Optimal Trajectory Design Using Attractive Set on Elliptical Orbit

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Abstract

We propose an optimal trajectory design method in boundary value problem connecting two points on an elliptical orbit. In Tschauner-Hempel equation, Attractive Set for optimal control based on the linear quadratic regulator theory is considered. Attractive Set is defined as a set of all initial states for reaching a desired state with respect to a performance index. Therefore, when an error is added to the velocity of the initial point, the characteristics of the performance index for transitioning to the state of the original terminal point are clarified. This research is applied to the estimation of the error correction amount of the trajectory transition to leave the moon and re-encounter the moon.

1. Introduction

Recently, a trajectory of spacecraft is sometimes designed by the re-encounter problem of a target celestial body and a spacecraft. Here, the target celestial body refers to a celestial body orbiting around a central celestial body. When the Earth orbiting around the Sun is considered, the Earth gravity assist method called EDVEGA was used in Hayabusa2 mission.[1] This is a problem of trajectory design to re-encounter the Earth after launched. Also, recently, the missions using the multiple lunar gravity assist such as DESTINY+, EQUULEUS have been considered. These trajectories are designed by continuously connecting the orbits to re-encounter on the Moon[2].[3] The trajectory used in re-encounter problem can be solved by the Lambert problem if assuming Keplerian dynamics with the central celestial body.[4] The solution of the Lambert problem gives velocities at the initial point and the final point uniquely. However, the initial velocity calculated by Lambert problem sometimes cannnot be used depending on launch conditions and gravity assist conditions. Especially, to find a trajectory using the multiple lunar gravity assist that satisfies the patched conics method is difficult.[5] If the transfer to the final state generated by the Lambert trajectory has to be achieved, the modification of the trajectory is necessary during flight.

The relative motion of a satellite (follower) with respect to the reference satellite (leader) in a circular orbit is described by autonomous nonlinear differential equations. The linearized equations at the origin are known as Hill-Clohessy-Wiltshire (HCW) equations.[6] The in-plane motion and the out-of plane motion are independent. The latter is a simple sinusoidal motion. If the reference orbit is elliptic, the equations of relative motion involve the true anomaly and the radius of the orbit, which are periodic functions. The linearized equations of motion at the origin are known as Tschauner-Hempel (TH) equation.[7] In many previous study, the characteristics of the formation flight of the reference satellite and the spacecraft and the solution of the encounter problem are clarified by solving the optimal control problem in the TH equation[8].[9]

The trajectory of the spacecraft which has the different initial velocity from the Lambert trajectory can be considered as the relative motion with respect to the Lambert trajectory. Furthermore, the motion of the spacecraft can be expressed as TH equations by linearizing around the Lambert trajectory. Under linearized assumptions and a quadratic performance index, an attractive set is considered for optimal control based on linear quadratic regulator theory.[10] The attractive set is defined as a set of all initial states that can reach the desired state with a given optimal control cost forms an ellipsoid. In the previous study, the attractive set is used for the rendezvous problem from a periodic orbit on an elliptic orbit.[11] This paper focuses on the attractive set for the velocity space. The optimal control problem is considered in the TH equation with the Lambert trajectory as the reference elliptical orbit. Characteristics of the attractive set in the velocity space is revealed and the attractive set is applied to the trajectory design.

The paper is structured as follows. In section2, the main theory including the generation of the attractive set is indicated. In section3, the characteristics of the attractive set and applications are stated as an example of a transfer from the Moon to the Moon. Section4 gives closing remarks.

2. Main Theory

In this section, the generation method of the reference elliptic orbit in order to solve the TH equation is reviewed. After that, the coordinate transformation method from the inertial coordinate system to TH coordinate system is stated and the method of generating the attractive set based on the optimal control problem in the TH equation is proposed.

