of the filter. In spite of these pitfalls, let us examine Eq. (14) to obtain a heuristic estimate of the best possible performance of the estimator.

Assume that the estimator is performing well. Thus, $\gamma(t)$ will be close to $\gamma(t) = \gamma_0$. Assume also that $P(T_2)$ is a good approximation of the mean square estimation error. Then, from Eq.(14) we have

$$P(T_2) = \frac{\left[\sum_{i=1}^{2}(1/N_i)\int_{T_1}^{T_2}\dot{s}(t-\phi_i(t,\hat{\gamma}(t)))\left(\frac{\partial\phi_i(t,\hat{\gamma}(t))}{\partial\hat{\gamma}}\right)^2\right]^{-1}}{1+\left[\sum_{i=1}^{2}(1/N_i)\int_{T_1}^{T_2}\dot{s}(t-\phi_i(t,\hat{\gamma}(t)))\left(\frac{\partial\phi_i(t,\hat{\gamma}(t))}{\partial\hat{\gamma}}\right)^2\right]^{-1}/\sigma^2_{\gamma_0}}$$

$$\leq \left[\sum_{i=1}^{2}(1/N_i)\int_{T_1}^{T_2}\dot{s}(t-\phi_i(t,\hat{\gamma}(t)))\left(\frac{\partial\phi_i(t,\hat{\gamma}(t))}{\partial\hat{\gamma}}\right)^2\right]^{-1}$$

(16)

Let γ_0 be the true value of the unknown angle. So, if $\gamma(t) \cong \gamma_0$, replacing $\gamma(t)$ by γ_0 in Eq. (16) shows that the upper bound on $P(T_2)$ is roughly twice the Cramer-Rao lower bound (Eq. (5)) on the optimum mean square error.

Thus from the above heuristic point of view, the best possible performance of the estimator is roughly within a factor of 2 from the Cramer-Rao lower bound of Eq. (5)

4. OPTIMUM THEORITICALLY ATTAINABLE ESTIMATION OERFORMANCE

As we noted previously the Cramer-Rao lower bound gives a lower bound on the optimum attainable angle mean square estimation error. In this section we shall examine the Cramer-Rao lower bound in a spesial case. In particular, we shall assume the following set of parameters:

$$d = 9\times10^8 \quad km$$
$$R_1 = R_2 = 6.5\times10^3 \quad km$$
$$v = 10 \quad km/sec$$
$$T_1 = T_2 = 30 \quad min$$

This set of parameters is consistent with the distances encounted in a Venus mission. We assume in addition that $N_1 = N_2$ for simplicity. We shall first analyze the effects of the relative angular positions and the rotation of the Earth on the Cramer-Rao lower bound (Eq. (5)). This, then, gives the dependence of the optimum attainable performance on these effects.

We first consider the effect of the angular position γ_0, ε_0 and η given in Fig. 1. Since the problem of estimating γ_0 is nonlinear, the minimum attainable estimation error would generally depend on γ_0. Consider first the case when $\varpi = v = 0$ for insight into dependence. Using the parameters in Eq. (17) we have from Eqs. (3) and (4) that

$$\phi_1(t,\gamma_0) = \left[d_0^2 + R_1^2 - 2d_0R_1\cos(\gamma_0-\varepsilon_0)\right]^{-\frac{1}{2}}/c$$
$$\cong \left[(d_0^2+R_1^2)^{1/2}/c\right]\left[1-(R_1d_0/(d_0^2+R_1^2))\cos(\gamma_0-\varepsilon_0)\right] \tag{18}$$

$$\phi_2(t,\gamma_0) = \left[d_0^2 + R_2^2 - 2d_0R_2\cos(\gamma_0-\varepsilon_0)\right]^{-\frac{1}{2}}/c$$
$$\cong \left[(d_0^2+R_2^2)^{1/2}/c\right]\left[1-(R_2d_0/(d_0^2+R_2^2))\cos(\gamma_0-\varepsilon_0-\eta)\right] \tag{19}$$