2. 1 Generation of Reference Elliptic Orbit by Lambert Problem

The target body which orbits around the central body as a circular orbit is considered. It is assumed that the spacecraft re-encounters the target body after departing from the target body. If the two-body problem with the central body is assumed, the trajectory of the spacecraft becomes elliptical. Such a two-point boundary value problem can be solved by the Lambert problem. In order to be solved by the Lambert problem, the inertial coordinate system centered the central body is defined. The x-axis is the direction from the central body to the target body at the initial point, y-axis is the direction of the velocity of the target body and z-axis is the outward of the orbital plane of the target body. The arguments of the Lambert problem are the position vectors of the target body at the initial point and the final point $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, and the transfer time t between them expressed as:

$$\boldsymbol{r}_1 = [1, 0, 0]^T \tag{1}$$

$$\boldsymbol{r}_2 = [\cos\theta, \sin\theta, 0]^T \tag{2}$$

$$t = 2\pi + \theta \tag{3}$$

where, θ indicates the phase angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The position vectors are nondimensionalized so that the distance between the central body and the target body is 1, and the transfer time is nondimensionalized so that the period of revolution is 2π. By using these, the velocities of the spacecraft at initial point and final point can be obtained and the trajectory can be decided uniquely. Fig.**1** shows the trajectory generated by the Lambert problem and in this paper this trajectory is assumed as the reference elliptic orbit. The angle θ_1,θ_2 from the periapsis $\boldsymbol{r}_p$ of the reference elliptic orbit to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are given as:

$$\theta_1 = \cos^{-1}(\boldsymbol{r}_1 \cdot \boldsymbol{r}_p) \tag{4}$$

$$\theta_2 = 2\pi - \cos^{-1}(\boldsymbol{r}_2 \cdot \boldsymbol{r}_p) \tag{5}$$

By using the distance of periapsis r_p and the distance of apoapsis r_a, the semi major axis a and the eccentricity e of the reference elliptic orbit are given as:

$$a = (r_a + r_p)/2 \tag{6}$$

$$e = (r_a - r_p)/(r_a + r_p) \tag{7}$$

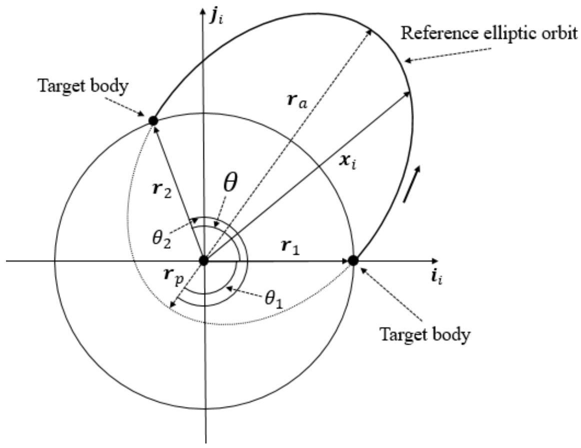

Fig. **1**: The reference elliptic orbit in the inertial coordinate system

2. 2 Equation of Motion in the TH Coordinate System

The radius of the reference elliptic orbit is $r_0 = p/(1 + e\cos\theta_0)$. Where, $p = a(1 - e^2)$ is the semilatus rectum, θ_0 is the true anomaly. The period of the orbit is $T = 2\pi(a^2/\mu)^{1/2}$. Newton's equation in the two-body problem yields:

$$\ddot{r}_0 - r_0\dot{\theta}_0^2 = -\frac{\mu}{r_0^3} \tag{8}$$

$$r_0\ddot{\theta}_0 + 2\dot{r}_0\dot{\theta}_0 = 0 \tag{9}$$

When the relative state vector of the spacecraft with respect to the reference elliptic orbit is $\boldsymbol{r}_t = [x_t, y_t, z_t, \dot{x}_t, \dot{y}_t, \dot{z}_t]^T$ is defined. This state vector is defined in the TH coordinate system. The TH coordinate system is defined as the state vector of the trajectory of the Lambert problem centered with the x-axis is the radius direction, y-axis is the flight direction and z-axis is the outward of the orbital plane of the trajectory of the Lambert Problem. Newton's equation can be divided into three equations as the following equations:

$$\begin{aligned} \ddot{x}_t - 2\dot{\theta}_0\dot{y}_t - \ddot{\theta}_0 y_t - \dot{\theta}_0^2 x_t - \frac{\mu}{r_0^2} &= -\frac{\mu}{r_t^3}(x + r_0) + u_x \\ \ddot{y}_t + 2\dot{\theta}_0\dot{x}_t + \ddot{\theta}_0 x_t - \dot{\theta}_0^2 y_t &= -\frac{\mu}{r_t^3} y_t + u_y \\ \ddot{z}_t &= -\frac{\mu}{r_t^3} z + u_z \end{aligned} \tag{10}$$

Where, $\boldsymbol{u} = [u_x, u_y, u_z]^T$ indicates inputs. The linearlized equation of Equ.(10) at the origin is given as:

$$\begin{aligned} \ddot{x}_t &= 2\dot{\theta}_0\dot{y}_t + \ddot{\theta}_0 y_t + \left(\dot{\theta}_0^2 + 2\frac{\mu}{r_0^3}x_t\right) + u_x \\ \ddot{y}_t &= -2\dot{\theta}_0\dot{x}_t - \ddot{\theta}_0 x_t - \left(\dot{\theta}_0^2 - \frac{\mu}{r_0^3}x_t\right) + u_y \\ \ddot{z}_t &= -\frac{\mu}{r_0^3} z + u_z \end{aligned} \tag{11}$$

These equations are called Tschauner-Hempel (TH) equations. The state equation of Equ.(11) is given as:

$$\dot{\boldsymbol{x}}_t = A(t)\boldsymbol{x}_t + B\boldsymbol{u} \tag{12}$$

Here, $A(t)$ and B indicate the linear time-varying system matrix and the input matrix given by:

$$A(t) = \begin{bmatrix} & O_{3\times3} & & & I_{3\times3} & \\ \dot{\theta}_0^2 + 2\frac{\mu}{r_0^3} & -2\dot{r}_0\frac{\dot{\theta}_0}{r_0} & 0 & 0 & 2\dot{\theta}_0 & 0 \\ 2\dot{r}_0\frac{\dot{\theta}_0}{r_0} & \dot{\theta}_0^2 - \frac{\mu}{r_0^3} & 0 & -2\dot{\theta}_0 & 0 & 0 \\ 0 & 0 & -\frac{\mu}{r_0^3} & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} O_{3\times3} \\ I_{3\times3} \end{bmatrix}$$

Here,

$$\dot{r}_0 = \frac{e\sin\theta_0}{\sqrt{a(1-e^2)}}$$
$$\dot{\theta}_0 = \frac{1+e\cos\theta_0}{r\sqrt{a(1-e^2)}}$$

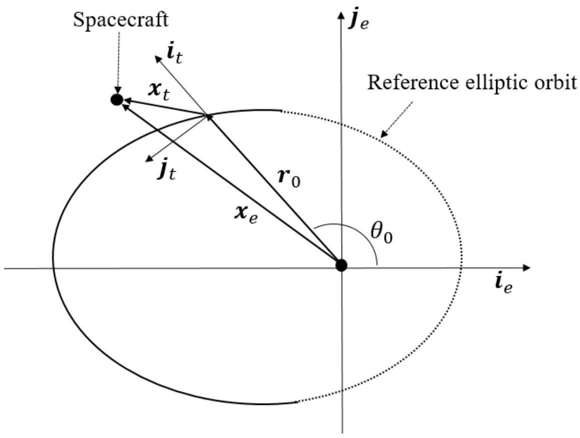

Fig. **2**: Reference Elliptic and TH coordinate system

2. 3 Coordinate Transformation

In order to consider the relative motion of the spacecraft with respect to the trajectory of the Lambert problem, it is necessary to transform the coordinate system from the inertial coordinate system to the TH coordinate system. First, let $\boldsymbol{r}_i$ and $\boldsymbol{r}_e$ be difined as the state vector, in the inertial coordinate system and in the reference elliptic coordinate system, respectively.

$$\boldsymbol{x}_i = [x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i]^T \tag{13}$$

$$\boldsymbol{x}_e = [x_e, y_e, z_e, \dot{x}_e, \dot{y}_e, \dot{z}_e]^T \tag{14}$$