so

$$\frac{\partial\phi_1(t,\gamma_0)}{\partial\gamma_0} \cong \left[R_1d_0/c(d_0^2+R_1^2)^{1/2}\right]\sin(\gamma_0-\varepsilon_0) \tag{20}$$

$$\frac{\partial\phi_2(t,\gamma_0)}{\partial\gamma_0} \cong \left[R_2d_0/c(d_0^2+R_2^2)^{1/2}\right]\sin(\gamma_0-\varepsilon_0-\eta) \tag{21}$$

Since $\left[R_id_0/c(d_0^2+R_2^2)^{1/2}\right] = 2.2\times10^{-2}$, $\phi_i(t,\gamma_0)$ is relatively independent of γ_0. Thus an approximation of the Cramer-Rao lower bound (Eq. (5)) in the case when $\varpi = v = 0$ is:

$$E\left[(\hat{\gamma}_0-\gamma_0)^2\right] \geq \left[\frac{2R_1^2d_0^2}{N_1c^2(d_0^2+R_1^2)}\int_{-T_1}^{T_2}\dot{s}^2\left(t-\frac{(d_0^2+R_1^2)^{1/2}}{c}\right)\sin(\gamma_0-\varepsilon_0)dt + \frac{2R_1^2d_0^2}{N_2c^2(d_0^2+R_2^2)}\int_{-T_1}^{T_2}\dot{s}^2\left(t-\frac{(d_0^2+R_2^2)^{1/2}}{c}\right)\sin^2(\gamma_0-\varepsilon_0-\eta)dt\right]^{-1} \tag{22}$$

Hence under the assumption that $R_1 = R_2$ and $N_1 = N_2$, Eq. (23) depends inversely on

$$f(\delta) = \sin^2\delta + \sin^2(\delta-\eta) = 1-\cos\eta\cos(2\delta-\eta) \tag{23}$$

where $\delta = \gamma_0 - \varepsilon_0$. Note that $f(\delta)$ is symmetric about $\delta = \eta/2$, which corresponds to when the spacecraft is halfway between the two stations (Fig.1). When $0 \leq \eta \leq 90^\circ$, $f(\delta)$ increase as δ deviates from $\eta/2$, or when the spacecraft moves toward either station from the midpoint. So, when $0 \leq \eta \leq 90^\circ$, the worst performance is when the spacecraft is exactly halfway between the two stations. When $\eta \geq 90^\circ$, the converse is true and the best performance is when the spacecraft is exactly halfway between two stations (Fig.1).

When $0 \leq \eta \leq 90^\circ$, $f(\delta)$ increase as δ deviates from $\eta/2$, or when the spacecraft moves toward either station from the midpoint. So, when $0 \leq \eta \leq 90^\circ$, the worst performance is when the spacecraft is exactly halfway between the two stations. When $\eta \geq 90^\circ$, the converse is true and the best performance is when the spacecraft is exactly halfway between two stations. Since $f(\delta)$ is independent of δ when $\eta = 90^\circ$, this is the best value of η from the viewpoint of uniformity of performance over a range of γ_0. An examination of Eq. (23) shows that for $80^\circ \leq \eta \leq 100^\circ$, the variation of performance is less than 20% for δ from 0 to η.

The previous consideration are when $\varpi = v = 0$. Let us now consider when $\omega \neq 0$ and $v \neq 0$. It is shown that approximate expressions for ϕ_1, ϕ_2, $\partial\phi_1/\partial\phi_2$ and $\partial\phi_2/\partial\gamma_0$ are:

$$\phi_1(t,\gamma_0) \cong -\left(1-\left(\frac{v}{c}\right)^2\right)^{-1}\left[\left(\frac{v}{c}\right)^2 + \frac{vd_0\cos\alpha}{[(d_0^2+R_1^2)(c^2-v^2)]^{1/2}}\right]t + \left(\frac{d_0^2+R_1^2}{c^2-v^2}\right)^{1/2} - \frac{vd_0\cos\alpha}{c^2-v^2} - \frac{R_1d_0}{[(d_0^2+R_1^2)(c^2-v^2)]^{1/2}}\cos(\gamma_0-\varepsilon_0-\omega t) \tag{24}$$

$$\phi_2(t,\gamma_0) \cong -\left(1-\left(\frac{v}{c}\right)^2\right)^{-1}\left[\left(\frac{v}{c}\right)^2+\frac{vd_0\cos\alpha}{[(d_0^2+R_2^2)(c^2-v^2)]^{1/2}}\right]t + \left(\frac{d_0^2+R_2^2}{c^2-v^2}\right)^{1/2} - \frac{vd_0\cos\alpha}{c^2-v^2} - \frac{R_2 d_0}{[(d_0^2+R_2^2)(c^2-v^2)]^{1/2}}\cos(\gamma_0-\varepsilon_0-\eta-\omega t) \quad (25)$$

$$\phi_1(t,\gamma_0) \cong \frac{R_1 d_0}{[(d_0^2+R_1^2)(c^2-v^2)]^{1/2}}\sin(\gamma_0-\varepsilon_0-\omega t) - \frac{vR_1}{c^2-v^2}\left[1-\left(\frac{d_0^2+R_1^2}{c^2-v^2}\right)^{1/2}\left(1-\left(\frac{v}{c}\right)^2\right)^{-1} t\right] \cdot\sin(\gamma_0-\varepsilon_0+\alpha-\omega t) \quad (26)$$

$$\phi_2(t,\gamma_0) \cong \frac{R_2 d_0}{[(d_0^2+R_2^2)(c^2-v^2)]^{1/2}}\sin(\gamma_0-\varepsilon_0-\eta-\omega t) - \frac{vR_2}{c^2-v^2}\left[1-\left(\frac{d_0^2+R_2^2}{c^2-v^2}\right)^{1/2}\left(1-\left(\frac{v}{c}\right)^2\right)^{-1} t\right] \cdot\sin(\gamma_0-\varepsilon_0+\alpha-\eta-\omega t) \quad (27)$$

In Eq. (26), the factor in front of sin($\gamma_0-\varepsilon_0+\alpha-\omega t$) is of the order 10^{-7} while the factor in front of sin($\gamma_0-\varepsilon_0-\omega t$) is of the order 10^{-2}. Hence, the second term in Eq. (26) can be neglected expect when sin($\gamma_0-\varepsilon_0+\alpha-\omega t$) is sufficiently larger than $\sin(\gamma_0-\varepsilon_0-\omega t)$. In an extreme case $\gamma_0-\varepsilon_0=0^\circ$ and α =90 °, the first term in Eq. (26) is zero at t=0. However, as t deviates sufficiently from 0, the first term will again dominate the second term. For example, if $|t|$=10sec, the first term is 10 times the second in Eq. (26). So, in instances when observation time interval T_1+T_2 is much larger 10 sec, the contribution of the second term in Eq. (26) to the Cramer-Rao lower bound will be negligibly small. The same conclusion can be drawn for the second term in Eq. (29). Hence, neglecting these terms results in the following approximation to the Cramer-Rao lower bound (Eq. (5)):

$$E\left[(\hat{\gamma}_0-\gamma_0)^2\right] \ge \left[\frac{2R_1^2 d_0^2}{N_1 c^2(d_0^2+R_1^2)}\int_{-T_1}^{T_2}\dot{s}^2\left(t-\phi_1(t,\gamma_0)\right)\sin^2(\gamma_0-\varepsilon_0-\omega t)dt + \frac{2R_2^2 d_0^2}{N_2(c^2-v^2)(d_0^2+R_2^2)}\int_{-T_1}^{T_2}\dot{s}^2\left(t-\phi_2(t,\gamma_0)\right)\sin^2(\gamma_0-\varepsilon_0-\eta-\omega t)dt\right]^{-1} \quad (28)$$

where ϕ_1 and ϕ_2 are given by Eqs. (24) and (25) respectively.