The reference elliptic coordinate system is defined as the central body centered with the x-axis is the radius direction at periapsis, y-axis is the flight direction at periapsis and z-axis is the outward of the orbital plane of reference elliptic orbit. Here, the rotation matrices about the x,y and z axes by an angle θ are defined by the following equations:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{15}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{16}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

First, the transformation formula from the inertial coordinate system to the reference elliptic coordinate system is indicated as:

$$\boldsymbol{x}_e = \begin{bmatrix} R_z(-\theta_k) & O_{3\times3} \\ O_{3\times3} & R_z(-\theta_k) \end{bmatrix} \begin{bmatrix} R_x(i) & O_{3\times3} \\ O_{3\times3} & R_x(i) \end{bmatrix} \boldsymbol{x}_i \tag{18}$$

Here, $k = 1, 2$ are the indexes that indicate the initial point and the final point. When the velocity of the Lambert trajectory at the initial point in the inertial coordinate system is given as $\boldsymbol{v} = [v_x, v_y, v_z]^T$, the inclination of the reference elliptic orbit i in the inertial coordinate system is given as;

$$i = \tan^{-1}\left(\frac{v_z}{\sqrt{v_x^2 + v_y^2}}\right) \tag{19}$$

Next, the transformation formula from the reference elliptic coordinate system to the TH coordinate system is indicated as:

$$\boldsymbol{x}_t = \begin{bmatrix} R(\theta_k) & O_{3\times3} \\ O_{3\times3} & R(\theta_k) \end{bmatrix} \boldsymbol{x}_e + \omega_k \begin{bmatrix} 0 \\ 0 \\ 0 \\ -x_e\sin\theta_k + y_e\cos\theta_k \\ -y_e\sin\theta_k + x_e\cos\theta_k \\ 0 \end{bmatrix} + \begin{bmatrix} -R_{0,k} \\ 0 \\ 0 \\ -\dot{R}_{0.k} \\ 0 \\ 0 \end{bmatrix}$$

Here, ω_k indicate the magnitude of the angular velocity of the reference elliptic orbit given as:

$$\omega_k = \frac{1+e\cos\theta_k}{\sqrt{a(1-e^2)}} \tag{20}$$

2. 4 Optimal Feedback Control

The optimal input is solved as the optimal control problem with the fixed final state vector and time. In this problem, the optimal control problem for minimizing the performance index J is considered.

$$J = \int_{t_0}^{t_f} (\boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{u}^T R\boldsymbol{u}) \tag{21}$$

The optimal input for minimizing Equ.(21) is given as:

$$u^* = -R^{-1}B^T\{S(t)\boldsymbol{x}W_0^{-1}U_0^T\boldsymbol{x}_0\} \tag{22}$$

Here, S, U, W are the solution of Riccati equation given as:

$$\begin{aligned} \dot{S} &= -SA^T - SA + SBR^{-1}B^T S - Q \\ \dot{U} &= -(A^T - SBR^{-1}B^T)U \\ \dot{W} &= U^T BR^{-1}B^T U \end{aligned} \tag{23}$$

The final conditions of Equ.(23) are given as:

$$S(t_f) = O_{6\times6}, U(t_f) = I_{6\times6}, W(t_f) = O_{6\times6} \tag{24}$$

By using Equ.(22), the minimum value of the performance index is given as:

$$J^* = \boldsymbol{x}_0^T(S_0 - U_0 W_0^{-1} U_0^T)\boldsymbol{x}_0 \tag{25}$$

Here, S_0, U_0, W_0 are the solution of Riccati equation, and it depends only on the reference elliptic orbit. Therefore, the performance index is expressed as n-dimensional ellipsoid depending on the error of the initial velocity with respect to the initial velocity of the Lambert trajectory. Hence, the attractive set is given as:

$$A(C) = \{\boldsymbol{x}_0 | \boldsymbol{x}_0^T(S_0 - U_0 W_0^{-1} U_0^T)\boldsymbol{x}_0\} \leq C \tag{26}$$

The optimal trajectory departing from the inside of this ellipsoid is guaranteed that the performance index is less than C. The major axis direction of this ellipsoid is the initial velocity direction having the smallest performance index with respect to the error magnitude(optimal direction). On the other hand, the minor axis direction is the initial velocity direction with the highest performance index(worst direction). This direction is determined by the eigenvector of $S_0 - U_0 W_0^{-1} U_0^T$ calculated by Equ.(23), the eigenvector corresponding to the smallest eigenvalue is the optimal direction, the eigenvector corresponding to the largest eigenvalue is the worst direction.

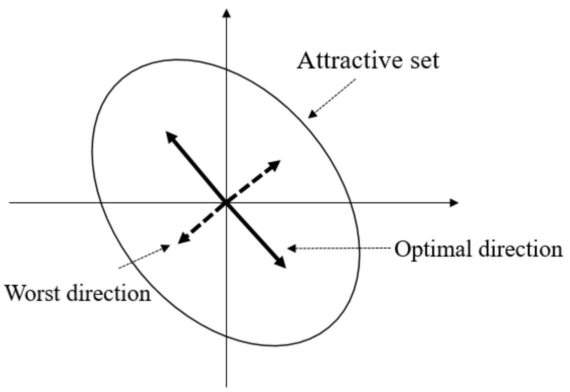

Fig. **3**: Optimal direction and worst direction

3. Analysis

3. 1 Shape of Attractive Set

The shape of the attractive set for the initial velocity of Lambert trajectory is discussed. In this analysis, the example of the phase angle is θ =120 deg is considered as the Lambert trajectory. Fig.**4**～Fig.**6** show the attractive set, the optimal direction and the worst direction as a contour, a blue arrow and a red arrow, respectively. The contour indicates the performance index. If a velocity error from the initial velocity is inside the ellipse, the trajectory can be modified below the energy indicated by the ellipse. And, the optimal direction is the major axis direction, the worst direction is the minor axis direction. The optimal direction and the worst direction exist in two opposite directions since the ellipse has symmetry. The transition of the attractive set with respect to the weight matrix Q is shown in the flow of Fig.**4** to Fig.**6**. As the weight matrix Q decreases, the shape of the attractive set extends in the major axis direction. And, the optimal direction and the worst direction are slightly changing. In order to make this transition more visible, a numerical result of the transition shown in Fig.**7** and Fig.**8**. Fig.**7** shows the transition of the angle of the optimal direction with respect to the weight matrix Q. The angle of the optimal direction is defined as the angle from the x-axis in the inertial coordinate system. As the weight matrix Q decreases, the optimal direction converges a constant value. Fig.**8** show the transition of the ratio of the maximum eigenvalue and the minimum eigenvalue of the solution of Riccati equation with respect to the weight matrix Q. As is the case with the result of the optimal direction, the ratio converges a constant value as the weight matrix Q decreases. It denotes that the shape of the attractive set is fixed by controlling to minimize energy consumption.

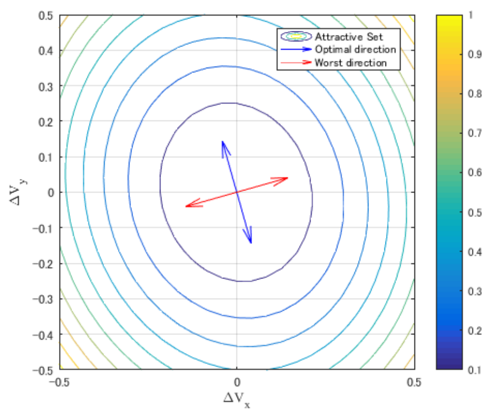

Fig. **4**: Attractive set on the xy-plane($Q = 10^0$)