Let us now compare Eq. (28) with Eq. (22) when $\omega=v=0$. From Eqs. (24) and (25) it can be seen that the dependence of ϕ_1 and ϕ_2 on γ_0 is small. We may assume that ϕ_1 and ϕ_2 are both essentially independent of γ_0 in Eq. (28). So from the viewpoint of dependence on γ_0, the essential difference in the structure of Eq. (28) to the structure of Eq. (22) is the $\sin^2(\gamma_0-\varepsilon_0-\omega t)$ and $\sin^2(\gamma_0-\varepsilon_0-\eta-\omega t)$ factors in the integrands in Eq. (28) versus the corresponding $\sin^2(\gamma_0-\varepsilon_0)$ and $\sin^2(\gamma_0-\varepsilon_0-\eta)$ factors in Eq. (22). Although the earth rotational angular velocity $\omega=7.27\times10^{-5}$ rad/sec, for t=30 minutes $\omega t=7.5^\circ$. Hence this difference is certainly not negligible. This points out a significant contribution to the estimation performance due to the rotation of the stations.

To assess the dependence of Eq. (28) on the angular position γ_0 we assume that ϕ_1 and ϕ_2 are essentially independent of γ_0 in Eq. (28). Under the assumption that $N_1=N_2$ and $R_1=R_2$, the integrand in Eq. (28) is directly proportional to

$$\sin^2(\gamma_0-\varepsilon_0-\omega t)+\sin^2(\gamma_0-\varepsilon_0-\eta-\omega t) = 1-\cos\eta\cos[2(\gamma_0-\varepsilon_0-\omega t)-\eta] \quad (29)$$

Comparing Eq. (29) to Eq. (23), we see that to a first order approximation, the conclusions regarding the dependence of performance on angular position $\gamma_0-\varepsilon_0$ in the case $\omega=v=0$ still hold here. In particular, it is clear from Eq. (29) that from a viewpoint of uniformity of performance over a range of γ_0, angular position near $\eta=90^\circ$ are desirable.

Finally, let us consider the effect of varying the observation duration T_1+T_2 on the optimum attainable estimation performance. We assume that T_1+T_2 is large compared to 10 seconds and that the other parameters are given as in Eq. (17). Then Eq. (28) is again a varid approximation of Eq. (5) with ϕ_1 and ϕ_2 approximated by Eqs. (24) and (25), respectively. We also assume that the frequency of the ranging signal $s(t)$ is much higher than $1/(T_1+T_2)$ and also much higher than $\omega/2\pi$ (ω=rotational angular velocity of the stations). It is still difficult to assess the dependence of Eq. (28) on T_1 and T_2

in general because of the $\sin^2(\gamma_0-\varepsilon_0-\omega t)$ and $\sin^2(\gamma_0-\varepsilon_0-\eta-\omega t)$ terms in the integrals in Eq. (28). These terms change the value of the integrands as T_1 and T_2 are varied. To a first order approximation it appears that right-hand side of Eq. (28) is iversely proportional to

$$(T_1+T_2)\Big[\sin^2(\gamma_0-\varepsilon_0-\omega T_2)-\sin^2(\gamma_0-\varepsilon_0+\omega T_1)$$
$$+\sin^2(\gamma_0-\varepsilon_0-\eta-\omega T_2)-\sin^2(\gamma_0-\varepsilon_0-\eta+\omega T_1)\Big]$$
$$=(T_1+T_2)\{2-\cos\eta\cos[2(\gamma_0-\varepsilon_0-\omega T_2)-\eta]$$
$$-\cos\eta\cos[2(\gamma_0-\varepsilon_0+\omega T_1)-\eta]\}$$

In the case when $\eta=90$ (deg.), Eq. (30) reduces to $2(T_1+T_2)$. Thus, when $\eta\cong 90$ (deg.), the optimum attainable root mean square error performance is approximately inversely proportional to $\sqrt{(T_1+T_2)}$. Finally, we consider a specific ranging signal $s(t)$ and perform numerical computations of the Cramer-Rao lower bound.