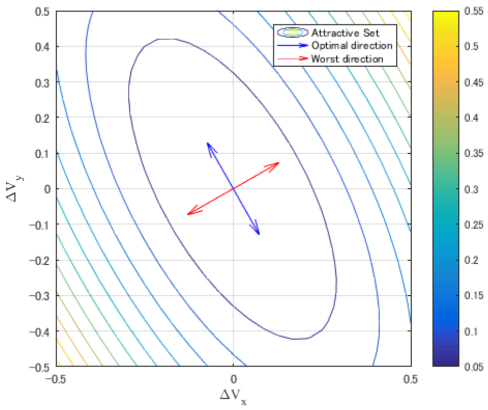

Fig. **5**: Attractive set on the zx-plane($Q = 10^{-4}$)

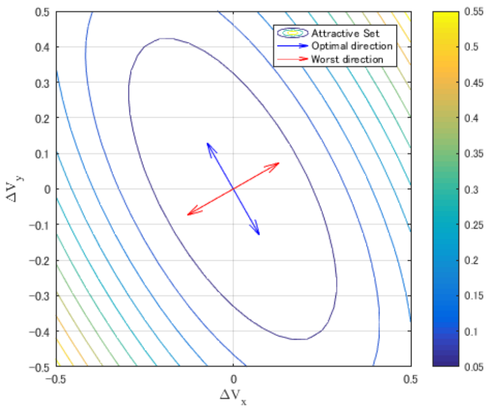

Fig. **6**: Attractive set on the zx-plane($Q = 10^{-8}$)

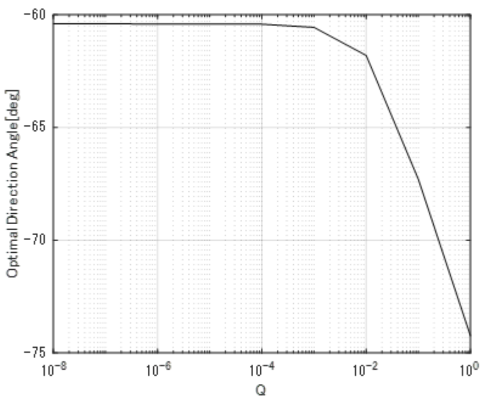

Fig. **7**: Transition of optimal direction angle with Q

3. 2 Application

In this section, the attractive set is applied to the trajectory design using the multiple lunar gravity assist (LGA). In order to design that, the patched conics method is often used. In this method, the trajectories

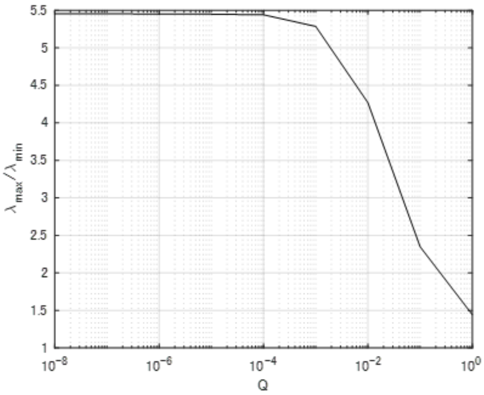

Fig. **8**: Transition of eigenvalue ratio with Q

before and after the LGA can be approximately connectable if the magnitude of the velocities of these trajectories with respect to the Moon are equal as shown in Equ.(27). Here, let $\boldsymbol{v}_\infty^-$ and $\boldsymbol{v}_\infty^+$ be the velocities before and after the LGA.

$$|\boldsymbol{v}_\infty^-| = |\boldsymbol{v}_\infty^+| \tag{27}$$

However, it is difficult to find trajectories with the same magnitude of $\boldsymbol{v}_\infty^-$ and $\boldsymbol{v}_\infty^+$. Therefore, it is assumed that two trajectories whose magnitudes of $\boldsymbol{v}_\infty^-$ and $\boldsymbol{v}_\infty^+$ are not equal are generated by the Lambert problem. Subsequently, $\boldsymbol{v}_\infty^+$ is changed so as to match the magnitude of $\boldsymbol{v}_\infty^-$. The initial velocity with the minimum energy satisfies the constraint is determined by the attractive set.

Fig.**9** show the trajectories whose magnitudes of $\boldsymbol{v}_\infty^-$ and $\boldsymbol{v}_\infty^+$ are not equal. The black lines indicate the trajectories generated by the Lambert problem. The red line indicates the orbit of the Moon. The magnitudes of $\boldsymbol{v}_\infty^-$ and $\boldsymbol{v}_\infty^+$ are 0.8215 km/s and 0.4940 km/s, respectively.

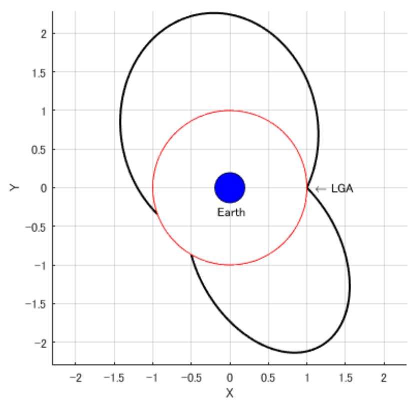

Fig. **9**: Trajectory generated by Lambert Problem

Next, the attractive set around the direction of the velocity of the trajectory after LGA is generated shown in

Fig.**10**. In Fig.**10**, the black short arrow indicates $\boldsymbol{v}_{\infty}^{+}$. The black circle indicates the magnitudes of $\boldsymbol{v}_{\infty}^{-}$. If $\boldsymbol{v}_{\infty}^{+}$ is changed to the new velocity on the black circle, the patched conics method is completed. The $\boldsymbol{v}_{\infty}^{+}$ that can transfer to the final state of the Lambert trajectory with minimum energy is selected from the black circle. The new $\boldsymbol{v}_{\infty}^{+}$ which enables the transfer with minimum energy is the point at which the smallest ellipse is in contact with the ellipse of the attractive set and the black circle. The long arrow is selected as a new initial velocity. It can minimize the energy consumption. The trajectory with the minimum energy satisfying the constraint of the magnitude of $\boldsymbol{v}_{\infty}$ is shown in Fig.**11**. It is confirmed that the trajectory after the LGA is modified in order to match the final state of the Lambert trajectory.

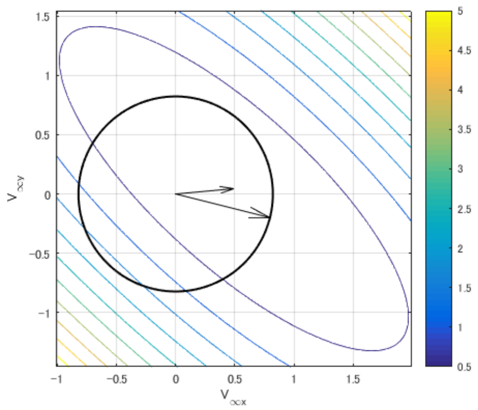

Fig. **10**: Attractive set around $\boldsymbol{v}_{\infty}^{+}$ satisfying the constraint of the magnitude of $\boldsymbol{v}_{\infty}^{-}$

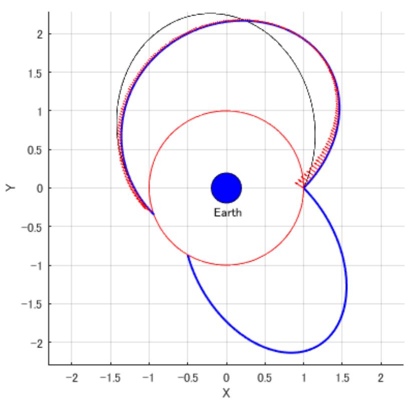

Fig. **11**: Trajectory with the minimum energy satisfying the constraint of the magnitude of $\boldsymbol{v}_{\infty}^{-}$

4. Conclusion

This paper presents the characteristics of the attractive set in the velocity space and application to the trajectory design. The Lambert trajectory is generated and the attractive set is considered around it. It is found that the shape of the attractive set is fixed when the input weight is large and the initial velocity and the worst direction are almost parallel. This method is very useful to estimate the direction of the initial velocity with minimum energy even when used under more detailed assumptions than Keplerian dynamics and the linear approximation.

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