5. RESULT

Consider a sinusoidal ranging signal of frequency f_c Hz. That is ,

$$s(t)=\sqrt{2S}\cos(2\pi f_c t)$$

Assume that for I=1,2, the demodulated ranging signal power to noise spectral density ratio is

$$S/N_i=10 \text{ dB}$$

This signal-to-noise ratio is consistent with X-band carrier, 20-dB spacecraft antenna gain and about 50 dB station antenna gains. And also, assume that

$$d=9\times10^8 \quad km$$
$$R_1=R_2=6.5\times10^3 \quad km$$
$$v=10 \quad km/sec$$

These parameters are consistent with that encountered in a Venus mission with ground-based stations. We also assume that $T_1=T_2$. Numerical Monte Carlo integration was used to compute the values of signal frequency f_c and observation time duration T_1+T_2. The numerical computations are within a 1% accuracy.

The numerical results are summarized in Table 1 and 2 below.

The listed angle estimation accuracies in these tables are the square root of the Cramer-Rao lower bound.

Table 1 shows that the optimum angle estimation accuracy is inversely proportional to the frequency of the sinusoidal ranging. Signal. Although this particular relation between estimation accuracy and signal frequency does not hold in general, it can be easily seen from Eq. (5) that signals of higher frequency give a smaller Cramer-Rao bound. Also note that Table 2 shows that the estimation accuracy is approximately inversely proportional T_1+T_2, as we would expect, since $\eta=90^\circ$.

Table 1 Estimation accuracy vs signal frequency

Signal frequency(fc)	Optimum angle est. accuracy
2 MHz	0.0180 μrad
5MHz	0.0072 μrad
10 MHz	0.0036 μrad
20 MHz	0.0018 μrad

Table 2 Estimation accuracy vs observation time

Obs. Time duration (T)	Optimum angle est. accuracy
10 min	0.046 μrad
30 min	0.026 μrad
60 min	0.018 μrad
90 min	0.015 μrad

Note that the above angle estimation accuracy was obtained usin the Cramer-Rao lower bound, Eq. (5) which is valid when γ_0 is an unknown but nonrandom parameter. Suppose instead that γ_0 is a random parameter and can a priori be assumed to be normally distributed. Then the relevant lower bound on mean square estimation error is Eq. (7). We claim that if the a priori variance of γ_0 is much larger than the lower bound Eq. (5), then the above estimation accuracy calculation is still valid. This follows because Eq. (5) is essentially independent of γ_0, in this case since $\eta=90^\circ$.

Hence, the expectation term in Eq. (7) is 1/(lower bound Eq. (5)).

6. CONCLUSION

This work has considered the problem of estimating the angular position of a spacecraft using two rotating stations. The optimum attainable angle mean square estimation error was derived along with an implementable suboptimal estimation algorithm.

A situation comparable to that encountered in a Venus mission was futher analyzed. In this situation it was shown that optimum angle between the two stations from a viewpoint of uniformity of estimation performance is 90°.

It was also shown that the optimum attainable estimation accuracy varies inversely with the distance of the stations from geocenter and approximately inversely with the square root of the observation time duration.

The optimum attainable angular estimation accuracy was numerically computed for a sinusoidal ranging signal.

These computations show that the optimum attainable estimation accuracy is 0.02 μrad for a 2Mhz signal and an observation time of one hour.

REFERENCE

[1] T. P. McGarty, Stochastic Systems and State estimation, John Wiley and Sons, New York, 1974.

[2] A. H. Jaswinski, Stochastic Processes and Filtering Theory, Academic Press, New York, 1970.